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Evaluation of special definite integrals related to Gamma Function

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ABSTRACT. In this note, an integral formula can be used to quickly evaluate certain integrals not expressible in terms of elementary functions. Furthermore, it is shown that the *Ramanujan's Master Theorem* can be obtained, as a special case, from this formula when n is a positive integer.

1. Introduction and lemma

In this note, we prove a new formula for the evaluation of definite integrals and use it in several interesting cases such that the Euler integral of the second kind [**1, 2**], integral representation of the beta function [**3, 4**], Gaussian integrals [**5, 6**], etc. Furthermore, it is shown that the *Ramanujan's Master Theorem (RMT)* when n is a positive integer [**7, 8, 9**] can be derived, as a special case, from this formula, and then we shall demonstrate that in certain cases this formula is a better tool and an effective procedure for the evaluation of certain difficult integrals.
To tackle this problem, we begin by considering the following Cauchy-Frullani integral [**10**]:

LEMMA 1. *Let f be a continuous function on any interval* $0 < A \leq x \leq B < \infty$ *and assume that both $f(\infty)$ and $f(0)$ exist. Then*

$$\int_0^\infty \frac{f(\alpha x) - f(\beta x)}{x} dx = (f(\infty) - f(0)) \ln \frac{\alpha}{\beta},\ \ \alpha,\ \beta > 0. \tag{1.1}$$

This formula was first published by Cauchy in 1823, and more completely in 1827 with a beautiful proof.
Let us consider $\beta = 1$ in Lemma 1. Thus

$$\int_0^\infty \frac{f(\alpha x) - f(x)}{x} dx = (f(\infty) - f(0)) \ln \alpha,\ \ \alpha > 0. \tag{1.2}$$

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Differentiating both sides of Eq.(1.2) $n-$times with respect to α, and using the chain rule $\frac{d}{d\alpha}f(\alpha x)=\frac{d}{d(\alpha x)}[f(\alpha x)]\times\frac{d(\alpha x)}{d\alpha}$, we obtain

$$\int_0^\infty x^{n-1}\frac{d^n}{d(\alpha x)^n}\left[f(\alpha x)\right]dx=(-1)^{n-1}\left[f(\infty)-f(0)\right]\frac{(n-1)!}{\alpha^n},\ \alpha>0. \tag{1.3}$$

The change of variable $t=\alpha x$ in the LHS of (1.3) yields

$$\frac{1}{\alpha^n}\int_0^\infty t^{n-1}\frac{d^n f(t)}{dt^n}dt=(-1)^{n-1}\left[f(\infty)-f(0)\right]\frac{(n-1)!}{\alpha^n},\ \alpha>0. \tag{1.4}$$

Thus

LEMMA 2. *Let $f\in\mathbb{C}^n[0,\infty)$ such that both $f(\infty)$ and $f(0)$ exist. Then*

$$\int_0^\infty x^{n-1}f^{(n)}(x)dx=(-1)^{n-1}\left[f(\infty)-f(0)\right]\Gamma(n),\ \Gamma(n)=(n-1)!. \tag{1.5}$$

This is a new helpful tool in proving the Ramanujan's Master Theorem [**7, 8, 9**] for a positive integer n and calculating special integrals related to gamma function.

2. Applications

In order to verify the accuracy of our present formula, we present some elementary examples.

2.1. Application 1: The Ramanujan's Master Theorem. The Ramanujan's Master Theorem [**7, 8, 9**] states that:

THEOREM 3. *If $F(x)$ is defined through the series expansion*

$$F(x)=\sum_{k=0}^\infty\phi(k)\frac{(-x)^k}{k!},\ \phi(0)\neq 0. \tag{2.1}$$

Then

$$\int_0^\infty x^{n-1}\sum_{k=0}^\infty\phi(k)\frac{(-x)^k}{k!}dx=\Gamma(n)\phi(-n), \tag{2.2}$$

where n is a positive integer.

It was widely used by the indian mathematician Srinivasa Ramanujan (1887-1920) to calculate definite integrals and infinite series.
Ramanujan asserts that his proof is legitimate with just simple assumptions [**7**]: (1) $F(x)$ can be expanded in a Maclaurin series; (2) $F(x)$ is continuous on $(0,\infty)$; (3) $n>0$; and (4) $x^nF(x)$ tends to 0 as x tends to ∞.

We note below and note that the Ramanujan's Master Theorem for a positive integer n can be derived as a special case from Lemma 2.

Proof. Assume that $f(x)$ is expanded in a Maclaurin series $f(x) = \sum_{k=0}^{\infty} \psi(k) \frac{(-x)^k}{k!}$ with $f(0) = \psi(0) \neq 0$ and $f(x)$ tends to 0 as x tends to ∞.
Then $f^{(n)}(x) = (-1)^n \sum_{k=0}^{\infty} \psi(n+k) \frac{(-x)^k}{k!}$. Substituting into (1.5), we obtain

$$\int_0^{\infty} x^{n-1} \sum_{k=0}^{\infty} \psi(n+k) \frac{(-x)^k}{k!} dx = f(0)\Gamma(n) = \psi(0)\Gamma(n). \tag{2.3}$$

We see that, in the notation of the Ramanujan's Master Theorem, $\phi(k) = \psi(n+k),\ k = 0, 1, ...$ and hence $\phi(-n) = \psi(0),\ n \in \mathbb{N}$. This is precisely formula (2.2), and the proof is complete. □

An immediate consequence of this is
Example 1 Let γ and $s \in \mathbb{R}$. Then,

$$\int_0^{\infty} x^{n-1} \left[\sum_{k=0}^{\infty} \Gamma(s+k) \gamma^{-k} \frac{(-x)^k}{k!} \right] dx = \Gamma(n)\Gamma(s-n)\gamma^n \tag{2.4}$$

is obtained by simply letting $f(x;\gamma) = \frac{1}{(\gamma+x)^m}$, where $m \in \mathbb{R}$, $f(\infty) = 0$ and $f(0) = \gamma^{-m}$.
Thus $f^{(n)}(x;\gamma) = (-1)^n m(m+1)...(m+n-1) \frac{1}{(\gamma+x)^{n+m}},\ n = 1, 2, ...$
Using the property of the gamma function:

$$\Gamma(m) = \frac{\Gamma(m+1)}{m} = \frac{\Gamma(m+2)}{m(m+1)} = ... = \frac{\Gamma(m+n)}{m(m+1)...(m+n-1)} \tag{2.5}$$

to obtain

$$\frac{1}{m(m+1)...(m+n-1)} = \frac{\Gamma(m)}{\Gamma(m+n)}, \tag{2.6}$$

and from the negative Binomial series:

$$\frac{1}{(\gamma+x)^s} = \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s)} \gamma^{-s-k} \frac{(-x)^k}{k!}, \quad s = m+n. \tag{2.7}$$

Thus, the $n-$th derivative $f^{(n)}(x;\gamma)$ becomes

$$f^{(n)}(x;\gamma) = \frac{(-1)^n}{\Gamma(s-n)} \sum_{k=0}^{\infty} \left[\Gamma(s+k) \gamma^{-(s+k)} \frac{(-x)^k}{k!} \right]. \tag{2.8}$$

Letting $f^{(n)}(x;\gamma)$ with $f(\infty) = 0$ and $f(0) = \gamma^{-m}$ in Eq.(1.5), where $m = s-n$, we obtain the desired result.
We see that, in the notation of the Ramanujan's Master Theorem, $\phi(k) = \Gamma(s+k)\gamma^{-k}$, which is consistent with this result.

2.2. Application 2: Integral representation of the beta function.

Definition 4. *The beta function $B(u;v)$ is also defined by means of an integral* [**3, 4**]*:*

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt,\ \Re(u) > 0,\ \Re(v) > 0. \tag{2.9}$$

This integral is often called the beta integral.

The connection between the beta function and the gamma function is given by the following theorem:

THEOREM 5.

$$B(u,v) == \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\ \Re(u) > 0,\ \Re(v) > 0. \tag{2.10}$$

From the definition and this theorem we easily obtain [**3, 4**]

THEOREM 6.

$$B(n,m) = \int_0^{\infty} x^{n-1}\frac{1}{(1+x)^{n+m}}dx = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)},\ m,n = 1,2,...,. \tag{2.11}$$

PROOF. This follows simply by letting $f(x) = \frac{1}{(1+x)^m},\ f(\infty) = 0,\ f(0) = 1$ and $f^{(n)}(x) = (-1)^n m(m+1)...(m+n-1)\frac{1}{(1+x)^{n+m}},\ n = 1,2,...$ in (1.5)(Lemma 2), and using the above property of the gamma function. □

2.3. Application 3: Integrals involving Hermite and Laguerre polynomials $L_n(x)$.

DEFINITION 7. *The Rodrigues formula for the Hermite polynomials:*

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right),\ n = 0,1,2,...,\ -\infty < x < +\infty. \tag{2.12}$$

The first few Hermite polynomials are:

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x, ...,. \tag{2.13}$$

DEFINITION 8. *The Laguerre Polynomials are:*

$$L_n(x) = \sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!(k!)^2},\ n = 0,1,2,...,\ 0 \leq x < +\infty. \tag{2.14}$$

Example 2 Consider the integral involving Hermite polynomials $H_n(x)$

$$\int_0^{\infty} x^{n-1}H_{n-1}(x)e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\Gamma(n). \tag{2.15}$$

This follows simply by letting $f(x) = erf(x)$ in (1.5) and using the Rodrigues formula for the Hermite polynomials:

$$\frac{d^n}{dx^n}\left[erf(x)\right] = (-1)^{n-1}\frac{2}{\sqrt{\pi}}H_{n-1}(x)e^{-x^2}. \tag{2.16}$$

Example 3 Consider the integral involving Laguerre polynomials $L_n(x)$

$$\int_0^{\infty} x^{n-1}L_n(x)e^{-x}dx = 0. \tag{2.17}$$

This follows simply by letting $f(x) = x^n e^{-x},\ f(\infty) = 0 = f(0)$ in (1.5) and using the Rodrigues formula for the Laguerre polynomials:

$$\frac{d^n}{dx^n}\left[x^n e^{-x}\right] = n!L_n(x)e^{-x}. \tag{2.18}$$

2.4. Application 4: Integrals involving other functions.

Example 4 Consider now other integrals involving special functions

$$\int_0^{\infty} x^{n-1}\left[\sum_{k=1}^{\infty}(-1)^{k-m-1}k^n e^{-kx}\right]dx=\frac{\pi}{2}\Gamma(n). \tag{2.19}$$

The evaluation of this integral follows directly from $f(x)=(1+e^x)^{-1}$ and

$$(-1)^n\frac{d^n}{dx^n}\left[(1+e^x)^{-1}\right]=\sum_{k=1}^{\infty}(-1)^{k-1}k^n e^{-kx}. \tag{2.20}$$

Example 5

$$\int_0^{\infty} x^{n-1}\frac{1}{(1+x^2)^{\frac{n}{2}}}\sin\left[n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right]dx=\frac{\pi}{2}. \tag{2.21}$$

The evaluation of this integral follows directly from $f(x)=\arctan(x),\ f(\infty)=\frac{\pi}{2},\ f(0)=0$ and

$$\frac{d^n}{dx^n}(\arctan x)=\frac{(-1)^{n-1}(n-1)!}{(1+x^2)^{\frac{n}{2}}}\sin\left[n\arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)\right]. \tag{2.22}$$

Example 6

$$\int_0^{\infty} x^{n-1}\left[e^{-\frac{1}{x}}\sum_{k=1}^{n}(-1)^k L(n,\ k)x^{-(n+k)}\right]dx=\Gamma(n), \tag{2.23}$$

where $L(n,\ k)$ are the Lah numbers defined by $L(n,\ k)=\frac{n!}{k!}\frac{(n-1)!}{(k-1)!(n-k)!},\ 1\leq k\leq n,\ L(0,0)=1$ [**11**].
The evaluation of this integral follows directly from $f(x)=e^{-\frac{1}{x}},\ f(\infty)=1,\ f(0)=0$ and the following explicit formula for computing the general derivative of the exponential function $f(x)=e^{-\frac{1}{x}}$ [**11**].

$$\frac{d^n}{dx^n}(e^{-\frac{1}{x}})=(-1)^n e^{-\frac{1}{x}}\sum_{k=1}^{n}(-1)^k L(n,\ k)x^{-(n+k)}. \tag{2.24}$$

Example 7

$$\int_0^{\infty} x^{n-1}\left[\sum_{k=n}^{\infty}(-1)^k B_k\frac{x^{k-n}}{(k-n)!}\right]dx=(-1)^n\frac{1-e}{e}\Gamma(n), \tag{2.25}$$

where B_k are the Bell numbers [**12**].
The evaluation of this integral follows directly from $f(x)=e^{e^{-x}},\ f(\infty)=1,\ f(0)=e$ and the following explicit formula for computing the general derivative [**12**]

$$\frac{d^n}{dx^n}(e^{e^{-x}})=e\sum_{k=n}^{\infty}(-1)^k B_k\frac{x^{k-n}}{(k-n)!}. \tag{2.26}$$

References

[1] T. H. Gronwall, The Gamma Function in the Integral Calculus, *Annals of Mathematics,* Second Series, Vol. 20, No. 2 (Dec., 1918), pp. 35-124.
[2] A. Jeffrey, and D. Hui-Hui, Handbook of Mathematical Formulas 4th Ed. Academic Press. ISBN 978-0-12-374288-9. pp. 234-235, 2008.
[3] E.T., Whittaker, and G.N. Watson, A Course in Modern Analysis, 4th ed. Cambridge, England: Cambridge University Press, 1990.
[4] G.E. Andrews, R. Askey and R. Roy, Special Functions. Cambridge, England: Cambridge University Press, 1999.
[5] A. L. Delgado, A Calculation of $\int_0^\infty e^{-x^2}dx$, *The College Math. J.,* 34, 321-323, 2003.
[6] Victor H. Moll, Special Integrals of Gradshteyn and Ryzhik: The Proofs-Volume II, CRC Press, Taylor and Francis Group, 2016.
[7] B.C. Berndt, Ramanujan's Notebooks: Part I. New York: Springer-Verlag, p. 298, 1985.
[8] G. H. Hardy, Ramanujan. Twelve lectures on subjects suggested by his life an work (Cambridge University Press, Cambridge, 1940).
[9] T. Amdeberhan and V. H. Moll, A formula for a quartic integral: a survey of old proofs and some new ones, *Ramanujan J.* 18, 91-102, 2009.
[10] R.P. Agnew, "Limits of Integrals," *Duke Math. J.*, 9, 10-19, 1942.
[11] Bai-Ni Guo and Feng Qi, Some integral representations and properties of Lah numbers, *Journal for Algebra and Number Theory Academia,* Volume 4, Issue 3, 2014, pp. 77-87.
[12] Bai-Ni Guo and Feng Qi, An explicit formula for Bell numbers in terms of Stirling numbers and hypergeometric functions, *Global Journal of Mathematical Analysis,* 2 (4) (2014) 243-248.

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