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Wallis m-integrals and their properties

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ABSTRACT. This paper investigates asymptotic behaviour of the m-Wallis integrals
$W(m,k) = \int_0^{\frac{\pi}{2}} x^m \cos^k x \, dx$, $m \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$.

1. Introduction

The purpose of this note is to establish several properties of the family of integrals

$$W(m,k) = \int_0^{\frac{\pi}{2}} x^m \cos^k x \, dx, \; m \in \mathbb{N} \cup \{0\}, \; k \in \mathbb{N}$$

In 2007 Tewodros Amdeberhan, Luis A. Medina, and Victor H. Moll [**1**] obtained many interesting properties of the whole family as well as of its subfamilies determined by fixing the value of one of the parameters. Those properties include recurrence formulae, for $k \geqslant 3$,

$$W(0,k) = \frac{k-1}{k} W(0,k-2), \; W(1,k) = \frac{k-1}{k} W(1,k-2) - \frac{1}{k^2},$$

and, for $m \geqslant 2$, and $k \geqslant 3$,

$$W(m,k) = \frac{k-1}{k} W(m,k-2) - \frac{m(m-1)}{k^2} W(m-2,k).$$

A straightforward application of integration by parts yields to each of the above recurrences.

The members of the subfamily $W(0,k) = W_k = \int_0^{\frac{\pi}{2}} \cos^k x dx$, $k \in \mathbb{N}$, are the well-known Wallis integrals. These integrals and their properties have been studied over a long period of time [**2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 14**]. Here are three properties that easily follow from the recurrence $W_k = \frac{k-1}{k} W_{k-2}$.

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PROPOSITION 1.1 ([**1, 11, 14**]). *Let* $\{W_k : k \in \mathbb{N}\}$ *be the family of the Wallis integrals. Then:*

(1) $W_k = \begin{cases} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} = \frac{\pi}{2^{2n+1}} \cdot \binom{2n}{n} & \text{if } k = 2n \\ \frac{(2n)!!}{(2n+1)!!} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} & \text{if } k = 2n+1 \end{cases}.$

(2) $\lim_{k\to\infty} \dfrac{W_{k+1}}{W_k} = 1.$

(3) $kW_kW_{k-1} = \frac{\pi}{2}$

From Proposition 1.1 it follows that for large values of k,

$$kW_k^2 \sim kW_{k+1}W_k \sim \frac{\pi}{2}.$$

In other words:

PROPOSITION 1.2. *Let* $\{W_k : k \in \mathbb{N}\}$ *be the family of the Wallis integrals. Then:*

(1) *For large values of* k, $W_k \sim \sqrt{\dfrac{\pi}{2\,k}}$.

(2) $\lim_{k\to\infty} W_k = 0.$

In this note we show that, for a fixed $m \geqslant 1$, the family $\{W(m,k) : k \geqslant 1\}$ has similar properties. Namely, in Section 2 we prove the following two theorems:

THEOREM 1.1. *For all* $m \geqslant 1$, $\lim_{k\to\infty} W(m,k) = 0.$

THEOREM 1.2. *For all* $m \geqslant 1$, $\lim_{k\to\infty} \dfrac{W(m,k+1)}{W_k(m,k)} = 1.$

In Section 3 we prove an analog to the first statement in Proposition 1.2:

THEOREM 1.3. *For large values of* k,

$$W(2m,k) \sim \frac{(2m)!}{k^m}\sqrt{\frac{\pi}{2k}} \text{ and } W(2m+1,k) \sim \frac{(2m+1)!}{k^{m+1}}.$$

Hence the following definition:

DEFINITION 1.1. For a fixed $m \geqslant 1$, the elements of the family $\{W(m,k) : k \in \mathbb{N}\}$ are called the Wallis m-integrals.

In Section 4 we prove:

THEOREM 1.4. *For any* $m \geqslant 0$ *and* $x \in [-1,1)$

$$\sum_{k=0}^{\infty} W(m,k)x^k = \int_0^{\frac{\pi}{2}} \frac{t^m}{1 - x\cos t}\, dt.$$

In addition, for $m \geqslant 2$,

$$\sum_{k=0}^{\infty} W(m,k) = \int_0^{\frac{\pi}{2}} \frac{t^m}{1 - \cos t}\, dt.$$

2. Two convergent sequences of Wallis m-integrals

We observe that Propositions 1.1 and 1.2 establish that both the sequence of Wallis integrals $\{W_k\}_{k\in\mathbb{N}}$ and the sequence of the ratios of the consecutive Wallis integrals $\left\{\frac{W_{k+1}}{W_k}\right\}_{k\in\mathbb{N}}$ converge. We prove that corresponding sequences of the Wallis m-integrals converge too.

Proof of Theorem 1.1. Note that, for any $k,m \geqslant 1$ and $x \in \left(0, \frac{\pi}{2}\right)$, $x^m \cos^k x > 0$. Hence,

$$0 < W(m,k) = \int_0^{\frac{\pi}{2}} x^m \cos^k x\, dx \leqslant \left(\frac{\pi}{2}\right)^m \int_0^{\frac{\pi}{2}} \cos^k x\, dx = \left(\frac{\pi}{2}\right)^m W_k.$$

Since $\lim_{k\to\infty} W_k = 0$ it follows, by the Squeeze Theorem, that $\lim_{k\to\infty} W(m,k) = 0$, for any $m \geqslant 1$. □

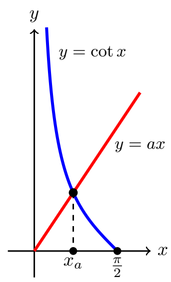
We note that, for $k \geqslant 1$,

$$\begin{aligned} 0 < \frac{W(m,k+1)}{W(m,k)} &= \frac{\int_0^{\frac{\pi}{2}} x^m \cos^{k+1} x dx}{\int_0^{\frac{\pi}{2}} x^m \cos^k x dx} = \frac{\int_0^{\frac{\pi}{2}} x^m \cos^k x \cos x dx}{\int_0^{\frac{\pi}{2}} x^m \cos^k x dx} \\ &\leqslant \frac{\int_0^{\frac{\pi}{2}} x^m \cos^k x dx}{\int_0^{\frac{\pi}{2}} x^m \cos^k x dx} = 1. \end{aligned}$$

It follows that, for a fixed $m \geqslant 0$, the sequence $\{W(m,k)\}_{k\in\mathbb{N}}$ is monotone decreasing and that the sequence $\left\{\frac{W(m,k+1)}{W(m,k)}\right\}_{k\in\mathbb{N}}$ is bounded from above by 1.

Our first step in establishing Theorem 1.2 is to find, for $m \geqslant 1$, the limit of the sequence $\left\{\frac{W(m,k)}{W(m,k-2)}\right\}_{k\geqslant 3}$. To do so we will use the following lemma:

Lemma 2.1. *Let $a > 0$ and let x_a be the unique solution of the equation $\cot x = ax$ in the interval $\left(0, \frac{\pi}{2}\right)$. Then $\lim_{a\to\infty} \sqrt{a}x_a = 1$.*



Proof. Observe that $\cos x_a = ax_a \sin x_a$ and $\lim_{a\to\infty} x_a = 0$.

From

$$\begin{aligned}\cos\left(\frac{1}{\sqrt{a}}-x_a\right) &= \sin x_a\left(ax_a\cos\left(\frac{1}{\sqrt{a}}\right)+\sin\left(\frac{1}{\sqrt{a}}\right)\right)\\ &= \frac{\sin x_a}{x_a}\cdot\left(ax_a^2\cos\left(\frac{1}{\sqrt{a}}\right)+x_a\cdot\sin\left(\frac{1}{\sqrt{a}}\right)\right),\end{aligned}$$

$\lim_{a\to\infty}\cos\left(\frac{1}{\sqrt{a}}-x_a\right)=\lim_{a\to\infty}\frac{\sin x_a}{x_a}=\lim_{a\to\infty}\cos\left(\frac{1}{\sqrt{a}}\right)=1$, and $\lim_{a\to\infty}x_a\cdot\sin\left(\frac{1}{\sqrt{a}}\right)=0$, it follows that $\lim_{a\to\infty}ax_a^2=1$.
Since $a>0$ and $x_a>0$, we conclude that $\lim_{a\to\infty}\sqrt{a}x_a=1$. □

Since the recurrence formulae for $W(1,k)$ and $W(m,k)$, $m\geqslant 2$, differ, when discussing the limit of the sequence $\left\{\frac{W(m,k)}{W(m,k-2)}\right\}_{k\geqslant 3}$ we distinguish two cases: $m=1$ and $m\geqslant 2$.
To proceed, we need two additional lemmas:

LEMMA 2.2. $\lim_{k\to\infty}\frac{1}{k^2W(1,k)}=0.$

PROOF. For $k\geqslant 1$, let $g_k(x)=x\cos^k x$. Since, for $k\geqslant 2$,

$$g_k'\left(\frac{1}{k}\right)=\sqrt{2}\cos\left(\frac{\pi}{4}+\frac{1}{k}\right)\cos^{k-1}\left(\frac{1}{k}\right)>0\ ,$$

by Lemma 2.1, the function g_k is increasing on the interval $\left[\frac{1}{k},\alpha_k\right]$, where α_k is the unique solution of the equation $\cot x=kx$ in the interval $\left(0,\frac{\pi}{2}\right)$. Hence, for $k\geqslant 2$,

$$\frac{1}{k}\cos^k\left(\frac{1}{k}\right)\leqslant g_k(x),\ x\in\left[\frac{1}{k},\alpha_k\right]$$

and consequently

$$0<\left(\alpha_k-\frac{1}{k}\right)\frac{1}{k}\cos^k\left(\frac{1}{k}\right)=\int_{\frac{1}{k}}^{\alpha_k}\frac{1}{k}\cos^k\left(\frac{1}{k}\right)\,dx\leqslant\int_0^{\frac{\pi}{2}}x\cos^k x\,dx=W(1,k)\ .$$

This, together with Lemma 2.1, implies that, for large k,

$$\begin{aligned}\frac{1}{k^2W(1,k)} &\leqslant \frac{1}{k^2}\cdot\frac{1}{\left(\alpha_k-\frac{1}{k}\right)\frac{1}{k}\cos^k\left(\frac{1}{k}\right)}=\frac{1}{k^2}\cdot\frac{k^2}{(k\alpha_k-1)\cos^k\left(\frac{1}{k}\right)}\\ &\sim \frac{1}{\left(k\cdot\frac{1}{\sqrt{k}}-1\right)\cos^k\left(\frac{1}{k}\right)}=\frac{1}{\sqrt{k}-1}\cdot\frac{1}{\cos^k\left(\frac{1}{k}\right)}\ .\end{aligned}$$

Since $\lim_{x\to\infty} \cos^x\left(\frac{1}{x}\right) = 1$, it follows that

$$0 \leqslant \lim_{k\to\infty} \frac{1}{k^2 W(1,k)} \leqslant \lim_{k\to\infty} \frac{1}{\sqrt{k}-1} \cdot \frac{1}{\cos^k\left(\frac{1}{k}\right)} = 0\ .$$

By the Squeeze Theorem, $\lim_{k\to\infty} \frac{1}{k^2 W(1,k)} = 0.$ □

LEMMA 2.3. *For any* $m \geqslant 2$,

$$\lim_{k\to\infty} \frac{1}{k^2} \cdot \frac{W(m-2,k)}{W(m,k-2)} = 0.$$

PROOF. Let $m \geqslant 2$ be fixed. Observe that for any $k \geqslant 2$

$$\begin{aligned} 0 \quad &< \quad \frac{W(m-2,k)}{W(m,k-2)} = \frac{\int_0^{\frac{\pi}{2}} x^{m-2}\cos^k x\ dx}{\int_0^{\frac{\pi}{2}} x^m \cos^{k-2} x\ dx} = \frac{\int_0^{\frac{\pi}{2}} x^{m-2}\cos^{k-2} x \cos^2 x\ dx}{\int_0^{\frac{\pi}{2}} x^m \cos^{k-2} x\ dx} \\ &\leqslant \quad \frac{\int_0^{\frac{\pi}{2}} x^{m-2}\cos^{k-2} x\ dx}{\int_0^{\frac{\pi}{2}} x^m \cos^{k-2} x\ dx} = \frac{W(m-2,k-2)}{W(m,k-2)}. \end{aligned}$$

Hence, it is enough to prove that $\lim_{k\to\infty} \frac{1}{k^2} \cdot \frac{W(m-2,k-2)}{W(m,k-2)} = 0.$

Let $f_{m,k}(x) = x^m \cos^k x,\ x \in \left[0, \frac{\pi}{2}\right]$. Then $f'_{m,k}(x) = x^{m-1}\cos^{k-1} x \cdot (m\cos x - kx\sin x)$. Since, the equation $\cot x = \frac{k}{m}x$ has a unique solution in the interval $\left(0, \frac{\pi}{2}\right)$, the function $f_{m,k}$ has a unique critical number $\alpha_{m,k} \in \left(0, \frac{\pi}{2}\right)$. Since $f_{m,k}(0) = f_{m,k}\left(\frac{\pi}{2}\right) = 0$ and since $f_{m,k}(x) > 0$ otherwise, it follows that $f_{m,k}(\alpha_{m,k}) = \alpha_{m,k}^m \cos^k \alpha_{m,k}$ is the absolute maximum value of $f_{m,k}$.

It follows that

$$W(m-2,k) = \int_0^{\frac{\pi}{2}} f_{m-2,k}(x)\ dx \leqslant \frac{\pi}{2} \cdot \alpha_{m-2,k}^{m-2} \cos^k \alpha_{m-2,k}\ .$$

We observe that, since $\frac{k}{m-2} > \frac{k}{m}$, the number $\alpha_{m-2,k}$, as the unique solution of the equation $\cot x = \frac{k}{m-2}x$, is smaller than the number $\alpha_{m,k}$, the unique solution of the equation $\cot x = \frac{k}{m}x$. This fact implies that $f_{m,k}(\alpha_{m-2,k}) < f_{m,k}(\alpha_{m,k})$.

It follows that

$$\begin{aligned} 0 \quad &< \quad (\alpha_{m,k} - \alpha_{m-2,k}) \cdot f_{m,k}(\alpha_{m-2,k}) = \int_{\alpha_{m-2,k}}^{\alpha_{m,k}} f_{m,k}(\alpha_{m-2,k})\ dx \\ &\leqslant \quad \int_0^{\frac{\pi}{2}} f_{m,k}(x)\ dx = W(m,k). \end{aligned}$$

Therefore, using the fact that $f_{m,k}(\alpha_{m-2,k}) = \alpha_{m-2,k}^m \cos^k \alpha_{m-2,k}$,

$$0 < \frac{W(m-2,k)}{W(m,k)} \leqslant \frac{\frac{\pi}{2}\cdot \alpha_{m-2,k}^{m-2}\cos^k \alpha_{m-2,k}}{(\alpha_{m,k}-\alpha_{m-2,k})\cdot \alpha_{m-2,k}^m \cos^k \alpha_{m-2,k}} = \frac{\pi}{2(\alpha_{m,k}-\alpha_{m-2,k})\cdot \alpha_{m-2,k}^2}.$$

By Lemma 2.1, for large k

$$\alpha_{m-2,k} \sim \sqrt{\frac{m-2}{k}} \text{ and } \alpha_{m,k} \sim \sqrt{\frac{m}{k}}.$$

It follows that for large k,

$$\frac{1}{(\alpha_{m,k}-\alpha_{m-2,k})\cdot \alpha_{m-2,k}^2} \sim \frac{1}{\left(\sqrt{\frac{m}{k}}-\sqrt{\frac{m-2}{k}}\right)\cdot \frac{m-2}{k}} = \frac{k^{\frac{3}{2}}}{(m-2)\cdot(\sqrt{m}-\sqrt{m-2})}.$$

Finally,

$$\begin{aligned} 0 &\leqslant \lim_{k\to\infty}\frac{1}{k^2}\frac{W(m-2,k)}{W(m,k)} \leqslant \lim_{k\to\infty}\frac{1}{k^2}\frac{\pi}{2(\alpha_{m,k}-\alpha_{m-2,k})\cdot \alpha_{m-2,k}^2} \\ &= \lim_{k\to\infty}\frac{1}{k^2}\frac{\pi k^{\frac{3}{2}}}{2(m-2)\cdot(\sqrt{m}-\sqrt{m-2})} = \frac{\pi}{2(m-2)\cdot(\sqrt{m}-\sqrt{m-2})}\lim_{k\to\infty}\frac{1}{\sqrt{k}} = 0. \end{aligned}$$

By the Squeeze Theorem, for any $m \geqslant 2$, $\lim_{k\to\infty}\frac{1}{k^2}\frac{W(m-2,k)}{W(m,k)} = 0$. □

PROPOSITION 2.1. *For any* $m \geqslant 1$*,* $\lim_{k\to\infty}\frac{W(m,k)}{W(m,k-2)} = 1$.

PROOF. If $m = 1$ then from $W(1,k) = \frac{k-1}{k}W(1,k-2) - \frac{1}{k^2}$ it follows, by Lemma 2.2,

$$\lim_{k\to\infty}\frac{W(1,k)}{W(1,k-2)} = \lim_{k\to\infty}\left(\frac{k-1}{k} - \frac{1}{k^2 W(1,k-2)}\right) = 1.$$

For $m \geqslant 2$, from $W(m,k) = \frac{k-1}{k}W(m,k-2) - \frac{m(m-1)}{k^2}W(m-2,k)$ it follows that

$$\frac{W(m,k)}{W(m,k-2)} = \frac{k-1}{k} - \frac{m(m-1)}{k^2}\frac{W(m-2,k)}{W(m,k-2)}.$$

By Lemma 2.3, $\lim_{k\to\infty}\frac{W(m,k)}{W(m,k-2)} = 1$.

□

PROOF OF THEOREM 1.2. Since the sequence $\{W(m,k)\}_{k\in\mathbb{N}}$ is monotone decreasing, it follows that, for $k \geqslant 1$, $W(m,k+2) \leqslant W(m,k+1) \leqslant W(m,k)$. In particular this, together with Proposition 2.1, implies that,

$$1 = \lim_{k\to\infty}\frac{W(m,k+2)}{W(m,k)} \leqslant \lim_{k\to\infty}\frac{W(m,k+1)}{W(m,k)} \leqslant 1\ .$$

Therefore, $\lim_{k\to\infty} \frac{W(m,k+1)}{W(m,k)} = 1$.

□

3. Wallis m-integrals and large values of k

In this section we present relatively simple functions that approximate the Wallis m-integrals $W(m,k)$ for large values of k. Our main tool will be the recurrence formulae for $W(m,k)$, $m,k \geqslant 1$.
We distinguish two cases based on the parity of the parameter m.

PROPOSITION 3.1. *For $m \geqslant 1$ and a large $k \in \mathbb{N}$,*

$$W(2m,k) \sim \frac{(2m)!}{k^m}\sqrt{\frac{\pi}{2k}}$$

and consequently

$$\lim_{k\to\infty} k^{m+\frac{1}{2}} W(2m,k) = (2m)!\sqrt{\frac{\pi}{2}}\,.$$

PROOF. Recall that by Proposition 2.1, for large $k \in \mathbb{N}$, $W(2m,k) \sim W(2m,k-2)$. This together with the recurrence

$$W(2m,k) = \frac{k-1}{k}W(2m,k-2) - \frac{2m(2m-1)}{k^2}W(2m-2,k)$$

implies that, for large k,

$$W(2m,k) \sim \frac{k-1}{k}W(2m,k) - \frac{2m(2m-1)}{k^2}W(2m-2,k)\,.$$

Hence, for large values of k,

$$\begin{aligned} W(2m,k) &\sim \frac{2m(2m-1)}{k}W(2m-2,k) \sim \frac{2m(2m-1)}{k}\frac{(2m-2)(2m-3)}{k}W(2m-4,k) \\ &\sim \cdots \sim \frac{(2m)!}{k^m}W_k\,. \end{aligned}$$

From $W_k \sim \sqrt{\frac{\pi}{2\,k}}$, it follows that $W(2m,k) \sim \frac{(2m)!}{k^m}\sqrt{\frac{\pi}{2\,k}}$.
Consequently,

$$\lim_{k\to\infty} k^{m+\frac{1}{2}} W(2m,k) = (2m)!\sqrt{\frac{\pi}{2}}\,.$$

□

MAPLE confirms the conclusion of Proposition 3.1 and provides an additional insight what happens when $2m$ and k change:

$\frac{(2m)!}{k^m}\sqrt{\frac{\pi}{2k}} - W(2m,k)$

	$k = 10$	$k = 100$	$k = 1000$
$2m = 2$	0.04422015870	0.001268861966	0.00003968277699
$2m = 8$	1.596101060	0.00005041239505	$1.593886245 \cdot 10^{-9}$
$2m = 14$	3455.160227	0.0001092616476	$3.455155769 \cdot 10^{-12}$
$2m = 20$	$9.642386939 \cdot 10^{7}$	0.003049190480	$9.642386937 \cdot 10^{-14}$

PROPOSITION 3.2. *For $m \geqslant 1$ and a large $k \in \mathbb{N}$,*

$$W(2m+1,k) \sim \frac{(2m+1)!}{k^{m+1}}$$

and consequently

$$\lim_{k\to\infty} k^{m+1} W(2m+1,k) = (2m+1)!\,.$$

PROOF. Recall that by Proposition 2.1, for large $k \in \mathbb{N}$, $W(2m+1,k) \sim W(2m+1,k-2)$. This together with the recurrence

$$W(2m+1,k) = \frac{k-1}{k} W(2m+1,k-2) - \frac{(2m+1)(2m)}{k^2} W(2m-1,k)$$

implies that, for large k,

$$W(2m+1,k) \sim \frac{k-1}{k} W(2m+1,k) - \frac{(2m+1)(2m)}{k^2} W(2m-1,k)\,,$$

which implies

$$\begin{aligned} W(2m+1,k) &\sim \frac{(2m+1)(2m)}{k} W(2m-1,k) \sim \frac{(2m+1)(2m)}{k}\frac{(2m-1)(2m-2)}{k} W(2m-3,k) \\ &\sim \cdots \sim \frac{(2m+1)!}{k^m} W(1,k)\,. \end{aligned}$$

Since, from $W(1,k) = \frac{k-1}{k} W(1,k-2) - \frac{1}{k^2}$ and, for large k, $W(1,k) \sim W(1,k-2)$, it follows that $W(1,k) \sim \frac{1}{k}$. Hence, $W(2m+1,k) \sim \frac{(2m+1)!}{k^{m+1}}$.
Consequently,

$$\lim_{k\to\infty} k^{m+1} W(2m+1,k) = (2m+1)!\,.$$

□

MAPLE confirms the conclusion of Proposition 3.2 and provides an additional insight what happens when $2m+1$ and k change:

$\frac{(2m+1)!}{k^{m+1}} - W(2m+1,k)$

	$k=10$	$k=100$	$k=1000$
$2m+1=3$	0.04354944490	0.0004039522813	$4.003995203 \cdot 10^{-6}$
$2m+1=9$	3.627293538	0.00003625323313	$3.624998186 \cdot 10^{-10}$
$2m+1=15$	13076.74293	0.0001307673859	$1.307673738 \cdot 10^{-12}$
$2m+1=21$	$5.109094217 \cdot 10^{8}$	0.005109094217	$5.109094217 \cdot 10^{-14}$

We observe that the claim of Theorem 1.3 summarizes Propositions 3.1 and 3.2.

4. Sums of Wallis m-integrals

For $m \geqslant 1$ we define $W(m,0) = \int_0^{\frac{\pi}{2}} x^m dx = \dfrac{\pi^{m+1}}{2^{m+1}(m+1)}$.

We note that for $x \in (-1,1)$ and $t \in \left[0, \frac{\pi}{2}\right]$, $0 \leqslant |x\cos t| \leqslant |x| < 1$. It follows that the geometric series $\sum_{k=0}^{\infty} x^k \cos^k t$ converges for all $x \in (-1,1)$ and $t \in \left[0, \frac{\pi}{2}\right]$ and that

$$\sum_{k=0}^{\infty} x^k \cos^k t = \frac{1}{1 - x\cos t}.$$

PROOF OF THEOREM 1.4. For $m \geqslant 0$ and $x \in (-1,1)$,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{t^m}{1 - x\cos t} dt &= \int_0^{\frac{\pi}{2}} t^m \left(\sum_{k=0}^{\infty} x^k \cos^k t \right) dt = \sum_{k=0}^{\infty} \left(\int_0^{\frac{\pi}{2}} t^m \cos^k t \, dt \right) x^k \\ &= \sum_{k=0}^{\infty} W(m,k) x^k . \end{aligned}$$

Since $\lim_{k\to\infty} W(m,k) = 0$ and $0 < W(m,k+1) < W(m,k)$, by the Alternating Series Test, the series $\sum_{k=0}^{\infty} (-1)^k W(m,k)$ is convergent. By Abel's Theorem **[12]**

$$\sum_{k=0}^{\infty} (-1)^k W(m,k) = \int_0^{\frac{\pi}{2}} \frac{t^m}{1 + \cos t} dt.$$

We observe that by Theorem 1.3, for large k,

$$kW(2m,k) \sim \frac{(2m)!}{k^{m-1}} \sqrt{\frac{\pi}{2k}} \quad \text{and} \quad kW(2m+1,k) \sim \frac{(2m+1)!}{k^m}.$$

It follows that, for $m \geqslant 2$, $\lim_{k\to\infty} kW(m,k) = 0$. By Tauber's criterion [**7, 15**], the series $\sum_{k=0}^{\infty} W(m,k)$ is convergent. By Abel's Theorem, for $m \geqslant 2$,

$$\sum_{k=0}^{\infty} W(m,k) = \int_0^{\frac{\pi}{2}} \frac{t^m}{1-\cos t}\,dt.$$

□

REMARK 4.1. We note that Propositions 1.1 and 3.2, together with the fact that the series $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k}}$ and $\sum_{k=0}^{\infty} \frac{1}{k}$ diverge, imply that both series $\sum_{k=0}^{\infty} W(0,k) = \sum_{k=0}^{\infty} W_k$ and $\sum_{k=0}^{\infty} W(1,k)$ diverge. In the view of Theorem 1.4, this is in the agreement with the fact that neither the integral $\int_0^{\frac{\pi}{2}} \frac{dt}{1-\cos t}$ nor the integral $\int_0^{\frac{\pi}{2}} \frac{tdt}{1-\cos t}$ exists.

5. Conclusion

In this article we have demonstrated that, for $m \geqslant 1$, the Wallis m-integrals posses properties that are analogue to some of of the basic properties of the Wallis integrals.This extends the list of properties of the Wallis m-integrals established by Amdeberhan, Medina, and Moll in [**1**].
To underline the connection between the Wallis m-integrals and the Wallis integrals, we reformulate part of one of, in our view, main results obtained in [**1**].

THEOREM 5.1 ([**1**], Theorem 2.11). *For* $m,k \in \mathbb{N}$,

$$W(m,2k) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} a_{m,k,j}\pi^{m+1-2j} + \delta_{odd,m} \cdot a^*_{m,k},$$

where

$$a_{m,k,j} = \frac{(-1)^j m! W_{2k}}{\pi 2^m (m+1-2j)!} \sum_{1\leqslant i_1 \leqslant \cdots \leqslant i_j \leqslant k} \frac{1}{i_1^2 \cdots i_j^2},$$

$\delta_{odd,m}$ *is Kronecker's delta function at the odd integers, and*

$$a^*_{m,k} = \frac{(-1)^{\lfloor \frac{m}{2} \rfloor} 2m! W_{2k}}{\pi} \sum_{1\leqslant i_1 \leqslant \cdots \leqslant i_m \leqslant k} \frac{1}{i_1^2 \cdots i_m^2} \sum_{j=1}^{i_m} \frac{(2j+1)W_{2j+1}}{j^2}.$$

A similar expression is established for $W(m,2k+1)$. In other words, Amdeberhan, Medina, and Moll showed that $W(m,k)$ may be expressed as polynomial in π with rational coefficients.

In particular they showed that

$$W(1,2k)=\frac{W_{2k}}{2\pi}\cdot\left(\frac{\pi^2}{2}-\sum_{j=1}^{k}\frac{2^{2j}}{j^2\binom{2j}{j}}\right)$$

and

$$W(1,2k+1)=W_{2k+1}\cdot\left(\frac{\pi}{2}-\sum_{j=1}^{k}\frac{\binom{2j}{j}}{2^{2j}(2j+1)}\right).$$

Amdeberhan, Medina, and Moll [**1**] note "a classical result:"

$$\frac{\pi^2}{2}=\sum_{j=1}^{\infty}\frac{2^{2j}}{j^2\binom{2j}{j}}\text{ and }\frac{\pi}{2}=\sum_{j=1}^{\infty}\frac{\binom{2j}{j}}{2^{2j}(2j+1)}.$$

We observe that the recurrence $W(2,k)=\frac{k-1}{k}W(2,k-2)-\frac{2}{k^2}W_k$, together with the initial conditions $W(2,1)=\frac{\pi^2}{4}-2$ and $W(2,0)=\frac{\pi}{2}$, gives

$$W(2,2k)=2W_{2k}\cdot\left(\frac{\pi^2}{24}-\sum_{j=1}^{k}\frac{1}{(2j)^2}\right)$$

and

$$W(2,2k+1)=2W_{2n+1}\cdot\left(\frac{\pi^2}{8}-\sum_{j=0}^{k}\frac{1}{(2j+1)^2}\right).$$

Recall that

$$\sum_{j=1}^{\infty}\frac{1}{(2j)^2}=\frac{\pi^2}{24}\text{ and }\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}=\frac{\pi^2}{8}.$$

We wonder for how long, if at all, this pattern continues, i.e., if, for some $m>2$, $W(m,2k)=cW_{2k}\cdot R_k$ and/or $W(m,2k+1)=dW_{2k+1}\cdot S_k$, where c and d are constants and R_k and S_k are remainders of two series that each sums to a fraction of a power of π.

Finally, we mention that the study of the Wallis integrals is closely related to the Gamma and Betta functions [**8, 13**]. We wonder if there is a relation between the Wallis m-integrals and some of the special functions.

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