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The integrals in Gradshteyn and Ryzhik. Part 32: Powers of trigonometric functions

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve powers of trigonometrical functions. A selected sample of these entries are discussed.

1. Introduction

The table of integrals by I. S. Gradshteyn and I. M. Ryzhik [**3**] contains a large selection of integrals. The present work is part of a project dedicated to proving all these evaluations and to provide context for them. This project started with [**4**]. The entries discussed here supplement those in [**1**] and [**2**].

The basic trigonometric functions $\cos x$ and $\sin x$ are encountered in the elementary courses. The obey the differentiation rules

$$\frac{d}{dx}\cos x = -\sin x \quad \text{and} \quad \frac{d}{dx}\sin x = \cos x. \tag{1.1}$$

Most of the entries established here follow from these two identities. Among the other identities employed in the proofs are the duplication formulas

$$\sin 2x = 2\sin x\,\cos x \quad \text{and} \quad \cos 2x = \cos^2 x - \sin^2 x \tag{1.2}$$

and the extensions to higher multiple angles. Naturally, the fundamental rule $\cos^2 x + \sin^2 x = 1$ is often used without mentioning it.

In this paper the integrand has the form $(\cos x)^a\,(\sin x)^b$ with $a,\, b \in \mathbb{Z}$. The entries evaluated here are those for which an explicit solution can be achieved. The table also contains a variety of recurrences and parametric finite sums. These will be analyzed in a future publication.

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2. **Section** 2.513

The entries evaluated in this section can be reduced to primitives of pure powers of sin or $\cos x$. All the examples given below can be obtained from the fact that such a power is a linear combination of sines and cosines of multiple angles. The identities

$$\sin^{2n} x = \frac{1}{2^{2n}} \left\{ 2 \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \tag{2.1}$$

and

$$\sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{2n-1}{k} \sin(2n-2k-1)x, \tag{2.2}$$

with similar expressions for powers of cosine. These identities have appeared in [**1**] and they will be analyzed in a future publication.

2.1. **Entry** 2.513.5.

$$\int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x \tag{2.3}$$

Proof. Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to obtain

$$\int \sin^2 x \, dx = \frac{1}{2} \left(\int 1 \, dx - \int \cos 2x \, dx \right) = -\frac{1}{4} \sin 2x + \frac{x}{2}. \tag{2.4}$$

The first formulation follows from $\sin(2x) = 2 \sin x \cos x$. $\square$

2.2. **Entry** 2.513.6.

$$\int \sin^3 x \, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x \tag{2.5}$$

Proof. Start with

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x (1 - \cos^2 x) \, dx = \int \sin x \, dx - \int \sin x \cos^2 x \, dx \\ &= -\cos x + \frac{1}{3} \cos^3 x. \end{aligned} \tag{2.6}$$

This gives the second form. To obtain the first one, use the identity

$$\cos(3x) = 4 \cos^3 x - 3 \cos x \tag{2.7}$$

which is obtained from the addition theorem (and it appears as Entry **1.335.2**). $\square$

2.3. **Entry** 2.513.7.

$$\begin{aligned} \int \sin^4 x \, dx &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \\ &= -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3x}{8} \end{aligned} \tag{2.8}$$

PROOF. Start with the relation $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3)$ which is obtained from the the double angle formulas (and it appears as Entry **1.321.3**) one obtains the first formula. The second one follows from $\sin(4x) = \cos x(4\sin x - 8\sin^3 x)$. $\square$

2.4. Entry 2.513.8**.**

$$\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos 3x - \frac{1}{80}\cos 5x \tag{2.9}$$
$$= -\frac{1}{5}\sin^4 x\cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x$$

PROOF. Integrate the identity $\sin^5 x = \frac{1}{16}\sin 5x - \frac{5}{16}\sin 3x + \frac{5}{8}\sin x$ to obtain the first expression. The second one comes by using $\cos 3x = 4\cos^3 x - 3\cos x$ and $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$. $\square$

2.5. Entry 2.513.9**.**

$$\int \sin^6 x\,dx = \frac{5x}{16} - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x \tag{2.10}$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5x}{16}$$

PROOF. Entry **1.321.5** states that

$$\sin^6 x = \frac{1}{32}\left(-\cos 6x + 6\cos 4x - 15\cos 2x + 10\right). \tag{2.11}$$

Integration produces the first answer. To obtain the second answer, write

$$\int \sin^6 x\,dx = \int \sin^4 x\,dx - \int \sin^4 x\,\cos^2 x\,dx \tag{2.12}$$

and integrate the second integral by parts:

$$\int \left(\sin^4 x\cos x\right)\,\cos x = \frac{1}{5}\sin^5 x\cos x + \frac{1}{5}\int \sin^6 x\,dx. \tag{2.13}$$

This gives

$$\int \sin^6 x\,dx = \frac{5}{6}\int \sin^4 x\,dx - \frac{1}{6}\sin^5 x\cos x. \tag{2.14}$$

The final step is to integrate $\sin^4 x$ by parts, following the same steps as above, to produce

$$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{3}{8}\cos x\sin x - \frac{1}{4}\sin^3 x\cos x. \tag{2.15}$$

Replacing this expression gives the stated formula. $\square$

2.6. Entry 2.513.10.

$$\int \sin^7 x\,dx = -\frac{35}{64}\cos x + \frac{7}{64}\cos 3x - \frac{7}{320}\cos 5x + \frac{1}{448}\cos 7x \tag{2.16}$$
$$= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x + \frac{8}{35}\cos^3 x - \frac{24}{35}\cos x$$

Proof. Integrate the relation **1.321.6**

$$\sin^7 x = \frac{1}{64}\left(-\sin 7x + 7\sin 5x - 21\sin 3x + 35\sin x\right) \tag{2.17}$$

to obtain the first answer. The second one comes by writing

$$\int \sin^7 x\,dx = \int (1-\cos^2 x)^3\,\sin x\,dx \tag{2.18}$$

and making the change of variables $u = \cos x$ to obtain

$$\int \sin^7 x\,dx = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x. \tag{2.19}$$

This can be reduced to the trigonometric form given as the second answer. □

2.7. Entry 2.513.11.

$$\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x\cos x + \frac{x}{2} \tag{2.20}$$

Proof. Integrate by parts to get

$$\int \cos^2 x\,dx = \cos x\sin x + \int \sin^2 x\,dx. \tag{2.21}$$

The result now follows by replacing $\sin^2 x$ by $1-\cos^2 x$. This gives the second formula. The first one follows from $\sin 2x = 2\sin x\cos x$. □

2.8. Entry 2.513.12.

$$\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x \tag{2.22}$$

Proof. Write the integrand as $\cos x(1-\sin^2 x)$ and make the change of integration $u = \sin x$ gives

$$\int \cos^3 x\,dx = \int (1-u^2)\,du = u - \tfrac{1}{3}u^3 \tag{2.23}$$

and this gives the first form. The second form is obtained from the relation $\sin 3x = 3\sin x - 4\sin^3 x$ (which appears as Entry **1.333.2**). □

2.9. Entry 2..

$$\int \cos^4 x\,dx = \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = \frac{3x}{8} + \frac{3}{8}\sin x\cos x + \frac{1}{4}\sin x\cos^3 x \tag{2.24}$$

PROOF. The identities

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) \quad \text{and} \quad \cos^4 x = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) \tag{2.25}$$

appear as entries **1.323.1** and **1.323.3**. Then integrate to produce the first form. The second form appears from the formulas

$$\sin 2x = 2\sin x\cos x \quad \text{and} \quad \sin 4x = \cos x(4\sin x - 8\sin^3 x), \tag{2.26}$$

appearing as entries **1.333.1** and **1.333.3**. □

2.10. Entry 2.513.14**.**

$$\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x = \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x\sin x \tag{2.27}$$

PROOF. Write the integrand as $(1-\sin^2 x)^2\cos x$ to get

$$\int \cos^5 x\,dx = \int \cos x\,dx - 2\int \sin^2 x\cos x\,dx + \int \sin^4 x\cos x\,dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x. \tag{2.28}$$

To obtain the second form of the answer write $\sin^5 x = \sin x(1-\cos^2 x)^2$ and expand. The first form comes from the second one using the expressions

$$\sin^3 x = \frac{1}{4}(-\sin 3x + 3\sin x) \quad \text{and} \quad \sin^5 x = \frac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$$

appearing as entries **1.321.2** and **1.321.4**, respectively. □

2.11. Entry 2.513.15**.**

$$\int \cos^6 x\,dx = \frac{5x}{16} + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x = \frac{5x}{16} + \frac{5}{16}\sin x\cos x + \frac{5}{24}\sin x\cos^3 x + \frac{1}{6}\sin x\cos^5 x \tag{2.29}$$

PROOF. Entry **1.323.5** gives $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$ and the first answer follows by integration. For the second form, start with

$$\int \cos^6 x\,dx = \int \cos^4 x(1-\sin^2 x)\,dx = \int \cos^4 x\,dx - \int \cos^4 x\sin^2 x\,dx. \tag{2.30}$$

The first integral was evaluated in Entry **2.513.13**. The second is obtained by integration by parts:

$$\int \left(\cos^4 x \sin x\right) \times \sin x \, dx = -\frac{1}{5}\cos^5 x \sin x + \frac{1}{5}\int \cos^6 x \, dx. \tag{2.31}$$

Now replace to obtain an equation for the integral of $\cos^6 x$. Solve to obtain the second form of the answer. □

2.12. Entry 2.513.16**.**

$$\begin{aligned} \int \cos^7 x \, dx &= \frac{35}{64}\sin x + \frac{7}{64}\sin 3x + \frac{7}{320}\sin 5x + \frac{1}{448}\sin 7x \\ &= \frac{24}{35}\sin x - \frac{8}{35}\sin^3 x + \frac{6}{35}\sin x \cos^4 x + \frac{1}{7}\sin x \cos^6 x \end{aligned} \tag{2.32}$$

Proof. Integrating the identity

$$\cos^7 x = \frac{1}{64}\left(\cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x\right) \tag{2.33}$$

(appearing as Entry **1.323.6**) gives the first answer. The second one is obtained by using

$$\begin{aligned} \sin 3x &= 3\sin x - 4\sin^3 x \\ \sin 5x &= 5\sin x - 20\sin^3 x + 16\sin^5 x \\ \sin 7x &= 7\sin x - 56\sin^3 x + 112\sin^5 x - 64\sin^7 x, \end{aligned} \tag{2.34}$$

appearing as entries **1.333.2**, **1.333.4** and **1.333.6**, respectively. □

2.13. Entry 2.513.17**.**

$$\begin{aligned} \int \sin x \cos^2 x \, dx &= -\frac{1}{4}\left(\frac{1}{3}\cos 3x + \cos x\right) \\ &= -\frac{1}{3}\cos^3 x \end{aligned} \tag{2.35}$$

Proof. The change of variables $u = \cos x$ transforms the integral to

$$\int \sin x \, \cos^2 x \, dx = -\int u^2 \, du = -\frac{1}{3}u^3 \tag{2.36}$$

and this gives the second expression. For the first one use $\cos 3x = 4\cos^3 x - 3\cos x$. □

2.14. Entry 2.513.18**.**

$$\int \sin x \cos^3 x \, dx = -\frac{1}{4}\cos^4 x \tag{2.37}$$

Proof. Let $u = \cos x$ to obtain the evaluation. □

2.15. Entry 2.513.19.

$$\int \sin x \cos^4 x \, dx = -\frac{1}{5} \cos^5 x \tag{2.38}$$

PROOF. The change of variables $u = \cos x$ gives the evaluation. □

2.16. Entry 2.513.20.

$$\int \sin^2 x \cos x \, dx = -\frac{1}{4} \left(\frac{1}{3} \sin 3x - \sin x \right) = \frac{\sin^3 x}{3} \tag{2.39}$$

PROOF. The change of variables $u = \sin x$ gives the second expression. To obtain the first one use $\sin(3x) = 3 \sin x - 4 \sin^3 x$. This last identity comes from the addition theorem for sine. □

2.17. Entry 2.513.21.

$$\int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left(\frac{1}{4} \sin 4x - x \right) \tag{2.40}$$

PROOF. Write the integrand as

$$\frac{1}{4} \sin^2(2x) = \frac{1}{8} - \frac{1}{8} \cos(4x) \tag{2.41}$$

and now integrate to produce the evaluation. □

2.18. Entry 2.513.22.

$$\begin{aligned} \int \sin^2 x \cos^3 x \, dx &= -\frac{1}{16} \left(\frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right) \\ &= \frac{\sin^3 x}{5} \left(\cos^2 x + \frac{2}{3} \right) = \frac{\sin^3 x}{5} \left(\frac{5}{3} - \sin^2 x \right) \end{aligned} \tag{2.42}$$

PROOF. Make the change of variables $t = \sin x$ and write $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\int \sin^2 x \cos^3 x \, dx = \int t^2 (1 - t^2) \, dt = \tfrac{1}{3} \sin^3 x - \tfrac{1}{5} \sin^5 x = \tfrac{1}{5} \sin^3 x \left(\tfrac{5}{3} - \sin^2 x \right). \tag{2.43}$$

This gives the last expression. The previous one follows by writing $\sin^2 x = 1 - \cos^2 x$. To obtain the first evaluation, use the relations

$$\sin^3 x = \tfrac{1}{4}(-\sin 3x + 3 \sin x) \quad \text{and} \quad \sin^5 x = \tfrac{1}{16}(\sin 5x - 5 \sin 3x + 10 \sin x) \tag{2.44}$$

which appear as entries **1.321.2** and **1.321.4**, respectively. □

2.19. Entry 2.513.23**.**

$$\int \sin^2 x \cos^4 x \, dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x \tag{2.45}$$

PROOF. Write the integrand as $(1-\cos^2 x)\cos^4 x = \cos^4 x - \cos^6 x$ and now use the expressions in entries **1.323.3** and **1.323.5**

$$\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3) \quad \text{and} \quad \cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$$

to obtain

$$\cos^4 x - \cos^6 x = -\frac{1}{32}\cos 6x - \frac{1}{16}\cos 4x + \frac{1}{32}\cos 2x + \frac{1}{16}. \tag{2.46}$$

The result follows by integration. □

2.20. Entry 2.513.24**.**

$$\int \sin^3 x \cos x \, dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{\sin^4 x}{4} \tag{2.47}$$

PROOF. Let $t = \sin x$ to obtain the second expression. The first one appears as Entry **1.321.3** as $\sin^4 x = \dfrac{1}{8}(\cos 4x - 4\cos 2x + 3)$. Recall that constants of integration are not included in the evaluations. □

2.21. Entry 2.513.25**.**

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right) \\ &= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x \end{aligned} \tag{2.48}$$

PROOF. Use $\sin^2 x = 1 - \cos^2 x$ to write the integrand as $\cos^2 x \sin x - \cos^4 x \sin x$ and integrate to produce the second answer. The first one follows from $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$ and $\cos^5 x = \frac{1}{16}(\cos 5x + 5\cos 3x + 10\cos x)$. □

2.22. Entry 2.513.26**.**

$$\int \sin^3 x \cos^3 x \, dx = \frac{1}{32}\left(\frac{1}{6}\cos 6x - \frac{3}{2}\cos 2x\right) \tag{2.49}$$

PROOF. Write the integrand as $\cos^3 x \sin x - \cos^5 x \sin x$. Now let $t = \cos x$ and integrate to produce $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x$. The result follows by using $\cos^6 x = \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10)$ and $\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3)$. As usual, constants of integration are not written. □

2.23. Entry 2.513.27**.**

$$\int \sin^3 x \cos^4 x \, dx = \frac{1}{7} \cos^3 x \left(-\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right) \tag{2.50}$$

Proof. Write the integrand as $\sin x \left(1 - \cos^2 x\right) \cos^4 x$ and make the change of variables $t = \cos x$. This gives

$$\int \sin^3 x \cos^4 x \, dx = -\int t^4 \, dt + \int t^6 \, dt = -\frac{1}{5} \cos^5 x + \frac{1}{7} \cos^7 x. \tag{2.51}$$

To match this answer with the one appearing in the table write it as

$$\tfrac{1}{7} \cos^3 x \left(\cos^4 x - \tfrac{7}{5} \cos^2 x \right). \tag{2.52}$$

Now use $\cos^2 x = 1 - \sin^2 x$ to express the previous result in terms of sine. This gives the form in the table. □

2.24. Entry 2.513.28**.**

$$\int \sin^4 x \cos x \, dx = \frac{1}{5} \sin^5 x \tag{2.53}$$

Proof. Let $u = \sin x$ to obtain the evaluation. □

2.25. Entry 2.513.29**.**

$$\int \sin^4 x \cos^2 x \, dx = \frac{x}{16} - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x \tag{2.54}$$

Proof. Start with

$$\int \sin^4 x \cos^2 x \, dx = \int \sin^4 x (1 - \sin^2 x) \, dx = \int \sin^4 x \, dx - \int \sin^6 x \, dx \tag{2.55}$$

and then use the identities

$$\begin{aligned} \sin^4 x &= \frac{1}{8} \left(\cos 4x - 4 \cos 2x + 3 \right) \\ \sin^6 x &= \frac{1}{32} \left(-\cos 6x + 6 \cos 4x - 15 \cos 2x + 10 \right). \end{aligned} \tag{2.56}$$

Replace this in (2.55) and integrate to obtain the result. □

2.26. Entry 2.513.30**.**

$$\int \sin^4 x \cos^3 x \, dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right) \tag{2.57}$$

Proof. Use $\sin^2 x = 1 - \cos^2 x$ to write the integral as

$$\int \cos^3 x \, dx - 2 \int \cos^5 x \, dx + \int \cos^7 x \, dx. \tag{2.58}$$

These integrals have been evaluated in Entries **2.513.12**, **2.513.14** and **2.513.16**, respectively. Replace their values to obtain the formula stated here. □

2.27. Entry 2.513.31**.**

$$\int \sin^4 x \cos^4 x \, dx = \frac{3x}{128} - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x \tag{2.59}$$

PROOF. Start with

$$\int \sin^4 x \cos^4 x \, dx = \int (1-\cos^2 x)^2 \cos^4 x \, dx = \int \left(\cos^4 x - 2\cos^6 x + \cos^8 x \right) \, dx$$

and use Entry **1.320.5**

$$\cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\} \tag{2.60}$$

in the special cases $2 \leqslant n \leqslant 4$

$$\begin{aligned} \cos^4 x &= \frac{1}{8}(\cos 4x + 4\cos 2x + 3) \\ \cos^6 x &= \frac{1}{32}(\cos 6x + 6\cos 4x + 15\cos 2x + 10) \\ \cos^8 x &= \frac{1}{128}(\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35) \end{aligned} \tag{2.61}$$

to conclude that

$$\cos^4 x - 2\cos^6 x + \cos^8 x = \frac{1}{128}\cos 8x - \frac{1}{32}\cos 4x + \frac{3}{128}. \tag{2.62}$$

Integrate to obtain the result. □

3. Entry 2.518

3.1. Entry 2.518.1**.**

$$\int \frac{\sin^p x}{\cos^2 x} \, dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-2} x \, dx \tag{3.1}$$

PROOF. Apply the standard method to integrate by parts and choose

$$u \, dv = \frac{\sin^p x}{\cos^2 x} \quad \text{and} \quad uv = \frac{\sin^{p-1} x}{\cos x}. \tag{3.2}$$

Divide these two relations to obtain $\dfrac{dv}{v} = \dfrac{\sin x}{\cos x} = -\dfrac{(-\sin x)}{\cos x}$ and integrate to get $v = 1/\cos x$. From here $u = (\sin x)^{p-1}$. Therefore

$$v \, du = \frac{1}{\cos x}(p-1)(\sin x)^{p-2} \cos x \, dx = (p-1)\sin^{p-2} x \, dx. \tag{3.3}$$

Integrate by parts to obtain the statement. □

4. Entry 2.523

4.1. Entry 2.523.

$$\int \frac{\cos^m x\,dx}{\sin^2 x} = -\frac{\cos^{m-1} x}{\sin x} - (m-1)\int \cos^{m-2} x\,dx \tag{4.1}$$

PROOF. In order to integrate by parts, choose u, v so that

$$u\,dv = \frac{\cos^m x\,dx}{\sin^2 x} \quad \text{and } u\,v = -\frac{\cos^{m-1} x}{\sin x}. \tag{4.2}$$

Dividing these two relations gives $\dfrac{dv}{v} = -\dfrac{\cos x}{\sin x}$. Integration gives $v = 1/\sin x$ and from here $u = -\cos^{m-1} x$. The statement now follows by integration by parts. □

5. Entry 2.526

5.1. Entry 2.526.1.

$$\int \frac{dx}{\sin x} = \ln \tan \frac{x}{2} \tag{5.1}$$

PROOF. The change of variables $u = \tan \frac{x}{2}$ (the so-called `Weierstrass substitution`) gives $\sin x = \dfrac{2u}{1+u^2}$ and $dx = \dfrac{du}{1+u^2}$.. Replace to obtain the result. □

5.2. Entry 2.526.2.

$$\int \frac{dx}{\sin^2 x} = -\cot x \tag{5.2}$$

PROOF. This follows from the elementary rule $\dfrac{d}{dx}\dfrac{\cos x}{\sin x} = -\dfrac{1}{\sin^2 x}$. □

5.3. Entry 2.526.3.

$$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\tan\frac{x}{2} \tag{5.3}$$

PROOF. Integrate by parts starting from

$$\int (\text{cosec } x)\frac{d}{dx}(-\cot x) = -\text{cosec } x \cot x - \int \cot^2 x \,\text{cosec } x. \tag{5.4}$$

Now use $\cot^2 x = \text{cosec}^2 x - 1$ to produce

$$\int \text{cosec}^3 x\,dx = -\frac{\cos x}{\sin^2 x} - \int \text{cosec}^3 x\,dx + \int \text{cosec}\, x\,dx. \tag{5.5}$$

Then use the result from Entry **2.526.1** to obtain the statement. □

5.4. Entry 2.526.4.

$$\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3}\cot x = -\frac{1}{3}\cot^3 x - \cot x \tag{5.6}$$

PROOF. Start with the rules $\text{cosec}^2 x = -\frac{d}{dx}\cot x$ and $1 + \cot^2 x = \text{cosec}^2 x$. Then

$$\int \frac{dx}{\sin^4 x} = \int \text{cosec}^4 x = -\int \text{cosec}^2 x \, d(\cot x).$$

The change of variables $u = \cot x$ gives

$$\int \frac{dx}{\sin^4 x} = -\int (1+u^2)du = -u - \frac{1}{3}u^3 \tag{5.7}$$

and this gives the second formula for the entry. To obtain from here the first relation, simply use

$$\cot^3 x = \left(1 - \sin^2 x\right)\frac{\cos x}{\sin^3 x}. \tag{5.8}$$

□

5.5. Entry 2.526.5.

$$\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3\cos x}{8\sin^2 x} + \frac{3}{8}\ln\tan\frac{x}{2} \tag{5.9}$$

PROOF. Integrate by parts with $u = 1/\sin x$ and $dv = 1/\sin^4 x$. Entry **2.526.4** gives

$$v = -\frac{\cos x}{3\sin^3 x} - \frac{2\cos x}{3\sin x} \quad \text{and } du = -\cot x \, \text{cosec}\, x \tag{5.10}$$

Therefore

$$\int \frac{dx}{\sin^5 x} = -\left(\frac{\cos x}{3\sin^4 x} + \frac{2\cos x}{3\sin^2 x}\right) + \frac{1}{3}\int \frac{\cos^2 x}{\sin^5 x}dx + \frac{2}{3}\int \frac{\cos^2 x}{\sin^3 x}dx. \tag{5.11}$$

The first integral on the right is

$$\frac{1}{3}\int \frac{\cos^2 x \, dx}{\sin^5 x} = \frac{1}{3}\int \frac{dx}{\sin^5 x} - \frac{1}{3}\int \frac{dx}{\sin^3 x}. \tag{5.12}$$

Move the new first integral to the left and evaluate the second one from Entry **2.526.4**. The second integral on the right is

$$\frac{2}{3}\int \frac{\cos^2 x}{\sin^3 x}dx = \frac{2}{3}\int \frac{dx}{\sin^3 x} - \frac{2}{3}\int \frac{dx}{\sin x}. \tag{5.13}$$

These last two integrals have been evaluated in Entry **2.526.3** and **2.526.1**, respectively. □

5.6. Entry 2.526.6**.**

$$\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\sin^5 x} - \frac{4}{15}\cot^3 x - \frac{4}{5}\cot x \tag{5.14}$$
$$= -\frac{1}{5}\cot^5 x - \frac{2}{3}\cot^3 x - \cot x$$

PROOF. The integrand is written as

$$\text{cosec}^6 x = -\text{cosec}^4 x \times \frac{d}{dx}(\cot x) = -(1+\cot^2 x)^2 \times \frac{d}{dx}(\cot x)$$

and the change of variables $u = \cot x$ gives

$$\int \frac{dx}{\sin^6 x} = -\int (1 + 2u^2 + u^4)\, du. \tag{5.15}$$

Integrating yields the second expression for the entry. Replace in here

$$\frac{\cos x}{\sin^3 x} = \cot \times \frac{1}{\sin^2 x} = \cot x(1+\cot^2 x) \tag{5.16}$$

to obtain the first form of the entry. □

5.7. Entry 2.526.7**.**

$$\int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6\sin^2 x}\left(\frac{1}{\sin^4 x} + \frac{5}{4\sin^2 x} + \frac{15}{8}\right) + \frac{5}{16}\ln\tan\frac{x}{2} \tag{5.17}$$

PROOF. Integrate by parts with $u = 1/\sin x$ and $dv = 1/\sin^6 x$. The function v has been evaluated in Entry **2.526.6**. Then the integral is reduced to the evaluation of

$$\int v\, du = \frac{1}{5}\int \frac{\cos^2 x}{\sin^7 x}\, dx + \frac{4}{15}\int \frac{\cos^4 x}{\sin^5 x} + \frac{4}{5}\int \frac{\cos^2 x}{\sin^3 x}\, dx. \tag{5.18}$$

Observe that all the powers of $\cos x$ are even. These can be reduced to powers of $\sin x$. After doing this reduction one obtains $1/5$ times the original problem (coming from the first integral). Move this unique term to the left (to get $4/5$ the original problem) to reduce the question to the evaluation of

$$\int \frac{dx}{\sin x}, \quad \int \frac{dx}{\sin^3 x} \quad \text{and} \int \frac{dx}{\sin^5 x}. \tag{5.19}$$

These appear in Entries **2.526.1**, **2.526.3** and **2.526.5**. This produces the stated result. □

5.8. Entry 2.526.8**.**

$$\int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7}\cot^7 x + \frac{3}{5}\cot^5 x + \cot^3 x + \cot x\right) \tag{5.20}$$

PROOF. Write the integrand as $\text{cosec}^8 x = -(1+\cot^2 x)^3 \dfrac{d}{dx}(\cot x)$ and then the change of variables $u = \cot x$ gives the result. □

5.9. Entry 2.526.9.

$$\int \frac{dx}{\cos x} = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \ln \cot\left(\frac{\pi}{4} - \frac{x}{2}\right) = \ln \sqrt{\frac{1+\sin x}{1-\sin x}} \tag{5.21}$$

PROOF. The change of variables $u = \tan\frac{x}{2}$ gives

$$dx = \frac{2\,du}{1+u^2} \quad \text{and} \quad \cos x = \frac{1-u^2}{1+u^2} \tag{5.22}$$

and this implies

$$\int \frac{dx}{\cos x} = \int \frac{2\,du}{1-u^2} = \int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du. \tag{5.23}$$

Integrating this last form gives

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) \tag{5.24}$$

The addition theorem for tangent shows that this is the first stated form. To match the other forms, use

$$\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} \tag{5.25}$$

and the half-angle formulas

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} \quad \text{and} \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}. \tag{5.26}$$

All the forms can be verified from here. □

5.10. Entry 2.526.10.

$$\int \frac{dx}{\cos^2 x} = \tan x \tag{5.27}$$

PROOF. This follows directly from the formula $\dfrac{d}{dx}\tan x = \sec^2 x$. □

5.11. Entry 2.526.11.

$$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{5.28}$$

PROOF. Write the integrand as $\sec x \times \sec^2 x$ and integrate by parts with $u = \sec x$ and $dv = \sec^2 x$. This gives

$$\int \sec^3 x = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx. \tag{5.29}$$

Now bring the original integral appearing on the right to the left-hand side and use the result of Entry **2.526.9**. □

5.12. Entry 2.526.12**.**

$$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}\tan x = \frac{1}{3}\tan^3 x + \tan x \tag{5.30}$$

Proof. The change of variables $u = \tan x$ gives $\int \sec^4 x\, dx = \int (1+u^2) du$ and integration produces the second expression. To obtain the first one use

$$\tan^3 x = \frac{\sin x(1-\cos^2 x)}{\cos^3 x} = \frac{\sin x}{\cos^3 x} - \tan x. \tag{5.31}$$

□

5.13. Entry 2.526.13**.**

$$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{8}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \tag{5.32}$$

Proof. In order to evaluate this entry, we produce a recurrence for the integral

$$S_n = \int \sec^n x\, dx. \tag{5.33}$$

The value S_1 appears in entry **2.526.9** and S_2 is given in entry **2.526.10**. Start with the relation

$$S_n = \int \sec^{n-2} x \times (\sec^2 x = \tan^2 x + 1)\, dx = \int \sec^{n-2} x \tan^2 x\, dx + S_{n-2}. \tag{5.34}$$

To evaluate the remaining integral, integrate by parts with $u = \sec^{n-3} x \tan x$ and $dv = \sec x \tan x$. Then $v = \sec x$ and $du = ((n-3)\sec^{n-4} x \tan^2 x + \sec^{n-1} x)dx$. Therefore $v\, du = (n-3)\sec^{n-2} x \tan^2 x + \sec^n x$. Replace in the original equation to obtain the recurrence

$$S_n = \frac{n-2}{n-1} S_{n-2} + \frac{1}{n-1}\sec^{n-2} x \tan x, \quad \text{for } n \geqslant 3. \tag{5.35}$$

For example, when $n = 4$, this recurrence gives

$$S_4 = \int \frac{dx}{\cos^4 x} = \frac{2}{3}\tan x + \frac{1}{3}\sec^2 x \tan x. \tag{5.36}$$

This confirms the value proved in Entry **2.526.12**. In the current problem, $n = 5$ and the recurrence gives

$$S_5 = \frac{3}{4} S_3 + \frac{1}{4}\sec^3 x \tan x. \tag{5.37}$$

Replacing the value of $S_3 = \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ appearing in Entry **2.526.11** confirms the stated value for S_5. □

5.14. Entry 2.526.14**.**

$$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5\cos^5 x} + \frac{4}{15}\tan^3 x + \frac{4}{5}\tan x \tag{5.38}$$
$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x$$

Proof. The recurrence (5.35) gives

$$S_6 = \int \frac{dx}{\cos^6 x} = \frac{8}{15}\tan x + \frac{4}{15}\sec^2 x \tan x + \frac{1}{5}\sec^4 x \tan x. \tag{5.39}$$

The second form of the current entry follows from here by converting the even powers of $\sec x$ into powers of $\tan x$. In order to obtain the first form, note that

$$\tan^5 x = \sin \times \frac{\sin^4 x}{\cos^5 x} = \frac{\sin x}{\cos^5 x} - 2\frac{\sin x}{\cos^3 x} + \tan x \tag{5.40}$$

and

$$\frac{\sin x}{\cos^3 x} = \frac{\sin x}{\cos x} \times \left(\sec^2 x = \tan^2 x + 1\right) = \tan^3 x + \tan x. \tag{5.41}$$

gives the first form of the evaluation. □

5.15. Entry 2.526.15**.**

$$\int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\cos^6 x} + \frac{5\sin x}{24\cos^4 x} + \frac{5\sin x}{16\cos^2 x} + \frac{5}{16}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \tag{5.42}$$

Proof. This follows from the recurrence (5.35). The details are left to the reader. □

5.16. Entry 2.526.16**.**

$$\int \frac{dx}{\cos^8 x} = \frac{1}{7}\tan^7 x + \frac{3}{5}\tan^5 x + \tan^3 x + \tan x \tag{5.43}$$

Proof. Write the integrand as $\sec^2 x \times \sec^6 x$. Then use $\sec^2 x = \tan^2 x + 1$ and expand $\sec^6 x = (\tan^2 x + 1)^3$. The change of variables $u = \tan x$ gives integrands that are polynomials in u. Integrate to obtain the result. □

5.17. Entry 2.526.17**.**

$$\int \frac{\sin x}{\cos x}\, dx = -\ln\cos x \tag{5.44}$$

Proof. Let $u = \cos x$ to obtain the evaluation. □

5.18. Entry 2.526.18**.**

$$\int \frac{\sin^2 x}{\cos x}\, dx = -\sin x + \ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{5.45}$$

Proof. Write the integrand as $\dfrac{1}{\cos x} - \cos x$ and use Entry **2.526.9** to evaluate the first integral. □

5.19. Entry 2.526.19**.**

$$\int \frac{\sin^3 x}{\cos x}\, dx = -\frac{1}{2}\sin^2 x - \ln\cos x = \frac{1}{2}\cos^2 x - \ln\cos x \tag{5.46}$$

PROOF. Write the integrand as

$$\frac{\sin^3 x}{\cos x} = \frac{\sin x(1-\cos^2 x)}{\cos x} = \frac{\sin x}{\cos x} - \sin x\cos x \tag{5.47}$$

and integrate to produce the second expression. The first one come from this one by using $\cos^2 = 1 - \sin^2 x$. □

5.20. Entry 2.526.20**.**

$$\int \frac{\sin^4 x}{\cos x}\, dx = -\frac{1}{3}\sin^3 x - \sin x + \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) \tag{5.48}$$

PROOF. Write the integrand as $1/\cos x - 2\cos x + \cos^3 x$. The first integral was evaluated in Entry **2.526.9**, the second one is elementary and the third one was evaluated in Entry **2.513.9**. □

5.21. Entry 2.526.21**.**

$$\int \sin^2 x\cos^2 x\, dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right) \tag{5.49}$$

PROOF. Write the integrand as $\frac{1}{4}\sin^2(2x)$. Now use $\sin^2(2x) = \frac{1}{2}(1-\cos(4x))$ and integrate to obtain the result. □

5.22. Entry 2.526.22**.**

$$\int \frac{\sin^2 x}{\cos^2 x}\, dx = \tan x - x \tag{5.50}$$

PROOF. Write $\dfrac{\sin^2 x}{\cos^2 x} = \tan^2 x = \sec^2 x - 1$ to obtain

$$\int \frac{\sin^2 x}{\cos^2 x}\, dx = \int \left(\sec^2 x - 1\right)\, dx = \tan x - x. \tag{5.51}$$

□

5.23. Entry 2.526.23**.**

$$\int \frac{\sin^3 x}{\cos^2 x}\, dx = \cos x + \frac{1}{\cos x} \tag{5.52}$$

PROOF. Write the integrand as

$$\frac{\sin^3 x}{\cos^2 x} = \frac{\sin x(1-\cos^2 x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} - \sin x. \tag{5.53}$$

The change of variables $u = \cos x$ evaluates the first integral. This completes the proof. □

5.24. Entry 2.526.24**.**

$$\int \sin^3 x \cos x \, dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{1}{4}\sin^4 x \tag{5.54}$$

Proof. Let $u = \sin x$ to obtain the second expression. Then use $\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2z + 3)$ (which appears as entry **1.321.3**) to obtain the first expression. Recall that constant of integration are not written. □

5.25. Entry 2.526.25**.**

$$\int \frac{\sin x \, dx}{\cos^3 x} = \frac{1}{2\cos^2 x} = \frac{1}{2}\tan^2 x \tag{5.55}$$

Proof. Write the integrand as $\dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos^2 x}$ and then let $u = \tan x$ to obtain the last expression. The first one comes from $\tan^2 x = \sec^2 x + 1$ (and recall that constants of integration are not written). □

5.26. Entry 2.526.26**.**

$$\int \frac{\sin^2 x \, dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{5.56}$$

Proof. Write the integrand as

$$\frac{\sin^2 x}{\cos^3 x} = \frac{1 - \cos^2 x}{\cos^3 x} = \frac{1}{\cos^3 x} - \frac{1}{\cos x}. \tag{5.57}$$

The first integral was evaluated in Entry **2.526.11** and the second one in Entry **2.526.9**. This completes the proof. □

5.27. Entry 2.526.27**.**

$$\int \frac{\sin^3 x}{\cos^3 x} dx = \frac{1}{2\cos^2 x} + \ln\cos x \tag{5.58}$$

Proof. Write the integrand as

$$\frac{\sin^3 x}{\cos^3 x} = \frac{\sin x(1 - \cos^2 x)}{\cos^3 x} = \frac{\sin x}{\cos^3 x} - \frac{\sin x}{\cos x}. \tag{5.59}$$

The change of variables $u = \cos x$ converts these two integrals into integrals of powers. The evaluation is finished. □

5.28. Entry 2.526.28**.**

$$\int \frac{\sin^4 x \, dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \sin x - \frac{3}{2}\ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \tag{5.60}$$

Proof. Write the integrand as

$$\frac{\sin^4 x}{\cos^3 x} = \frac{(1 - \cos^2 x)^2}{\cos^3 x} = \frac{1}{\cos^3 x} - \frac{2}{\cos x} + \cos x. \tag{5.61}$$

The integral of the first term appears in Entry **2.526.11**, the second integral is in Entry **2.526.9** and the third one is $\sin x$. □

5.29. Entry 2.526.29.

$$\int \frac{\sin x}{\cos^4 x}\, dx = \frac{1}{3\cos^3 x} \tag{5.62}$$

PROOF. The change of variables $u = \cos x$ gives the result. □

5.30. Entry 2.526.30.

$$\int \frac{\sin^2 x}{\cos^4 x}\, dx = \frac{1}{3}\tan^3 x \tag{5.63}$$

PROOF. Write the integrand as $\dfrac{\sin^2 x}{\cos^2 x} \times \dfrac{1}{\cos^2 x}$ and make the change of variables $u = \tan x$ to obtain the result. □

5.31. Entry 2.526.31.

$$\int \frac{\sin^3 x}{\cos^4 x}\, dx = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x} \tag{5.64}$$

PROOF. Write the integrand as

$$\frac{\sin^3 x}{\cos^4 x} = \frac{\sin x(1-\cos^2 x)}{\cos^4 x} = \frac{\sin x}{\cos^4 x} - \frac{\sin x}{\cos^2 x}. \tag{5.65}$$

Now let $u = \cos x$ and evaluate the resulting integrals. □

5.32. Entry 2.526.32.

$$\int \frac{\sin^4 x}{\cos^4 x}\, dx = \frac{1}{3}\tan^3 x - \tan x + x \tag{5.66}$$

PROOF. Write the integral as

$$\begin{aligned} \int \frac{\sin^4 x}{\cos^4 x}\, dx &= \int \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^2 x} \times \frac{dx}{\cos^2 x} \\ &= \int \left(\frac{1}{\cos^2 x} - 2 \right) \frac{dx}{\cos^2 x} + \int 1\, dx. \end{aligned} \tag{5.67}$$

Let $t = \tan x$ in the first integral and use $1/\cos^2 x = t^2 + 1$ to obtain the result. □

5.33. Entry 2.526.33.

$$\int \frac{\cos x}{\sin x}\, dx = \ln \sin x \tag{5.68}$$

PROOF. The change of variables $u = \sin x$ gives the result.

□

5.34. Entry 2.526.34**.**

$$\int \frac{\cos^2 x}{\sin x}\,dx = \cos x + \ln\tan\frac{x}{2} \tag{5.69}$$

Proof. Write the integrand as $\dfrac{1-\sin^2 x}{\sin x} = \dfrac{1}{\sin x} - \sin x$ and use the statement in Entry **2.526.1**. ☐

5.35. Entry 2.526.35**.**

$$\int \frac{\cos^3 x}{\sin x}\,dx = \frac{1}{2}\cos^2 x + \ln\sin x \tag{5.70}$$

Proof. Write the integrand as $\dfrac{\cos x(1-\sin^2 x)}{\sin x} = \dfrac{\cos x}{\sin x} - \cos x \sin x$ to reduce the problem to two simple integrals. ☐

5.36. Entry 2.526.36**.**

$$\int \frac{\cos^4 x}{\sin x}\,dx = \frac{1}{3}\cos^3 x + \cos x + \ln\tan\frac{x}{2} \tag{5.71}$$

Proof. Write the integrand as

$$\frac{(1-\sin^2 x)^2}{\sin x} = \frac{1}{\sin x} - 2\sin x + \sin^3 x. \tag{5.72}$$

The integral of the first term appears in Entry **2.526.1**, the second one integrates to $2\cos x$ and the third one has been evaluated as Entry **2.513.6**. ☐

5.37. Entry 2.526.37**.**

$$\int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x} \tag{5.73}$$

Proof. The change of variables $u = \sin x$ gives the result. ☐

5.38. Entry 2.526.38**.**

$$\int \frac{\cos^2 x}{\sin^2 x}\,dx = -\cot x - x \tag{5.74}$$

Proof. Write the integrand as $\dfrac{1}{\sin^2 x} - 1$. Both integrals are now direct. ☐

5.39. Entry 2.526.39**.**

$$\int \frac{\cos^3 x}{\sin^2 x}\,dx = -\sin x - \frac{1}{\sin x} \tag{5.75}$$

Proof. Write the integrand as $\dfrac{\cos x(1-\sin^2 x)}{\sin^2 x} = \dfrac{\cos x}{\sin^2 x} - \cos x$. Now let $t = \sin x$ in the first integral. ☐

5.40. Entry 2.526.40**.**

$$\int \frac{\cos^4 x}{\sin^2 x}\, dx = -\cot x - \frac{1}{2}\sin x \cos x - \frac{3x}{2} \tag{5.76}$$

Proof. Write the integrand as $\dfrac{1}{\sin^2 x} - 2 + \sin^2 x$. Now integrate using $\sin^2 x = \dfrac{1-\cos(2x)}{2}$. □

5.41. Entry 2.526.41**.**

$$\int \frac{\cos x}{\sin^3 x}\, dx = -\frac{1}{2\sin^2 x} \tag{5.77}$$

Proof. The change of variables $u = \sin x$ gives the result. □

5.42. Entry 2.526.42**.**

$$\int \frac{\cos^2 x}{\sin^3 x}\, dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\tan\frac{x}{2} \tag{5.78}$$

Proof. Use $\cos^2 x = 1 - \sin^2 x$ to write

$$\int \frac{\cos^2 x}{\sin^3 x}\, dx = \int \frac{dx}{\sin^3 x} - \int \frac{dx}{\sin x}. \tag{5.79}$$

The first integral is evaluated in Entry **2.526.3** and the second one appears in Entry **2.526.1**. This completes the evaluation. □

5.43. Entry 2.526.43**.**

$$\int \frac{\cos^3 x}{\sin^3 x}\, dx = -\frac{1}{2\sin^2 x} - \ln\sin x \tag{5.80}$$

Proof. Use $\cos^2 x = 1 - \sin^2 x$ to write the integral as

$$\int \frac{\cos^3 x\, dx}{\sin^3 x} = \int \frac{\cos x}{\sin^3 x}\, dx - \int \frac{\cos x}{\sin x}. \tag{5.81}$$

and now let $t = \sin x$ to see that the entry is the integral of t^{-3} minus the integral of t^{-1} with respect to t. That is the result. □

5.44. Entry 2.526.44**.**

$$\int \frac{\cos^4 x}{\sin^3 x}\, dx = -\frac{\cos x}{2\sin^2 x} - \cos x - \frac{3}{2}\ln\tan\frac{x}{2} \tag{5.82}$$

Proof. Use $\cos^2 x = 1 - \sin^2 x$ to write the integrand as

$$\int \frac{\cos^4 x}{\sin^3 x}\, dx = \int \frac{dx}{\sin^3 x} - 2\int \frac{dx}{\sin x} + \int \sin x\, dx. \tag{5.83}$$

The first integral appears in Entry **2.526.3**, the second one in Entry **2.526.1** and third one is $-\cos x$. □

5.45. Entry 2.526.45**.**

$$\int \frac{\cos x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x} \tag{5.84}$$

Proof. The change of variables $u = \sin x$ gives the result. ☐

5.46. Entry 2.526.46**.**

$$\int \frac{\cos^2 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x \tag{5.85}$$

Proof. Write the integrand as $\cot^2 x \times \text{cosec}^2 x$ and make the change of variables $u = \cot x$ to obtain the result. ☐

5.47. Entry 2.526.47**.**

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x} \tag{5.86}$$

Proof. Use $\cos^3 x = \cos x(1-\sin^2 x)$ to write the integral as

$$\int \frac{\cos^3 x}{\sin^4 x}\,dx = \int \frac{\cos x}{\sin^4 x}\,dx - \int \frac{\cos x}{\sin^2 x}\,dx. \tag{5.87}$$

The change of variables $u = \sin x$ evaluates the last two integrals to produce the result. ☐

5.48. Entry 2.526.48**.**

$$\int \frac{\cos^4 x}{\sin^4 x}\,dx = -\frac{1}{3}\cot^3 x + \cot x + x \tag{5.88}$$

Proof. Use $\cot^2 x = \text{cosec}^2 x - 1$ and write the problem as

$$\int (\cot^2 x)\cot^2 x\,dx = \int \cot^2 x \times \text{cosec}^2 x\,dx - \int \frac{1}{\sin^2 x} + \int 1\,dx. \tag{5.89}$$

Every integral is now elementary and the result follows fro their evaluation. ☐

5.49. Entry 2.526.49**.**

$$\int \frac{dx}{\sin x \cos x} = \ln \tan x \tag{5.90}$$

Proof. Write the integrand as $\frac{1}{2}\sin(2x)$ and make the change of variables $t = 2x$ to obtain

$$\int \frac{dx}{\sin x \cos x} = \int \frac{dt}{\sin t}. \tag{5.91}$$

The result now follows from Entry **2.526.1**. ☐

5.50. Entry 2.526.50**.**

$$\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln \tan \frac{x}{2} \tag{5.92}$$

Proof. Write the integrand as $\dfrac{\sin x}{(1-\cos^2 x)\cos^2 x}$. Then make the change of variables $t = \cos x$ to obtain

$$\int \frac{dx}{\sin x \cos^2 x} = -\int \frac{dt}{t^2(1-t^2)}. \tag{5.93}$$

Now use the partial fraction expansion $\dfrac{1}{t^2(1-t^2)} = \dfrac{1}{t^2} + \dfrac{1}{2(1+t)} + \dfrac{1}{2(1-t)}$. and integrate to produce the answer

$$\frac{1}{\cos x} - \frac{1}{2}\ln\left(\frac{1+\cos x}{1-\cos x}\right). \tag{5.94}$$

This agrees with the stated formula by using the formulas for half-angle. □

5.51. Entry 2.526.51**.**

$$\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2\cos^2 x} + \ln \tan x \tag{5.95}$$

Proof. Write the integrand as $\dfrac{\sin x}{(1-\cos^2 x)\cos^3 x}$. The change of variables $t = \cos x$ gives a rational integrand with partial fraction expansion

$$\frac{-1}{t^3(1-t^2)} = -\frac{1}{2(1-t)} - \frac{1}{t^3} - \frac{1}{t} + \frac{1}{2(1+t)} \tag{5.96}$$

Integration gives

$$-\int \frac{dt}{t^3(1-t^2)} = \frac{1}{2}\ln(1-t) + \frac{1}{2t^2} - 2\ln t + \frac{1}{2}\ln(1+t) \tag{5.97}$$

and going back to $t = \cos x$ gives the result.

□

5.52. Entry 2.526.52**.**

$$\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln \tan \frac{x}{2} \tag{5.98}$$

Proof. Write the integrand as $\dfrac{\sin x}{\sin^2 x \, \cos^4 x}$ and make the change of variables $t = \cos x$ to get

$$\int \frac{dx}{\sin x \cos^4 x} = -\int \frac{dt}{(1-t^2)t^4}. \tag{5.99}$$

Now use the partial fraction expansion

$$-\frac{1}{(1-t^2)t^4} = -\frac{1}{2(1-t)} - \frac{1}{t^4} - \frac{1}{t^2} - \frac{1}{2(1+t)} \tag{5.100}$$

and integrate to get

$$\frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)+\frac{1}{3\cos^3 x}+\frac{1}{\cos x}. \tag{5.101}$$

This can be written in the form given in the table using $\cos x = 2\cos^2\frac{x}{2}-1$. □

5.53. Entry 2.526.53.

$$\int \frac{dx}{\sin^2 x\cos x} = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - \operatorname{cosec} x \tag{5.102}$$

Proof. Write the integrand as $\dfrac{\cos x}{\sin^2 x(1-\sin^2 x)}$. Now let $t=\sin x$, expand the resulting integrand in partial fractions to produce the result. The identity $\dfrac{1+\sin x}{\cos x} = \dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$ is used to bring the result to the form stated here. □

5.54. Entry 2.526.54.

$$\int \frac{dx}{\sin^2 x\cos^2 x} = -2\cot 2x \tag{5.103}$$

Proof. The integrand is $4/\sin^2(2x)$. The change of variables $u=2x$ gives

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{2\,du}{\sin^2 u} = -2\cot u. \tag{5.104}$$

This is the proof. □

5.55. Entry 2.526.55.

$$\int \frac{dx}{\sin^2 x\cos^3 x} = \left(\frac{1}{2\cos^2 x}-\frac{3}{2}\right)\frac{1}{\sin x}+\frac{3}{2}\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) \tag{5.105}$$

Proof. Write the integrand as $\dfrac{\cos x}{\sin^2 x(1-\sin^2 x)^2}$ and make the change of variables $t=\sin x$ to obtain

$$\int \frac{dx}{\sin^2 x\cos^3 x} = \int \frac{dt}{t^2(1-t^2)^2}. \tag{5.106}$$

Integrate the partial fraction expansion

$$\frac{1}{t^2(1-t^2)^2} = \frac{1}{4(1-t)^2}+\frac{3}{4(1-t)}+\frac{1}{t^2}+\frac{1}{4(1+t)^2}+\frac{3}{4(1+t)} \tag{5.107}$$

to produce

$$\frac{1}{2t}\left(-3+\frac{1}{1-t^2}\right)+\frac{3}{4}\ln\left(\frac{1+t}{1-t}\right). \tag{5.108}$$

Now let $t=\sin x$ to produce the result. □

5.56. Entry 2.526.56**.**

$$\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3 \sin x \cos^3 x} - \frac{8}{3} \cot 2x \tag{5.109}$$

Proof. Write the integrand as $\dfrac{\sec^2 x}{\sin^2 x \, \cos^2 x}$ and use make the change of variables $t = \tan x$ and the expressions

$$\sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x} \quad \text{and} \quad \cos^2 x = \frac{1}{1 + \tan^2 x} \tag{5.110}$$

to obtain

$$\int \frac{dx}{\sin^2 x \cos^4 x} = \int \frac{1 + 2t^2 + t^4}{t^2} \, dt = \int \left(t^{-2} + 2 + t^2 \right) \, dt \tag{5.111}$$

and integration produces

$$-\frac{1}{\tan x} + 2 \tan x + \frac{1}{3} \tan^3 x = \frac{1 + 4 \cos^2 x - 8 \cos^4 x}{3 \sin x \cos^3 x}. \tag{5.112}$$

This matches the expression given in the table. □

5.57. Entry 2.526.57**.**

$$\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2 \sin^2 x} + \ln \tan x \tag{5.113}$$

Proof. Write the integrand as $\dfrac{\sin x}{\cos x (1 - \sin^4 x)}$ and make the change of variables $t = \sin x$ to obtain

$$\int \frac{dx}{\sin^3 x \cos x} = -\int \frac{dt}{t(1 - t^2)^2}. \tag{5.114}$$

Integrate the partial fraction expansion

$$-\frac{1}{t(1 - t^2)^2} = -\frac{1}{4(1 - t)^2} - \frac{1}{2(1 - t)} - \frac{1}{t} + \frac{1}{4(1 + t)^2} + \frac{1}{2(1 + t)} \tag{5.115}$$

to produce $\dfrac{1}{2} \ln \left(\dfrac{1 - t^2}{t^2} \right) - \dfrac{1}{2(1 - t^2)}$. Finally let $t = \sin x$ to produce the stated form. □

5.58. Entry 2.526.58**.**

$$\int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x} \left(\frac{1}{2 \sin^2 x} - \frac{3}{2} \right) + \frac{3}{2} \ln \tan \frac{x}{2} \tag{5.116}$$

Proof. Write the integrand as

$$\frac{\sin x}{\sin^4 x \cos^2 x} = \frac{\sin x}{(1 - \cos^2 x)^2 \, \cos^2 x} \tag{5.117}$$

and make the change of variables $t = \sin x$ to obtain a rational integrand with partial fraction expansion

$$-\frac{1}{t^2(1 - t^2)^2} = -\frac{1}{4(1 - t)^2} - \frac{3}{4(1 - t)} - \frac{1}{t^2} - \frac{1}{4(1 + t)^2} - \frac{3}{4(1 + t)}. \tag{5.118}$$

Now integrate and simplify to produce

$$-\frac{1}{t}\left(\frac{1}{1-t^2}-\frac{3}{2}\right)+\frac{3}{4}\ln\left(\frac{1-t}{1+t}\right), \tag{5.119}$$

and write back $t=\cos x$ to obtain the result. □

5.59. Entry 2.526.59**.**

$$\int\frac{dx}{\sin^3 x\cos^3 x}=-\frac{2\cos 2x}{\sin^2 2x}+2\ln\tan x \tag{5.120}$$

Proof. Write the integrand as $\dfrac{\sin x}{\sin^4 x\cos^3 x}$ and let $t=\cos x$ to obtain a rational integrand with partial fraction expansion

$$\frac{-1}{4(1-t)^2}-\frac{1}{1-t}-\frac{1}{t^3}-\frac{2}{t}+\frac{1}{4(1+t)^2}+\frac{1}{1+t}. \tag{5.121}$$

Integrate and simplify to get

$$-\frac{2t^2-1}{2t^2(1-t^2)}+\ln\left(\frac{1-t^2}{t^2}\right). \tag{5.122}$$

Finally, write $t=\cos x$ back to get the result. □

5.60. Entry 2.526.60**.**

$$\int\frac{dx}{\sin^3 x\,\cos^4 x}=\frac{2}{\cos x}+\frac{1}{3\cos^3 x}-\frac{\cos x}{2\sin^2 x}+\frac{5}{2}\ln\tan\frac{x}{2} \tag{5.123}$$

Proof. Write the integrand as

$$\frac{\sin x}{\cos^4 x(1-\cos^2 x)^2} \tag{5.124}$$

then the change of variables $u=\cos x$ produces a rational integrand with partial fraction decomposition

$$-\frac{1}{4(1-u)^2}-\frac{5}{4(1-u)}-\frac{1}{u^4}-\frac{2}{u^2}-\frac{1}{4(1+u)^2}-\frac{5}{4(1+u)}. \tag{5.125}$$

Each of these terms can be integrated directly and simplification produces the stated answer. □

5.61. Entry 2.526.61**.**

$$\int\frac{dx}{\sin^4 x\,\cos x}=-\frac{1}{\sin x}-\frac{1}{3\sin^3 x}+\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) \tag{5.126}$$

Proof. Write the integrand as $\dfrac{\cos x}{\sin^4 x(1-\sin^2 x)}$. The change of variables $u=\sin x$ then produces a rational integrand with partial fraction decomposition

$$\frac{1}{2(1-u)}+\frac{1}{u^4}+\frac{1}{u^2}+\frac{1}{2(1+u)}. \tag{5.127}$$

Each of these terms can be integrated directly and simplification produces the stated answer. □

5.62. Entry 2.526.62**.**

$$\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3\cos x \sin^3 x} - \frac{8}{3}\cot 2x \tag{5.128}$$

Proof. Write the integrand as $8(\cos(2x)+1)/\sin^4(2x)$ and make the change of variables $t = 2x$ to obtain

$$\int \frac{dx}{\sin^4 x \cos^2 x} = 4\int \sin^{-4} t \cos t\, dt + 4\int \frac{dt}{\sin^4 t}. \tag{5.129}$$

The first integral is elementary and the second one was evaluated in Entry **2.526.4**, This produces the answer

$$-\frac{4}{3\sin^3 t} - \frac{4}{3}\cot^3 t - 4\cot t \tag{5.130}$$

and this can be written in the form stated in (5.128). □

5.63. Entry 2.526.63**.**

$$\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2}\ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) \tag{5.131}$$

Proof. Write the integrand in the form $\cos x/(\sin^4 x(1-\sin^2 x))$ and make the change of variables $t = \sin x$ to obtain a rational integrand with partial fraction decomposition

$$\frac{1}{2(1-t)} + \frac{1}{t^4} + \frac{1}{t^2} + \frac{1}{2(1+t)}. \tag{5.132}$$

Each of these terms can be integrated in elementary form. The stated answer follows by simplificatiion. □

5.64. Entry 2.526.64**.**

$$\int \frac{dx}{\sin^4 x \cos^4 x} = -8\cot 2x - \frac{8}{3}\cot^3 2x \tag{5.133}$$

Proof. The change of variables $t = 2x$ gives

$$\int \frac{dx}{\sin^4 x \cos^4 x} = 16\int \frac{dx}{\sin^4(2x)} = 8\int \frac{dt}{\sin^4 t}. \tag{5.134}$$

Now use the result of Entry **2.526.4**. □

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