SCIENTIA
Series A: *Mathematical Sciences*, Vol. 29 (2019), 29–36
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

About some identities for Bessel polynomials

Christophe Vignat

Abstract. In the paper [1], Yu. A. Brychkov derived a set of identities for multiple sums of special functions, using generating functions. Among these identities, a particularly interesting one involves multiple sums of Bessel I_ν functions with half-integer indices. We derive here some equivalent identities that involve different kinds of Bessel polynomials, using a probabilistic approach based on the properties of the Generalized Inverse Gaussian probability density.

1. Introduction

In the paper [**1**], using generating functions, Yu. A. Brychkov derived the following identity

$$\sum_{k_1+\cdots+k_m=n}\prod_{i=1}^{m}\frac{1}{k_i!}\left[I_{-k_i-\frac{1}{2}}(z)-I_{k_i+\frac{1}{2}}(z)\right]=(-1)^n\frac{m^{\frac{1}{2}}}{n!}\pi^{\frac{1-m}{2}}\left(\frac{z}{2}\right)^{\frac{1-m}{2}-n} \tag{1.1}$$
$$\times\sum_{k=0}^{n}\binom{n}{k}\left(\frac{m-1}{2}\right)_{n-k}\left(-\frac{mz}{2}\right)^k\left[I_{-k-\frac{1}{2}}(mz)-I_{k+\frac{1}{2}}(mz)\right].$$

where $I_k(z)$ is the modified Bessel function of the first kind. Our aim here is to show that this formula is equivalent to several simple identities involving various kinds of Bessel polynomials, and to provide some extended versions of it. Our approach involves a probabilistic interpretation of these identities but does not require the computation of any generating function.

Let us first remark that, from [**3**, p. 675],

$$I_{-k-\frac{1}{2}}(z)-I_{k+\frac{1}{2}}(z)=\frac{2}{\pi}(-1)^k K_{k+\frac{1}{2}}(z)$$

where $K_k(z)$ is the modified Bessel function of the third kind or Macdonald function. For k integer, this function is related to the Bessel polynomial $q_k(z)$ of degree k as

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2000 *Mathematics Subject Classification.* Primary 33C10, Secondary 33B99,60B05.

Key words and phrases. Bessel polynomials, Bessel functions, Turán-type inequality, Generalized Inverse Gaussian distribution.

follows

$$\exp(-z)\, q_k(z) = \frac{2^{\frac{1}{2}-k}}{\Gamma\left(k+\frac{1}{2}\right)} z^{k+\frac{1}{2}} K_{k+\frac{1}{2}}(z)\,;\ z \geqslant 0 \tag{1.2}$$

where the Bessel polynomial $q_k(z)$ is defined by

$$q_k(z) = \sum_{l=0}^{k} \frac{\binom{k}{l}}{\binom{2k}{l}} \frac{(2z)^l}{l!}.$$

First examples of these Bessel polynomials are

$$q_0(z) = 1,\ q_1(z) = 1+z,\ q_2(z) = 1+z+\frac{z^2}{3}.$$

These polynomials satisfy the normalization constraint

$$q_n(0) = 1.$$

Replacing the Bessel functions $I_k(z)$ by their expression in terms of Bessel polynomials yields, after some elementary algebra, the equivalent and more compact version of identity (1.1)

$$\sum_{k_1+\cdots+k_m=n} \prod_{i=1}^{m} \binom{2k_i}{k_i} q_{k_i}(z) = \sum_{k=0}^{n} \binom{2k}{k} 2^{2n-2k} \frac{\left(\frac{m-1}{2}\right)_{n-k}}{(n-k)!} q_k(mz)\,. \tag{1.3}$$

Moreover, replacing the Bessel functions $I_k(z)$ by their expression in terms of Bessel $K_k(z)$ functions yields the equivalent identity

$$\sum_{k_1+\cdots+k_m=n} \prod_{i=1}^{m} \frac{z^{k_i+\frac{1}{2}} K_{k_i+\frac{1}{2}}(z)}{k_i!} = \left(\frac{\sqrt{\pi}}{2}\right)^{m-1} \sum_{k=0}^{n} \frac{\left(\frac{m-1}{2}\right)_{n-k}}{(n-k)!} \frac{\left(\frac{mz}{2}\right)^{k+\frac{1}{2}} K_{k+\frac{1}{2}}(mz)}{k!}.$$

2. A probabilistic approach to identity (1.3)

We show here that identity (1.3) can be interpreted in a probabilistic setting; this interpretation relies on properties of the generalized inverse Gaussian distribution, defined as [**2**]

$$f(x;\psi,\chi,\lambda) = \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\psi\chi}\right)} x^{\lambda-1} \exp\left(-\frac{\chi}{2}x^{-1} - \frac{\psi}{2}x\right),\ x,\psi,\chi > 0,\ \lambda > -1. \tag{2.1}$$

In order to explicit this probabilistic interpretation, we need the following preliminary results. This first lemma can be found in [**2**].

Lemma 2.1. [**2**] *Let $X_{-\frac{1}{2},z_1}$ and $X_{-\frac{1}{2},z_2}$ two independent random variables with generalized inverse distribution as in (2.1) with parameters $\psi = 1,\ \chi = z^2$ and $\lambda = -\frac{1}{2}$, then*

$$X_{-\frac{1}{2},z_1} + X_{-\frac{1}{2},z_2} \sim X_{-\frac{1}{2},z_1+z_2}. \tag{2.2}$$

where the sign $\sim$ denotes identity in distribution. Moreover,

$$X_{-\frac{1}{2},z} + X_{\frac{1}{2},0} \sim X_{\frac{1}{2},z}. \tag{2.3}$$

From these two results, we can deduce the following identity, that can also be found in [**2**]

LEMMA 2.2. *Let* $X_{\frac{1}{2},z_1}$ *and* $X_{\frac{1}{2},z_2}$ *two independent random variables with generalized inverse Gaussian distribution as in (2.1) with parameters* $\psi = 1,\ \chi = z^2,\ \lambda = \frac{1}{2}$, *then the identity in distribution*

$$X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2} \sim X_{\frac{1}{2},0} + X_{\frac{1}{2},z_1+z_2} \tag{2.4}$$

holds, where $X_{\frac{1}{2},z_1+z_2}$ *is independent of* $X_{\frac{1}{2},0}$.

PROOF. We notice from (2.3) that

$$X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2} \sim \left(X_{-\frac{1}{2},z_1} + X_{\frac{1}{2},0}\right) + \left(X_{-\frac{1}{2},z_2} + \tilde{X}_{\frac{1}{2},0}\right)$$

where the random variables on the right-hand side are mutually independent. By the stability property (2.2), we obtain

$$\begin{aligned} X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2} &\sim X_{-\frac{1}{2},z_1+z_2} + X_{\frac{1}{2},0} + \tilde{X}_{\frac{1}{2},0} \\ &\sim X_{\frac{1}{2},z_1+z_2} + X_{\frac{1}{2},0} \end{aligned}$$

where we have used again the property (2.3), hence the result. □

We note that the probability density of $X_{\frac{1}{2},0}$ is the χ^2 density with one degree of freedom, or equivalently the Gamma density with scale parameter 2 and shape parameter 1/2:

$$f_{X_{\frac{1}{2},0}}(x) = \begin{cases} \frac{1}{2\sqrt{\pi}} \left(\frac{x}{2}\right)^{-\frac{1}{2}} \exp\left(-\frac{x}{2}\right), & x \geqslant 0 \\ 0, & \text{else.} \end{cases}$$

The link between the Generalized Inverse Gaussian random variables and the Bessel polynomials is characterized as follows.

LEMMA 2.3. *Given* $\nu \in \mathbb{R}$, *the* $\nu-$*th moment of* $X_{\frac{1}{2},z}$ *is*

$$\mathbb{E}X_{\frac{1}{2},z}^{\nu} = \sqrt{\frac{2}{\pi}} \exp(z)\, z^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(z). \tag{2.5}$$

When $\nu = n$ *is a positive integer, this formula simplifies to*

$$\mathbb{E}X_{\frac{1}{2},z}^{n} = \frac{1}{2^n}\frac{(2n)!}{n!} q_n(z) \tag{2.6}$$

where $q_n(z)$ *is the Bessel polynomial of degree* n.

PROOF. The $\nu-$th moment is easily computed as the integral

$$\int_0^{+\infty} x^{\nu} f\left(x; 1, z^2, \frac{1}{2}\right) dx = \sqrt{\frac{2}{\pi}} \exp(z)\, z^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(z).$$

In the case where $\nu = n$ is an integer, using the expression (1.2) of the Bessel function $K_{n+\frac{1}{2}}$ in terms of the Bessel polynomial q_n, we obtain

$$\begin{aligned}
\mathbb{E}X^n_{\frac{1}{2},z} &= \sqrt{\frac{2}{\pi}}\exp(z)\, z^{n+\frac{1}{2}} K_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\exp(z)\,\Gamma\left(n+\frac{1}{2}\right) 2^{n-\frac{1}{2}}\exp(-z)\, q_n(z) \\
&= \frac{2^n}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right) q_n(z)
\end{aligned}$$

with $\frac{\Gamma(n+1/2)}{\Gamma(1/2)} = \frac{1}{2^{2n}}\frac{(2n)!}{n!}$, hence the result. □

We now have the necessary tools to prove the following extension of identity (1.3).

THEOREM 2.1. *If $m \geqslant 2$ and $\{z_i,\ 1 \leqslant i \leqslant m\}$ are complex numbers then, with $z = \sum_{i=1}^m z_i$,*

$$\sum_{k_1+\cdots+k_m=n}\prod_{i=1}^m \binom{2k_i}{k_i} q_{k_i}(z_i) = \sum_{k=0}^n \binom{2k}{k}\frac{2^{2n-2k}\left(\frac{m-1}{2}\right)_{n-k}}{(n-k)!} q_k(z). \tag{2.7}$$

The special case where all z_i are equal is Brychkov's identity (1.3).

PROOF. We consider first the case $m = 2$. As a consequence of Lemma 2.2, the moments of $X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2}$ are the same as the moments of $X_{\frac{1}{2},0} + X_{\frac{1}{2},z_1+z_2}$. Using the binomial expansion, the moment of order n of $X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2}$ yields

$$\begin{aligned}
\mathbb{E}\left(X_{\frac{1}{2},z_1} + X_{\frac{1}{2},z_2}\right)^n &= \sum_{k=0}^n \binom{n}{k}\mathbb{E}\left(X_{\frac{1}{2},z_1}\right)^k \mathbb{E}\left(X_{\frac{1}{2},z_2}\right)^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k}\frac{1}{2^k}\frac{(2k)!}{k!} q_k(z_1)\frac{1}{2^{n-k}}\frac{(2n-2k)!}{(n-k)!} q_{n-k}(z_2) \\
&= \frac{n!}{2^n}\sum_{k=0}^n \binom{2k}{k}\binom{2n-2k}{n-k} q_k(z_1)\, q_{n-k}(z_2).
\end{aligned}$$

The same approach applied to $X_{\frac{1}{2},0} + X_{\frac{1}{2},z_1+z_2}$ gives

$$\begin{aligned}
\mathbb{E}\left(X_{\frac{1}{2},0} + X_{\frac{1}{2},z_1+z_2}\right)^n &= \sum_{k=0}^n \binom{n}{k} 2^k \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{1}{2^{n-k}}\frac{(2n-2k)!}{(n-k)!} q_{n-k}(z_1+z_2) \\
&= \frac{1}{2^n}\sum_{k=0}^n \binom{n}{k} 2^{2k}\frac{1}{2^{2k}}\frac{(2k)!}{k!}\frac{(2n-2k)!}{(n-k)!} q_{n-k}(z_1+z_2) \\
&= \frac{n!}{2^n}\sum_{k=0}^n \binom{2k}{k}\binom{2n-2k}{n-k} q_k(z_1+z_2),
\end{aligned}$$

which yields the result. The extension to the case of an arbitrary integer value $m > 2$ is left to the reader, using the following elementary extension of identity (2.4):

LEMMA 2.4. *If $m \geqslant 2$ and $\{z_i,\, 1 \leqslant i \leqslant m\}$ are real positive numbers and $X_{\frac{1}{2},z_i}$ are independent Generalized Inverse Gaussian random variables, then the following identity in distribution holds*

$$\sum_{i=1}^{m} X_{\frac{1}{2},z_i} \sim X_{\frac{1}{2},\sum_{i=1}^{m} z_i} + X_{\frac{m-1}{2},0}.$$

□

We notice that $X_{\frac{m-1}{2},0}$ is distributed as a Gamma random variable with scale parameter 2 and shape parameter $\frac{m-1}{2}$ (or equivalently a χ^2 random variable with $m-1$ degrees of freedom).

3. Links to Laguerre polynomials

We note that the Laguerre polynomials $L_n^{(\mu)}(z)$ are related to the Bessel polynomials $q_n(z)$ as

$$q_n(z) = \frac{(-1)^n}{\binom{2n}{n}} L_n^{(-2n-1)}(2z);$$

as a consequence, an equivalent statement of identity (2.7) in terms of Laguerre polynomials reads, with $z = \sum_{i=1}^{m} z_i$,

$$\sum_{k_1+\dots+k_m=n} \prod_{i=1}^{m} L_{k_i}^{(-2k_i-1)}(z_i) = \sum_{k=0}^{n} 2^{2n-2k} \frac{\left(\frac{m-1}{2}\right)_{n-k}}{(n-k)!} (-4)^{n-k} L_k^{(-2k-1)}(z).$$

The special case $m = 2$ and $z_1 = -z_2 = z$ of this identity can be found in [**4**, 4.4.2.9] as

$$\sum_{k=0}^{n} L_k^{(-2k-1)}(z)\, L_{n-k}^{(-2n+2k-1)}(-z) = (-4)^n.$$

4. Links to other Bessel polynomials

In this section, we show that two other families of Bessel polynomials can be interpreted as moments in the same way as in identity (2.6). We deduce from these representations some identities equivalent to (2.7) but with a more elementary form.

4.1. Bessel θ_n polynomials. In his textbook [**5**], Grosswald considers the following Bessel polynomials, called *reverse Bessel polynomials*

$$\theta_n(z) = \frac{(2n)!}{n!} 2^{-n} q_n(z). \tag{4.1}$$

For example,

$$\theta_0(z) = 1,\ \theta_1(z) = 1 + z,\ \theta_2(z) = 3 + 3z + z^2.$$

From identity (2.6), we deduce

$$\theta_n(z) = \mathbb{E} X_{\frac{1}{2},z}^n.$$

Moreover, the identity (2.7) reads in terms of these polynomials

$$\sum_{k_1+\cdots+k_m=n}\prod_{i=1}^{m}\frac{\theta_{k_i}(z_i)}{k_i!}=\sum_{k=0}^{n}\frac{2^{n-k}\left(\frac{m-1}{2}\right)_{n-k}}{(n-k)!}\frac{\theta_k(z)}{k!}.$$

with $z=\sum_{i=1}^{m}z_i$. The case $m=2$ appears in [**6**, eqn. (5.4)].

4.2. Bessel f_n polynomials. The third family of Bessel polynomials f_n is defined by L. Carlitz [**6**] as

$$f_n(z)=z\theta_{n-1}(z)\,;\ n\geqslant 1, \tag{4.2}$$

and $f_0(z)=1$. First examples are

$$f_1(z)=z;\ f_2(z)=z+z^2;\ f_3(z)=3z+3z^2+z^3.$$

Note that as in the case of the reverse Bessel polynomials, these polynomials have their highest degree coefficient equal to 1.

Since it can be easily checked from (2.1) that the Generalized Inverse Gaussian density satisfies the functional equation

$$xf\left(x;1,z^2,-\frac{1}{2}\right)=zf\left(x;1,z^2,+\frac{1}{2}\right),$$

we deduce

$$z\mathbb{E}X_{\frac{1}{2},z}^{n-1}=\mathbb{E}X_{-\frac{1}{2},z}^{n}$$

so that

$$f_n(z)=\mathbb{E}X_{-\frac{1}{2},z}^{n},\ \forall n\geqslant 1.$$

Since moreover $f_0(z)=1$, this representation holds in fact $\forall n\geqslant 0$. By the stability property (2.2), we deduce that the Bessel polynomials f_n satisfy the multinomial property

$$\sum_{k_1+\cdots+k_m=n}\prod_{i=1}^{m}\frac{f_{k_i}(z_i)}{k_i!}=\frac{f_n(z)}{n!} \tag{4.3}$$

with $z=\sum_{i=1}^{m}z_i$. The case $m=2$ is given in [**6**, eqn. (2.7)].

Moreover, since, by (4.2), (4.1) and (1.2), the polynomials f_n are related to the Bessel functions K_n as

$$f_n(z)=\sqrt{\frac{2}{\pi}}z^{n+\frac{1}{2}}K_{n-\frac{1}{2}}(z),$$

we deduce the equivalent version of (4.3)

$$\sum_{k_1+\cdots+k_m=n}\prod_{i=1}^{m}\frac{z_i^{k_i+\frac{1}{2}}}{k_i!}K_{k_i-\frac{1}{2}}(z_i)=\left(\frac{2}{\pi}\right)^{\frac{1-m}{2}}\frac{(z)^{n+\frac{1}{2}}}{n!}K_{n-\frac{1}{2}}(z)$$

with $z=\sum_{i=1}^{m}z_i$, which can be found in [**3**, 5.18.1.3].

5. Conclusions

We have shown that an identity introduced by Brychkov using generating functions can be interpreted as a multiplication identity for several types of Bessel polynomials. Moreover, we have given a probabilistic background to this identity and exhibited its relationship with the Generalized Inverse Gaussian density. Other tools, such as generating functions, can certainly replace this probabilistic approach; however, we found that it is particularly convenient in this context.

As a final illustration of the efficiency of this tool, we derive a quick proof of a famous Turán type inequality for Bessel K_ν functions: in [**9**], Ismail and Muldoon proved that the function

$$\nu \mapsto \frac{K_{\nu+b}\left(z\right)}{K_\nu\left(z\right)}$$

is increasing, implying that the function

$$\nu \mapsto K_\nu\left(z\right)$$

is log-convex. As a consequence, the following Turán-type inequality - as named by Karlin and Szegö - for Bessel functions holds: for $\frac{x}{p}$ and $\frac{y}{q} > -\frac{1}{2}$, and with $\frac{1}{p}+\frac{1}{q}=1$,

$$K_{\frac{x}{p}+\frac{y}{q}}\left(z\right) \leqslant K_x\left(z\right)^{\frac{1}{p}} K_y\left(z\right)^{\frac{1}{q}}.$$

The case $p=q=2$ of this inequality is also derived in [**8**] from the following generalization of the Cauchy-Schwarz inequality

$$\int_a^b g(t)f^m(t)dt \;\times \int_a^b g(t)f^n(t)dt \geqslant \left(\int_a^b g(t)f^{\frac{m+n}{2}}(t)dt\right)^2$$

The probabilistic approach based on the moment representation (2.5) allows to derive a straightforward proof of this result that does not require such a refinement: choosing a random variable $X_{\frac{1}{2},z}$ and applying the standard Hölder inequality

$$\mathbb{E}\left(Z_1Z_2\right) \leqslant \left[\mathbb{E}Z_1^p\right]^{\frac{1}{p}}\left[\mathbb{E}Z_2^q\right]^{\frac{1}{q}}$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $Z_1 = X_{\frac{1}{2},z}^{\nu/2-1/4}$ and $Z_2 = X_{\frac{1}{2},z}^{\mu/2-1/4}$ produces the desired result.

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Department of Mathematics, Tulane University, New Orleans, LA 70118 and L.S.S. Supelec, Université Paris-Sud Orsay, France

E-mail address: cvignat@tulane.edu

Received 09 05 2019 revised 28 08 2019