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Two Theorems with consequences for Bessel Function Integrals.

M.L. Glasser

Abstract. Differential equations satisfied by the class of Laplace transforms

$$\int_0^\infty e^{-xt}\frac{f(t)}{a^k+t^k}dt,\ k=1,2$$

are solved and applied to various Bessel and related functions.

1. Two Theorems

Theorem 1 Let $f:\mathcal{R}\to\mathcal{R}$ be any function for which the integral

$$F(x)=\int_0^\infty e^{-xt}\frac{f(t)}{a+t}dt \tag{1}$$

exists and let

$$\phi(x)=\int_0^\infty e^{-xt}f(t)dt \tag{2}$$

be its Laplace transform. Then F satisfies the first order differential equation

$$F'+aF=-\phi(x) \tag{3}$$

Therefore

$$F(x)=e^{-ax}F(0)-\int_0^x e^{-a(x-t)}\phi(t)dt \tag{4}$$

The proof, by inspection, is elementary.

Theorem 2 Let f and ϕ be as in theorem 1. Then the integral

$$G(x)=\int_0^\infty e^{-xt}\frac{f(t)}{a^2+t^2}dt \tag{5}$$

obeys the second order differential equation

$$G''+a^2G=\phi(x) \tag{6}$$

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Therefore

$$G(x) = G(0)\cos(ax) + \frac{1}{a}G'(0)\sin(ax) + \frac{1}{a}\int_0^x \sin[a(x-t)]\phi(t)dt. \tag{7}$$

The proof is simply the elementary variation of parameters applied to the homogeneous equation which has solutions $\sin(ax)$, $\cos(ax)$ with Wronskian $W = 1$.

2. Examples

Here we apply the theorems in Section 1 to the Bessel functions bounded at ∞: $J_0, Y_0, K_0, \mathbf{H}_0$ From Theorem 1 one finds

1)

$$\int_0^x e^{-a\sinh(t)}dt = \frac{\pi}{2}[\mathbf{H}_0(a) - Y_0(a)] - \int_a^\infty \frac{e^{-\sinh(x)t}}{t}J_0(t-a)dt$$

2)

$$\int_0^x te^{-a\sinh(t)}dt = C + \frac{\pi}{2}\int_a^\infty \frac{e^{-\sinh(x)t}}{t}Y_0(t-a)dt$$

$$C = \frac{1}{4}G_{13}^{31}\left(\begin{matrix} & 0 & \\ 0 & 0 & 0\end{matrix}\,\Big|\frac{a^2}{4}\right)$$

3)

$$\int_0^x te^{-a\cosh(t)}dt = C - \int_a^\infty \frac{e^{-\cosh(x)t}}{t}K_0(t-a)dt$$

$$C = \frac{1}{4\pi}G_{24}^{42}\left(\begin{matrix} & 0 & 1/2 & \\ 0 & 0 & 0 & 1/2\end{matrix}\,\Big|\frac{a^2}{4}\right)$$

4)

$$\int_x^\infty t\,\mathrm{csch}(t)e^{-a\mathrm{csch}(t)}dt = C - \frac{\pi}{2}\int_a^\infty \frac{e^{-\mathrm{csch}(x)t}}{t}\mathbf{H}_0(t-a)dt$$

$$C = \frac{1}{2a}G_{24}^{31}\left(\begin{matrix} & 1 & 1/2 & \\ 1 & 1 & 1 & 1/2\end{matrix}\,\Big|\frac{a^2}{4}\right)$$

Here one finds the relation between a finite range (in essence the antiderivative) hyperbolic exponential and an infinite Bessel integral. These can be manipulated in various ways, particularly by taking derivatives with respect to x and a, to obtain further results. From the application of Theorem 2, one obtains similar results relating Laplace transforms to finite convolutions.

5)

$$\int_0^\infty e^{-xt}\frac{J_0(t)}{t^2+a^2}dt = \frac{\cos(ax)}{2a}[\pi I_0(a) - \mathbf{L}_0(a)] - \frac{\sin(ax)}{a}K_0(a) - \frac{1}{a}\int_0^x \frac{\sin[a(x-t)]}{\sqrt{t^2+1}}dt$$

6)

$$\int_0^\infty e^{-xt}\frac{Y_0(t)}{t^2+a^2}dt = -C\sin(ax) - \frac{\cos(ax)}{a}K_0(a) - \frac{2}{\pi a}\int_0^x \frac{\sin[a(x-t)]}{\sqrt{t^2+1}}\sinh^{-1}t\,dt$$

$$C = \frac{2}{a^3}G_{24}^{31}\left(\begin{array}{cccc} & 1 & 1/2 & \\ 1 & 1 & 1 & 1/2 \end{array} \,|\frac{a^2}{4}\right)$$

7)

$$\int_0^\infty e^{-xt}\frac{K_0(t)}{t^2+a^2}dt = \frac{\pi^2}{4a}\cos(ax)[\mathbf{H}_0(a)-Y_0(a)] - \frac{\sin(ax)}{a^3}C + \frac{1}{a}\int_0^x \sin[a(x-t)]\frac{\sinh^{-1}\sqrt{t^2-1}}{\sqrt{t^2-1}}dt$$

$$C = G_{13}^{31}\left(\begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \end{array} \,|\frac{a^2}{4}\right)$$

8)

$$\int_0^\infty e^{-xt}\frac{\mathbf{H}_0(t)}{a^2+t^2}dt = -\frac{\cos(ax)}{4\pi}C - \frac{\pi}{2a}[I_0(a)-\mathbf{L}_0(a)]\sin(ax) + \frac{2}{\pi a}\int_0^x \frac{\sin[a(x-t)]]}{\sqrt{t^2+1}}\ln[(1+\sqrt{t^2+1})/t]dt$$

$$C = Re\left[G_{24}^{32}\left(\begin{array}{cccc} & 0 & 0 & \\ -1/2 & 0 & 0 & -1/2 \end{array} \,|\frac{1+ia}{2}\right)\right]$$

3 Discussion

These formulas have been obtained from a short Mathematica algorithm, so the selection of examples was limited by the mathematical functions available in that system. An examination of the tables [1,2] turned up only a few special cases as Stieltjes transforms.

Although the procedures used here rely on elementary considerations, they can be useful in more sophisticated circumstances as illustrated by the following example. One obtains from Theorem 1 with $f(t) = W_{0,k}(t)$

9) For $|k| < 3/2$,

$$\int_0^\infty \frac{(1-e^{-(a+t)x})}{a+t}W_{0,k}(t)dt = \frac{\pi}{4}\frac{1-4k^2}{\cos(\pi k)}\int_0^x e^{-as}\,{}_2F_1(3/2-k, 3/2+k; 2, s-1/2)ds$$

From this one can conclude that for $s > 1/2$

$$\int_0^\infty e^{-st}W_{0,k}(t)dt = \frac{\pi}{4}\frac{1-4k^2}{\cos(\pi k)}\,{}_2F_1(3/2-k, 3/2+k, 2; 1/2-s) \qquad (8)$$

which appears to be incompatible with the value of this Laplace transform given in [2; I (4.22(16)]. Numerical evaluation suggests that (8) is correct.

References

[1] A.P.Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and Series, Special Functions,Vol 2*[Gordon and Breach Publishers, New York,1985]
[2] A.Érdelyi et al.,*Tables of Integral Transforms, Vol.1,2*, [McGraw-Hill Publishers, New York, 1954]

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Department of Physics,
Clarkson University, Potsdam, NY, U.S.A.
E-mail address: M Glasser <lglasser@clarkson.edu>