SCIENTIA
Series A: *Mathematical Sciences*, Vol. 28 (2017-2018), 55–62
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Sums of four or fewer squares

Aashita Kesarwani and V. H. Moll

Abstract. A classical theorem of Lagrange states that every positive integer is the sum of four squares. Properties of the sets obtained by fixing three, two or one of these squares are discussed.

1. Introduction

The classical theorem of Lagrange states that every positive integer is a sum of four squares. Thus, the set

$$SQ \equiv SQ(x,y,z,w) = \{x^2+y^2+z^2+w^2 : x,\, y,\, z,\, w \in \mathbb{Z}\} \tag{1.1}$$

is $\mathbb{N}$. Therefore, given an arbitrary $m \in \mathbb{N}$, the set SQ contains elements divisible by 2^m. This fact can be expressed in terms of valuations. Recall that, for a prime p, the p-adic valuation of $a \in \mathbb{Z}$ is the highest power of p that divides a. This is denoted by $\nu_p(a)$.

Lagrange theorem implies that the valuations $\nu_p(a)$, when a runs over $SQ(x,y,z,w)$, achieve every positive integer. The question considered here deals with the range of valuations when some of the variables in $SQ(x,y,z,w)$ are kept fixed. The result will involve the class of triangular numbers; that is, integers of the form $n = j(j+1)/2$, with $j \in \mathbb{N}$.

Several statements refer to the probability of $A \subset \mathbb{N}$ of cardinality $|A|$. This is defined here as the limiting case of the uniform distribution: $P(A) = \lim\limits_{N\to\infty} \dfrac{|A|}{N}$.

2. Fix three variables

In this first case, fix the values of y, z, w and let $n = y^2+z^2+w^2$. Then SQ reduces to

$$SQ_3(n) = \{x^2 + n : x \in \mathbb{Z}\}. \tag{2.1}$$

2000 *Mathematics Subject Classification.* Primary 11A25, Secondary 11P99.

Key words and phrases. Squares, Lagrange theorem, representations by squares.

The numbers n appearing in this reduction are sums of three squares. Legendre's theorem [**2, 4**] states that such n is not of the form $4^a(8\ell+7)$. Introduce the function

$$\lambda_3(n) = \sup_{x\in\mathbb{Z}} \left\{ \nu_2(x^2+n) \right\}. \tag{2.2}$$

Theorem 2.1 below determines its values. It is interesting that the missing case $4^a(8\ell+7)$ is the only case when λ_3 is not finite.

The analysis of valuations of polynomials has been discussed in [**6**] and the special case of quadratic polynomials appears in [**3**]. A special result required in the discussion presented here is reproduced next.

Theorem 2.1. The function λ_3 satisfies

(1) $\lambda_3(4^r k) = 2r + \lambda_3(k)$,
(2) $\lambda_3(4k+1) = 1$. Moreover, $\nu_2(x^2+4k+1) = 1$ if and only if $x \equiv 1 \bmod 2$,
(3) $\lambda_3(4k+2) = 1$. Moreover, $\nu_2(x^2+4k+2) = 1$ if and only if $x \equiv 0 \bmod 2$,
(4) $\lambda_3(8k+3) = 2$. Moreover, $\nu_2(x^2+4k+1) = 2$ if and only if $x \equiv 1 \bmod 2$,
(5) $\lambda_3(8k+7) = \infty$.

Proof. If $x = 2m$, then $x^2+4k = 4(m^2+k)$ implies $\nu_2(x^2+4k) = 2+\nu_2(m^2+k)$. On the other hand, if $x = 2m+1$, then $x^2+4k = 4(m^2+m+k)+1$ showing that $\nu_2(x^2+4k) = 0$. This proves (1) in the case $r = 1$. The general result follows by induction. The proof of parts (2), (3), (4) are elementary. The last statement is established in [**3, 6**]. $\square$

Corollary 2.1. Assume n is not of the form $4^r(8k+1)$. Then

$$\lambda_3(n) = \nu_2(n) + \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ 0 & \text{if } n \equiv 2 \bmod 4, \\ \lambda_3(2k+1) & \text{if } n \equiv 0 \bmod 4, n = 4^r(2k+1) \\ 2 & \text{if } n \equiv 3 \bmod 8. \end{cases} \tag{2.3}$$

Corollary 2.2. Assume $(a,b,c) \neq (0,0,0)$. Then

$$\lambda_3(a^2+b^2+c^2) \leqslant 2 + \nu_2(a^2+b^2+c^2). \tag{2.4}$$

Proof. Legendre's theorem states that a number that is the sum of three squares is not of the form $4^r(8k+1)$. The result is a restatement of Corollary 2.1. $\square$

Corollary 2.3. Assume n is not of the form $4^r(8k+7)$. The values of $x \in \mathbb{Z}$ where $\nu_2(x^2+n)$ achieves its maximum $\lambda_3(n)$ is a periodic sequence, with a period that is a power of 2.

Proof. This follows directly from Theorem 2.1. $\square$

Two auxiliary lemmas are stated next.

Lemma 2.2. Assume n is not of the form $4^r(8k+7)$. Then

(1) $\lambda_3(n) \geqslant 2r+1$ if and only if $\nu_2(n) \geqslant 2r$; that is, $n = 4^r k$ with $k \in \mathbb{N}$,
(2) $\lambda_3(n) \geqslant 2r$ if and only if n has one of the forms $4^r k$ or $4^{r-1}(8k+3)$.

PROOF. Assume the statement (1) holds with r replaced by $r-1$. Then if $\lambda_3(n) \geqslant 2r+1$, it follows that $\lambda_3(n) \geqslant 3$. Theorem 2.1 shows that n must be divisible by 4, say $n = 4\ell$. Then $\lambda_3(4\ell) = 2 + \lambda_3(\ell)$ implies $\lambda_3(\ell) \geqslant 2r-1$. The result now follows by induction. For the converse, if $n = 4^r k$, then $\lambda_3(n) = 2+\lambda_3(4^{r-1}k) \geqslant 2+2(r-1)+1 = 2r+1$. The proof of (1) is complete. The proof of (2) is similar. □

LEMMA 2.3. A number of the form $8n+3$ is a sum of three squares if and only if n is a sum of three triangular numbers.

PROOF. If $8n+3$ is a sum of three squares, these three squares must be odd integers. Then

$$8n+3 = (2k+1)^2 + (2\ell+1)^2 + (2m+1)^2 \tag{2.5}$$

is equivalent to

$$n = \frac{k(k+1)}{2} + \frac{\ell(\ell+1)}{2} + \frac{m(m+1)}{2}. \tag{2.6}$$

□

Note 2.4. Gauss' Eureka theorem [**1**] states that every number is the sum of three triangular numbers, hence every number of the form $8n+3$ is a sum of three squares.

THEOREM 2.4. Given $v \in \mathbb{N}$, there are infinitely many triples (a,b,c) such that the sequence $SQ_3(a^2+b^2+c^2) = \{x^2+a^2+b^2+c^2 : x \in \mathbb{Z}\}$ has infinitely many elements divisible by 2^v.

PROOF. Given $v > 0$, let r be defined in the following manner: if v is even, then let $r = v/2$, else let $r = (v-1)/2$ so that in either case $2r \geqslant v$. Since there are infinitely many triangular numbers, Lemma 2.3 implies there are infinitely many triplets (a,b,c) such that their sum $a^2+b^2+c^2$ equals $4^{r-1}(8k+3)$. From Theorem 2.1, for any k, $\lambda_3(4^{r-1}(8k+3)) = 2r \geqslant v$ and there are infinitely many x periodically spaced in the number line such that $x^2 + 4^{r-1}(8k+3)$ is divisible by 2^v. Hence, proved. □

THEOREM 2.5. For an arbitrary, but fixed, $(a,b,c) \neq (0,0,0)$, define $n = a^2 + b^2 + c^2$. Then the probability that the set $SQ_3(n) = \{x^2+n : x \in \mathbb{Z}\}$ has an element divisible by 2^v is $2^{-f(v)}$, where $f(v) = \left\lfloor \frac{3v}{2} \right\rfloor - 1$. That is,

$$\lim_{L\to\infty} \frac{|\{s \in SQ_3(n) :\ 2^v | s \text{ and } |s| \leqslant L\}|}{|\{s \in SQ_3(n) :\ |s| \leqslant L\}|} = 2^{-f(v)}. \tag{2.7}$$

PROOF. The existence of $x \in \mathbb{Z}$ such that 2^v divides $x^2+a^2+b^2+c^2$ is equivalent to $\lambda_3(n = a^2+b^2+c^2) \geqslant v$. Assume first that v is odd, say $v = 2r+1$. Lemma 2.3 implies that $n = 4^r k$. Then the probability in question is

$$P\left(a^2+b^2+c^2 \equiv 0 \bmod 4^r\right) \tag{2.8}$$

where a, b, c are taken independently over $\mathbb{Z}$. Since $a^2+b^2+c^2 \equiv 0 \bmod 2^{2r}$ is equivalent to $a \equiv 0$, $b \equiv 0$, $c \equiv 0$ modulo 2^r, it follows that

$$P\left(\lambda_3(a^2+b^2+c^2) \geqslant 2r\right) = \frac{1}{2^{3r}} = \frac{1}{2^{f(v)}}. \tag{2.9}$$

In the case v is even, say $v = 2r$, Lemma 2.3 shows that the corresponding probability is

$$\begin{aligned} P\left(\lambda_3(a^2+b^2+c^2) \geqslant 2r\right) &= P\left(\lambda_3(a^2+b^2+c^2) \equiv 0 \bmod 4^r\right) \\ &\qquad + P\left(\lambda_3(a^2+b^2+c^2) \equiv 3 \cdot 4^{r-1} \bmod 4^r\right) \\ &= P\left(a \equiv 0,\ b \equiv 0,\ c \equiv 0 \bmod 2^r\right) \\ &\quad + P\left(a \equiv 2^{r-1},\, b \equiv 2^{r-1}\, c \equiv 2^{r-1} \bmod 2^r\right) \\ &= \left(\frac{1}{2^r}\right)^3 + \left(\frac{1}{2^r}\right)^3 = \frac{1}{2^{3r-1}} = \frac{1}{2^{f(v)}}. \end{aligned}$$

The proof is complete. □

3. Fix two variables

Now fix two variables and define $n = z^2 + w^2$. Then SQ reduces to

$$SQ_2(n) = \{x^2 + y^2 + n : x,\, y \in \mathbb{Z}\}, \tag{3.1}$$

and define

$$\lambda_2(n) = \sup_{x,y \in \mathbb{Z}} \left\{\nu_2(x^2+y^2+n)\right\}. \tag{3.2}$$

Since n is a sum of two squares, it follows that $n = 2^r(4k+1)$, for some $r,\, k \in \mathbb{N}$; see [**5**]. The converse is also true: every number of the form $2^r(4k+1)$ is a sum of two squares.

The analysis of the function $\lambda_2(n)$ involves the sequence of triangular numbers $T_j = \frac{1}{2}j(j+1)$. The next elementary statement will be used in this analysis.

Lemma 3.1. A number of the form $4\ell+1$ is a sum of two squares if and only if ℓ is a triangular number or a sum of two such numbers.

Proof. If $\ell = T_n$, then $4\ell+1 = n^2+(n+1)^2$. On the other hand if $\ell = T_n + T_m$, then $4\ell+1 = (m-n)^2+(m+n+1)^2$. To establish the converse write $4\ell+1 = c^2+d^2$. Then c, d must be of opposite parity. Choose integers m, n such that $c = m-n$ and $d = m+n+1$. Define

$$\ell_1 = T_m = \tfrac{1}{8}\left((c+d)^2 - 1\right) \quad \text{and} \quad \ell_2 = T_n = \tfrac{1}{8}\left((d-c)^2 - 1\right). \tag{3.3}$$

Then $\ell = \ell_1 + \ell_2$ is a sum of two triangular numbers if $\ell_1,\, \ell_2 \neq 0$. Otherwise it reduces to one such number. □

Theorem 3.2. Let $n \in \mathbb{N}$. Then

$$\lambda_2(n) = \begin{cases} \nu_2(n)+1 & \text{if } n = 2^r(4k+1), \\ \infty & \text{if } n = 2^r(4k+3). \end{cases} \tag{3.4}$$

Proof. Assume $n = 2^r(4k+1)$, write $x^2+y^2 = 2^s(4\ell+1)$ so that

$$x^2+y^2+n = 2^s(4\ell+1) + 2^r(4k+1). \tag{3.5}$$

If $r \neq s$, then $\nu_2(x^2+y^2+n) = \min(r,s) \leqslant r$. On the other hand, if $r = s$, one obtains $x^2+y^2+n = 2^{r+1}(2\ell+2k+1)$ and $\nu_2(x^2+y^2+n) = r+1 = \nu_2(n)+1$. Thus, given the factorization $n = 2^r(4k+1)$, form the number $2^r(4\ell+1)$ and find (x, y) such that $x^2+y^2 = 2^r(4\ell+1)$. Then the value $r+1$ is achieved for the valuation $\nu_2(x^2+y^2+n)$ and it follows that $\lambda_2(n) = r+1$.

Now if $n = 2^r(4k+3)$ and writing $x^2+y^2 = 2^s(4\ell+1)$ as before,

$$\nu_2(x^2+y^2+n) = \nu_2(2^s(4\ell+1)+2^r(4k+3)) = \min(r, s) \quad \text{if } r \neq s. \tag{3.6}$$

So, in this case, $\nu_2(x^2+y^2+n) \leqslant r$ is bounded. On the other hand, if $r = s$,

$$\nu_2(x^2+y^2+n) = \nu_2(2^{r+2}(\ell+k+1)) = r+2+\nu_2(\ell+k+1). \tag{3.7}$$

Now choose x, y so that in the factorization $x^2+y^2 = 2^s(4\ell+1)$, the integer ℓ is a triangular number. The result now follows from the next statement. $\square$

Lemma 3.3. Let $r \in \mathbb{N}$ be fixed and T_j be the triangular number. Then the sequence $\{\nu_2(T_j+r) : j \in \mathbb{N}\}$ is unbounded.

Proof. Write $T_j = \frac{1}{2}j(j+1)$ and let $m = 2j+1$. Then $T_j = \frac{1}{8}(m^2-1)$, with m odd. Therefore

$$\nu_2(T_j+r) = \nu_2(\tfrac{1}{2}(m^2-1)+r) = \nu_2(m^2+8r-1)-3. \tag{3.8}$$

The result now follows from part (5) in Theorem 2.1, since $8r-1 \equiv 7 \bmod 8$. $\square$

Theorem 3.4. Given $v \in \mathbb{N}$, there are infinitely many pairs (a, b) such that $SQ_2(a^2+b^2) = \{x^2+y^2+a^2+b^2 : x, y \in \mathbb{Z}\}$ have infinitely many elements divisible by 2^v.

Proof. Take $v > 0$. The existence of infinitely many triangular numbers and Lemma 3.1 give infinitely many pairs (a, b) such that $a^2+b^2 = 2^{v-1}(4k+1)$. Theorem 3.2 shows $\lambda_2(2^{v-1}(4k+1)) = v$ for any k. Thus there are infinitely many $x, y \in \mathbb{Z}$ such that $x^2+y^2+2^{v-1}(4k+1)$ is divisible by 2^v. This completes the proof. $\square$

Theorem 3.5. Fix $(a, b) \neq (0, 0)$. Define $n = a^2+b^2$. Then the probability that the set $SQ_2(n) = \{x^2+y^2+n : x, y \in \mathbb{Z}\}$ has an element divisible by 2^v is $1/2^{v-1}$. That is,

$$\lim_{L\to\infty} \frac{|\{s \in SQ_2(n) : 2^v|s \text{ and } |s| \leqslant L\}|}{|\{s \in SQ_2(n) : |s| \leqslant L\}|} = \frac{1}{2^{v-1}}. \tag{3.9}$$

The proof requires an elementary result.

Lemma 3.6. For $x, y \in \mathbb{Z}$ and $r > 0$:

(1) $x^2+y^2 \equiv 0 \pmod{2^{2r}} \iff x \equiv y \equiv 0 \pmod{2^r}$.
(2) $x^2+y^2 \equiv 0 \pmod{2^{2r+1}} \iff x \equiv y \equiv 0 \text{ or } 2^r \pmod{2^{r+1}}$.

Proof of Theorem 3.5. The probability is the probability that $\lambda_2(a^2+b^2) \geqslant v$. This is computed according to the parity of v. Theorem 2.1 and Lemma 3.6 are used in the proof.

(1) For v odd, say $2r+1$:

$$\begin{aligned}
\text{Prob}\,\big(\lambda_2(a^2+b^2) \geqslant 2r+1\big) &= \text{Prob}\,\big(\nu_2(a^2+b^2) \geqslant 2r\big) \\
&= \text{Prob}\,\big(a^2+b^2 \equiv 0 \pmod{2^{2r}}\big) \\
&= \text{Prob}\,\big(a \equiv 0 \;\&\; b \equiv 0 \pmod{2^r}\big) \\
&= \frac{1}{2^r}\cdot\frac{1}{2^r} = \frac{1}{2^{2r}} = \frac{1}{2^{v-1}}
\end{aligned}$$

(2) For v even, say $2r+2$:

$$\begin{aligned}
\text{Prob}\,\big(\lambda_2(a^2+b^2) \geqslant 2r+2\big) &= \text{Prob}\,\big(\nu_2(a^2+b^2) \geqslant 2r+1\big) \\
&= \text{Prob}\,\big(a^2+b^2 \equiv 0 \pmod{2^{2r+1}}\big) \\
&= \text{Prob}\,\big(a \equiv 0 \;\&\; b \equiv 0 \pmod{2^{r+1}}\big) + \\
&\quad\ \text{Prob}\,\big(a \equiv 2^r \;\&\; b \equiv 2^r \pmod{2^{r+1}}\big) \\
&= \left(\frac{1}{2^{r+1}}\right)^2 + \left(\frac{1}{2^{r+1}}\right)^2 = \frac{1}{2^{2r+1}} = \frac{1}{2^{v-1}}
\end{aligned}$$

This completes the argument. □

4. Fix one variable

In the last case, we fix one of the variables in SQ, say w and consider the set

$$SQ_1(n) = \{x^2+y^2+z^2+n : \, x,\, y,\, z \in \mathbb{Z}\}. \tag{4.1}$$

Now let $\lambda_1(n)$ to be the highest power of 2 that divides an element of $SQ_1(n)$; that is,

$$\lambda_1(n) = \sup_{x,y,z\in\mathbb{Z}} \left\{\nu_2(x^2+y^2+z^2+n)\right\}. \tag{4.2}$$

The function $\lambda_1(n)$ is now determined for *arbitrary* $n \in \mathbb{N}$. The special original case, when n is a square, is described at the end of this section.

LEMMA 4.1. Assume $n = 4^r(8k+1)$ for some $r,\, k \in \mathbb{N}$. Then $\lambda_1(n) = 2r+2$.

PROOF. The number $x^2+y^2+z^2$ has the forms

$$2^{2s+1}(2\ell+1), \quad 4^s(4\ell+1) \quad \text{or} \quad 4^s(8\ell+3).$$

In the first case,

$$\nu_2(x^2+y^2+z^2+n) = \nu_2(2^{2s+1}(2\ell+1)+4^r(8k+1)) = \min(2r, 2s+1) \leqslant 2r;$$

in the second case

$$\begin{aligned}
\nu_2(x^2+y^2+z^2+n) &= \nu_2(4^s(4\ell+1)+n) \\
&= \begin{cases} \min(2r,2s) \leqslant 2r & \text{if } r \neq s \\ 2r + \nu_2(8k+1+4\ell+1) = 2r+1 & \text{if } r = s. \end{cases}
\end{aligned}$$

Finally, in the third case,

$$\nu_2(x^2+y^2+z^2+n) = \begin{cases} \min(2r,2s) \leqslant 2r & \text{if } r \neq s \\ 2r + \nu_2(8k+1+8\ell+3) = 2r+2 & \text{if } r = s. \end{cases}$$

Therefore $\nu_2(x^2 + y^2 + z^2 + n) \leqslant 2r + 2$. The final step of the argument requires to find x, y, $z \in \mathbb{N}$ such that $x^2 + y^2 + z^2 = 4^r(8\ell + 3)$. This is simple: choose variables X, Y, Z such that $x = 2^r X$, $y = 2^r Y$, $z = 2^r Z$ and then pick integers to satisfy $X^2 + Y^2 + Z^2 = 8\ell + 3$. This produces infinitely many integers (x, y, z) such that $x^2 + y^2 + z^2 + n$ is divisible by 2^{2r+2}. □

LEMMA 4.2. Assume $n \neq 4^r(8k + 1)$ for any r, $k \in \mathbb{N}$. Then $\lambda_1(n) = \infty$.

PROOF. The number n must have one of the forms

$$2^{2r+1}(2k + 1), \quad 4^r(4k + 3), \quad 4^r(8k + 5).$$

In the first case, $n = 2^{2r+1}(2k + 1)$, assume there are integers x, y, z such that

$$x^2 + y^2 + z^2 = 2^{2r+1}(2\ell + 1). \tag{4.3}$$

Then,

$$\begin{aligned} \nu_2(x^2 + y^2 + z^2 + n) &= \nu_2(2^{2r+1}(2\ell + 1) + 2^{2r+1}(2k + 1)) \\ &= 2r + 2 + \nu_2(\ell + k + 1). \end{aligned} \tag{4.4}$$

It is now shown that, by choosing x, y, $z \in \mathbb{Z}$, the value of ℓ in (4.3) may be chosen so that $\nu_2(\ell+k+1)$ is arbitrarily large. Let $a \in \mathbb{N}$ and take $\ell = 2^a b - k - 1$ with $b \in \mathbb{N}$. Then $\nu_2(\ell+k+1) = a$ and (4.3) is solvable since $2^{2r+1}(2\ell+1) = 4^r(2^{a+2}b - 4k - 2) \neq 4^r(8t+7)$ and so it is a sum of three squares. It follows that $\lambda_1(n)$ is not finite. A similar argument works for the other forms of n. □

In summary,

$$\lambda_1(n) = \sup_{x,y,z \in \mathbb{Z}} \left\{ \nu_2(x^2 + y^2 + z^2 + n) \right\} = \begin{cases} \nu_2(n) + 2 & \text{if } n \text{ is of the form } 4^r(8k + 1), \\ \infty & \text{if not.} \end{cases}$$

LEMMA 4.3. A number of the form $8k+1$ is a square if and only if k is a triangular number; that is, $k = j(j + 1)/2$ for some $j \in \mathbb{N}$.

PROOF. The expression $k = T_n$ is equivalent to $8k + 1 = (2n + 1)^2$. □

THEOREM 4.4. Assume n is a square. Then $\lambda_1(n)$ is always finite.

PROOF. This follows from Lemmas 4.1 and 4.2. □

In the original setting, the number n is a square. Therefore, the next statements deal with the set $SQ_1(a^2) = \{x^2 + y^2 + z^2 + a^2 : x, y, z \in \mathbb{Z}\}$.

THEOREM 4.5. Given $v \in \mathbb{N}$, there are infinitely many $a \in \mathbb{Z}$ such that $SQ_1(a^2)$ have infinitely many elements divisible by 2^v.

PROOF. Given $v > 0$ let $r = \lfloor v/2 \rfloor$, so that $2r \geqslant v$. Lemma 4.3 and the existence of infinitely many triangular numbers give infinitely many a's of the form $a^2 = 4^{r-1}(8k + 1)$. Lemma 4.1 yields $\lambda_1(4^r(8k+1)) = 2r \geqslant v$, for any k. Therefore there are infinitely many $x, y, z \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 + 4^r(8k + 1)$ is divisible by 2^v. □

THEOREM 4.6. Let $a \neq 0$ be fixed. Then the probability that $SQ_1(a^2)$ has an element divisible by 2^{2v} is $1/2^{v-1}$. That is,

$$\lim_{L\to\infty} \frac{\left|\{s \in SQ_1(n) :\ 2^{2v}|s \text{ and } |s| \leqslant L\}\right|}{|\{s \in SQ_1(n) :\ |s| \leqslant L\}|} = \frac{1}{2^{v-1}}. \tag{4.5}$$

PROOF. The requested probability is the probability that $\lambda(a^2) \geqslant 2v$. Lemma 4.1 and 4.2 imply that , $\lambda_1(a^2) = 2v$ is equivalent to $a^2 = 4^{v-1}(8k+1)$ and also equivalent to $\nu_2(a) = v-1$. It follows that $\lambda_1(a^2) \geqslant 2v$ occurs precisely when $a \equiv 0 \pmod{2^{v-1}}$. Therefore, $\text{Prob}(\lambda_1(a^2) \geqslant 2v) = \text{Prob}(a \equiv 0 \pmod{2^{v-1}}) = \frac{1}{2^{v-1}}$. □

5. Conclusions

According to a classical theorem of Lagrange, the range of the function $x^2+y^2+z^2+w^2$ as x, y, z, w run over all integers is the set $\mathbb{N}$ of non-negative integers. The results presented here involve restrictions of this function when some of variables are fixed.

References

[1] G. E. Andrews. EUREKA! num $= \delta + \delta + \delta$. *Jour. Number Theory*, 23:285–293, 1986.
[2] N. C. Ankeny. Sums of three squares. *Proc. Amer. Math. Soc.*, 8:316–319, 1957.
[3] A. Byrnes, J. Fink, G. Lavigne, I. Nogues, S. Rajasekaran, A. Yuan, L. Almodovar, X. Guan, A. Kesarwani, L. Medina, E. Rowland, and V. Moll. A closed-form solution might be given by a tree. The valuation of quadratic polynomials. *Submitted for publication*, 2017.
[4] E. Grosswald. *Representation of integers as sums of squares.* Springer-Verlag, New York, 1985.
[5] K. Ireland and M. Rosen. *A classical introduction to Number Theory.* Springer Verlag, 2nd edition, 1990.
[6] L. Medina, V. Moll, and E. Rowland. Periodicity in the p-adic valuation of a polynomial. *Journal of Number Theory*, 180:139–153, 2017.

Received 21 05 2018 revised 29 10 2018

DEPARTMENT OF MATHEMATICS,
TULANE UNIVERSITY, NEW ORLEANS, LA 70118.
E-mail address: akesarwa@tulane.edu

DEPARTMENT OF MATHEMATICS,
TULANE UNIVERSITY, NEW ORLEANS, LA 70118.
E-mail address: vhm@tulane.edu