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The Arithmetic-Harmonic mean and Landen transformations

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Abstract. An elementary transformation on the coefficients of a rational function of degree 2, preserving the value of its integral, is presented.

1. Introduction

Given positive real numbers a, b, the most common means found in the literature are the arithmetic, geometric and harmonic, defined respectively by

$$A(a,b) = \frac{a+b}{2}, \qquad G(a,b) = \sqrt{ab}, \qquad H(a,b) = \left(\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b} \right) \right)^{-1}. \tag{1.1}$$

Two of these, A and G, make a remarkable appearance in a classical problem explained below. The goal of this note is to present a situation in which the pair A and H play a similar role.

The classical problem mentioned above is connected to the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{1.2}$$

This is easily expressed in its trigonometric form: for $0 < b < a$, define

$$\mathcal{G}(a,b) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}}. \tag{1.3}$$

A simple calculation gives $K(k) = a\mathcal{G}(a,b)$, with $k = \sqrt{a^2-b^2}/a$. Gauss [**2**] found the remarkable result that $\mathcal{G}$ remains invariant under the transformation

$$(a,b) \mapsto (A(a,b), G(a,b)), \tag{1.4}$$

that is

$$\mathcal{G}(a,b) = \mathcal{G}(A(a,b), G(a,b)). \tag{1.5}$$

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Iteration of the map (1.4) produces a double sequence $\{(a_n, b_n)\}$ of real numbers, with initial conditions $a_0 = a$, $b_0 = b$, such that

$$\mathcal{G}(a, b) = \mathcal{G}(a_n, b_n). \tag{1.6}$$

It is simple to show that a_n, b_n have a common limit. This is famous arithmetic-geometric mean of a and b, denoted by $\text{AGM}(a, b)$. Passage to the limit in (1.5) produces the relation

$$\text{AGM}(a, b) = \frac{\pi}{2\mathcal{G}(a, b)}. \tag{1.7}$$

The reader will find in [**1**] more information about these ideas. The initial geometric version of the invariance (1.4) was obtained by J. Landen [**3, 4**], therefore the map (1.4) is known as a Landen transformation. In terms of the original elliptic integral, this is

$$K(k) = \frac{1}{1+k} K\left(\frac{2\sqrt{k}}{1+k}\right). \tag{1.8}$$

A survey of these type of transformations appears in [**7**].

The goal of this note is to present an elementary version of this result. It is part of the *rational Landen transformations* described in [**5**]. In the case discussed here, another pair of means makes their appearance. The integral in (1.9) below, is of course elementary.

THEOREM 1.1. *Let*

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_{-\infty}^{\infty} \frac{a_0\, dx}{b_0 + b_2 x^2}, \tag{1.9}$$

and denote by $A(a, b)$, $H(a, b)$ the arithmetic and harmonic mean of a and b, respectively. Then $\mathcal{I}_2$ is invariant under the transformation

$$a_0 \mapsto a_0, \quad b_0 \mapsto A(b_0, b_2), \quad b_2 \mapsto H(b_0, b_2); \tag{1.10}$$

that is,

$$\mathcal{I}_2(a_0; b_0, b_2) = \mathcal{I}_2(a_0; A(b_0, b_2), H(b_0, b_2)). \tag{1.11}$$

Note 1.1. The iteration defined by

$$b_0^{n+1} = A(b_0^n, b_2^n), \quad b_2^{n+1} = H(b_0^n, b_2^n), \tag{1.12}$$

starting at $b_0^0 = b_0$, $b_2^0 = b_2$, can be shown to converge quadratically to a common limit. This is the arithmetic-harmonic mean $AH(b_0, b_2)$ of the title. Passing to the limit in (1.9) and evaluating the (elementary) integral $\mathcal{I}_2(a_0; b_0, b_2)$ gives

$$AH(b_0, b_2) = \sqrt{b_0 b_2}. \tag{1.13}$$

This observation appears as Exercise 3, Chapter 1 of [**1**].

2. The proof

Start with the integral

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_{-\infty}^{\infty} \frac{a_0}{b_0 + b_2 x^2} \, dx \tag{2.1}$$

The proof is divided in a sequence of elementary steps.

`Step 1. Symmetrization of denominator`. Multiply the denominator by $b_0 x^2 + b_2$ to produce

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_{-\infty}^{\infty} \frac{a_0(b_0 x^2 + b_2)}{b_0 b_2 + (b_0^2 + b_2^2)x^2 + b_0 b_2 x^4} \, dx \tag{2.2}$$

The new denominator, $D(x)$, is a palindromic polynomial; that is, $D(x) = x^4 D(x^{-1})$.

`Step 2. Trigonometric version of the integral over` $[0, 2\pi]$. Make the change of variables $x = \cot\theta$ and use the formulas for the double-angle of trigonometric functions to produce, with $\varphi = 2\theta$,

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_0^{2\pi} \frac{a_0\left[(b_0 + b_2) + (b_0 - b_2)\cos\varphi\right]}{(b_0 + b_2)^2 - (b_0 - b_2)^2 \cos^2\varphi} \, d\varphi. \tag{2.3}$$

`Step 3. Reduction of the integral to` $[0, \pi]$. Since the integrand in (2.3) is a function of $\cos\varphi$, symmetry with respect to $\varphi = \pi$ shows that

$$\int_0^{2\pi} f(\cos\varphi) \, d\varphi = 2 \int_0^{\pi} f(\cos\varphi) \, d\varphi \tag{2.4}$$

and (2.3) becomes

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_0^{\pi} \frac{2a_0\left[(b_0 + b_2) + (b_0 - b_2)\cos\varphi\right]}{(b_0 + b_2)^2 - (b_0 - b_2)^2 \cos^2\varphi} \, d\varphi. \tag{2.5}$$

`Step 4. Reduction of the integral to` $[0, \pi/2]$ `and the vanishing of parts of the integral`. Now use symmetry with respect to $\varphi = \pi/2$ to observe that part of the integral in (2.5) vanishes. Indeed, if f is an odd function, then splitting the domain of integration at $\varphi = \pi/2$ gives

$$\int_0^{\pi} f(\cos\varphi) \, d\varphi = 0. \tag{2.6}$$

On the other hand, if f is an even function, the same process gives

$$\int_0^{\pi} f(\cos\varphi) \, d\varphi = 2 \int_0^{\pi/2} f(\cos\varphi) \, d\varphi. \tag{2.7}$$

Then the part of the integral in (2.5) with $\cos\varphi$ in the numerator vanishes and (2.5) becomes

$$\mathcal{I}_2(a_0; b_0, b_2) = \int_0^{\pi/2} \frac{4a_0(b_0 + b_2)}{(b_0 + b_2)^2 - (b_0 - b_2)^2 \cos^2\varphi} \, d\varphi. \tag{2.8}$$

`Step 5. Mapping back to an integral over` $[0,\pi]$. Since the integrand in (2.8) is an even function of $\cos\varphi$, the integral over $[\pi/2,\pi]$ is the same as that over $[0,\pi/2]$. Therefore

$$\mathcal{I}_2(a_0;b_0,b_2) = \int_0^{\pi} \frac{2a_0(b_0+b_2)}{(b_0+b_2)^2-(b_0-b_2)^2\cos^2\varphi}\,d\varphi. \tag{2.9}$$

`Step 6. Mapping back to an integral over` $(-\infty,\infty)$. The change of variables $y=\cot\varphi$ now brings back the integral to the whole line. Using the relations

$$\cos^2\varphi = \frac{\cot^2\varphi}{1+\cot^2\varphi} \quad \text{and} \quad d\varphi = -\frac{dy}{1+y^2} \tag{2.10}$$

in (2.9), a simple calculation gives

$$\mathcal{I}_2(a_0;b_0,b_2) = \int_{-\infty}^{\infty} \frac{a_0}{A(b_0,b_2)+H(b_0,b_2)y^2}\,dy, \tag{2.11}$$

as claimed.

Note 2.1. The procedure described above works on every even rational function. Unfortunately the transformations of the coefficients become increasingly complex. For instance, the reader is invited to verify that the integral

$$\mathcal{I}_4(a_0,a_2;b_0,b_2,b_4) = \int_{-\infty}^{\infty} \frac{a_0+a_2x^2}{b_0+b_2x^2+b_4x^4}\,dx \tag{2.12}$$

remains invariant under the change of paramerets

$$\begin{aligned}
a_0 &\mapsto 2(a_0+a_2)(b_0+b_2+b_4)\\
a_2 &\mapsto 8(a_2b_0+a_0b_4)\\
b_0 &\mapsto (b_0+b_2+b_4)^2\\
b_2 &\mapsto 4(b_0b_2+4b_0b_4+b_2b_4)\\
b_4 &\mapsto 16b_0b_4.
\end{aligned}$$

Note 2.2. The rational Landen transformations described in [**5, 6**] apply only to `even` rational integrands. The existence of a map on the coefficients of a general rational function which preserves the value of the integral remains an open question. This is true even for the simplest situation considered here. This difference might be explained from the nature of the answer: recall the discriminant

$$D = b_1^2 - 4b_0b_2, \tag{2.13}$$

then integration over the whole line gives an algebraic function of the coefficients:

$$\int_{-\infty}^{\infty} \frac{dx}{b_0+b_1x+b_2x^2} = \frac{2\pi}{\sqrt{-D}}, \tag{2.14}$$

and on the half-line one obtains

$$\int_0^{\infty} \frac{dx}{b_0+b_1x+b_2x^2} = \frac{\pi - 2\tan^{-1}\left(b_1/\sqrt{-D}\right)}{\sqrt{-D}} \tag{2.15}$$

so the answer is a transcendental function of the coefficients.

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