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Note on the paper: A table of definite integrals from the marriage of power and Fourier series

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Abstract. This note develops a new technique to exactly evaluate definite integrals for a wide class of analytic integrands in the complex plane which are neither easily calculated nor conveniently tabulated or readily referenced. We examine the real and imaginary parts in particular after expanding these integrands in their corresponding Laurent series. This note may be considered as an extension of the referenced work [**1**].

1. Introduction

In [**1**], the authors proposed a useful technique for evaluating the definite integrals of the forms $\int_{-\pi}^{\pi} f(x)dx$, $\int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)dx$, where n is a positive integer. For example, for $f(z) = \frac{1}{a-z}$ the authors obtained the following definite integrals:

$$\int_{-\pi}^{\pi} u(r,\theta)d\theta = \frac{2\pi}{a}, \tag{1.1}$$

$$\int_{-\pi}^{\pi} u(r,\theta)\cos(n\theta)d\theta = \frac{\pi r^n}{a^{n+1}} \tag{1.2}$$

and

$$\int_{-\pi}^{\pi} v(r,\theta)\sin(n\theta)d\theta = \frac{\pi r^n}{a^{n+1}}, \tag{1.3}$$

where

$$u(r,\theta) = \frac{a - r\cos\theta}{a^2 + r^2 - 2ar\cos\theta}, \ v(r,\theta) = \frac{r\sin\theta}{a^2 + r^2 - 2ar\cos\theta} \tag{1.4}$$

are the real and imaginary parts of $f(z)$, respectively. The evaluation of these integrals is obtained by comparing the coefficients of related power series of this function and Fourier series, under a suitable condition $r \in [0, a)$.

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Similarly, for the analytic functions $f(z) = \frac{e^z}{1-z}$ and $f(z) = \left(\frac{1}{1-z}\right)^2$ they obtained the values for definite integrals of the above forms under $r \in [0,1]$ and $r \in [0,1)$, respectively; see No. (25)-(27) and No. (31)-(33) in [**1**].
A question which arises naturally is what are the values of these definite integrals when $r > a$ for $f(z) = \frac{1}{a-z}$ and $r > 1$ for $f(z) = \frac{e^z}{1-z}$ and $f(z) = \left(\frac{1}{1-z}\right)^2$?
On the other hand, it is easy to see from [**1**] that this technique is useful only in the case where the function $f(z)$ can be represented by the Taylor series as $f(z) = \sum_{n=0}^{\infty} c_n z^n$. However, when $f(z)$ is expressed as a series of terms with negative powers of z, in this case and in many such cases, this technique cannot be applied to such functions. As a simple example, $f(z) = \cos \frac{1}{z},\ f(z) = \sin \frac{1}{z}, f(z) = 1 + \frac{1}{z}$, etc..
This note may be considered as an extension of the referenced work [**1**]. In particular, we are concerned with these questions, where the general object of this note and [**1**] is the same.
In a similar way, let $f(z)$ be an analytic function in the region

$$\Omega = \{z : r_1 <\mid z - z_0 \mid< r_2\}. \tag{1.5}$$

Then $f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{1.6}$$

Letting $z - z_0 = re^{i\theta}$ into Eq.(1.6), we obtain

$$f(z) = a_0 + \sum_{n=1}^{\infty}(a_n r^n + b_n r^{-n}) \cos(n\theta) + i \sum_{n=1}^{\infty}(a_n r^n - b_n r^{-n}) \sin(n\theta). \tag{1.7}$$

Now, decompose $f(z)$ into real and imaginary parts as

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{1.8}$$

In view of Eq.(1.7) and Eq.(1.8), we obtain

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty}(a_n r^n + b_n r^{-n}) \cos(n\theta) \tag{1.9}$$

and

$$v(r, \theta) = \sum_{n=1}^{\infty}(a_n r^n - b_n r^{-n}) \sin(n\theta). \tag{1.10}$$

From the Fourier cosine and sine series on the interval $(-\pi, \pi)$, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r, \theta) d\theta, \tag{1.11}$$

$$a_n r^n + b_n r^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos(n\theta) d\theta,\ n = 1, 2, 3, ..., \tag{1.12}$$

$$a_n r^n - b_n r^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} v(r, \theta) \sin(n\theta) d\theta,\ n = 1, 2, 3, ... \tag{1.13}$$

These formulas are a new helpful tool in calculating the definite integrals.

2. Examples

The reader will find in the following examples different definite integrals of the above forms which are neither always easily calculated nor conveniently tabulated for [**1**] or readily referenced.

EXAMPLE 1. (No. (1)-(3)[**1**]). Consider $f(z) = \frac{1}{a-z}$, $z \neq a$, we have

$$u(r,\theta) = \frac{a - r\cos\theta}{a^2 + r^2 - 2ar\cos\theta}, \ v(r,\theta) = \frac{r\sin\theta}{a^2 + r^2 - 2ar\cos\theta} \tag{2.1}$$

and the Laurent series of $f(z)$ is given by $f(z) = -\sum_{n=1}^{\infty} \frac{a^{n-1}}{z^n}, \mid z \mid > a$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} \frac{a-r\cos\theta}{a^2+r^2-2ar\cos\theta} d\theta &= 0, \\ \int_{-\pi}^{\pi} \frac{a-r\cos\theta}{a^2+r^2-2ar\cos\theta} \cos(n\theta) d\theta &= -\pi \frac{a^{n-1}}{r^n}, \ n = 1,2,3,..., \\ \int_{-\pi}^{\pi} \frac{r\sin\theta}{a^2+r^2-2ar\cos\theta} \sin(n\theta) d\theta &= \pi \frac{a^{n-1}}{r^n}, \ n = 1,2,3,..., \end{cases} \tag{2.2}$$

where $r \in (a, \infty)$.

EXAMPLE 2. (No. (31)-(33)[**1**]). Let $f(z) = \left(\frac{1}{1-z}\right)^2, z \neq 1$, we have

$$u(r,\theta) = \frac{r^2\cos(2\theta) - 2r\cos\theta + 1}{(1 + r^2 - 2r\cos\theta)^2}, \ v(r,\theta) = \frac{2r\sin\theta - r^2\sin(2\theta) + 1}{(1 + r^2 - 2r\cos\theta)^2} \tag{2.3}$$

and $f(z) = \sum_{n=1}^{\infty} \frac{n}{z^{n+1}}, \mid z \mid > 1$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} \frac{r^2\cos(2\theta)-2r\cos\theta+1}{(1+r^2-2r\cos\theta)^2} d\theta &= 0, \\ \int_{-\pi}^{\pi} \frac{r^2\cos(2\theta)-2r\cos\theta+1}{(1+r^2-2r\cos\theta)^2} \cos(n\theta) d\theta &= \pi \frac{n-1}{r^n}, \ n = 1,2,3,..., \\ \int_{-\pi}^{\pi} \frac{2r\sin\theta-r^2\sin(2\theta)+1}{(1+r^2-2r\cos\theta)^2} \sin(n\theta) d\theta &= -\pi \frac{n-1}{r^n}, \ n = 1,2,3,..., \end{cases} \tag{2.4}$$

where $r \in (1, \infty)$.

EXAMPLE 3. (No. (25)-(27)[**1**]). Let $f(z) = \frac{e^z}{z-1}, z \neq 1$, we have

$$u(r,\theta) = \frac{e^{r\cos\theta}[\cos(r\sin\theta)(1 - r\cos\theta) - r\sin\theta\sin(r\sin\theta)]}{1 + r^2 - 2r\cos\theta}, \tag{2.5}$$

$$v(r,\theta) = \frac{e^{r\cos\theta}[\cos(r\sin\theta)r\sin\theta + (1 - r\cos\theta)\sin(r\sin\theta)]}{1 + r^2 - 2r\cos\theta} \tag{2.6}$$

and the Laurent series of $f(z)$ is given by $f(z) = \sum_{n=0}^{\infty} \left(\sum_{p=n+1}^{\infty} \frac{1}{p!}\right) z^n + \sum_{n=1}^{\infty} \frac{e}{z^n}$, $\mid z \mid > 1$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} u(r,\theta) \ d\theta &= 2\pi \sum_{p=n+1}^{\infty} \frac{1}{p!}, \\ \int_{-\pi}^{\pi} u(r,\theta)\cos(n\theta) \ d\theta &= \pi(r^n \sum_{p=n+1}^{\infty} \frac{1}{p!} + er^{-n}), \\ \int_{-\pi}^{\pi} v(r,\theta)\sin(n\theta) \ d\theta &= \pi(r^n \sum_{p=n+1}^{\infty} \frac{1}{p!} - er^{-n}), \end{cases} \tag{2.7}$$

where $r \in (1, \infty)$.

EXAMPLE 4. Consider $f(z) = e^{\frac{1}{z}},\ z \neq 0$, we have

$$u(r,\theta) = e^{\frac{\cos\theta}{r}} \cos\left(\frac{\sin\theta}{r}\right),\ v(r,\theta) = -e^{\frac{\cos\theta}{r}} \sin\left(\frac{\sin\theta}{r}\right) \tag{2.8}$$

and the Laurent series of $f(z)$ is given by $f(z) = \sum_{n=0}^{\infty} \frac{1}{n! z^n}, \mid z \mid > 0$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} e^{\frac{\cos\theta}{r}} \cos\left(\frac{\sin\theta}{r}\right) d\theta &= 2\pi, \\ \int_{-\pi}^{\pi} e^{\frac{\cos\theta}{r}} \cos\left(\frac{\sin\theta}{r}\right) \cos(n\theta) d\theta &= \pi\frac{1}{n! r^n}, \\ \int_{-\pi}^{\pi} e^{\frac{\cos\theta}{r}} \sin\left(\frac{\sin\theta}{r}\right) \sin(n\theta) d\theta &= \pi\frac{1}{n! r^n}, \end{cases} \tag{2.9}$$

where $r \in (0, \infty)$.

EXAMPLE 5. Consider $f(z) = \sin\left(\frac{1}{z}\right), z \neq 0$, we have

$$u(r,\theta) = \sin\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right),\ v(r,\theta) = -\sinh\left(\frac{\sin\theta}{r}\right) \cos\left(\frac{\cos\theta}{r}\right) \tag{2.10}$$

and $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n+1}}, \mid z \mid > 0$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} \sin\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right) d\theta &= 0, \\ \int_{-\pi}^{\pi} \sin\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right) \cos(2n+1)\theta d\theta &= \pi\frac{(-1)^n (2n+1)!}{r^{2n+1}}, \\ \int_{-\pi}^{\pi} \sinh\left(\frac{\sin\theta}{r}\right) \cos\left(\frac{\cos\theta}{r}\right) \sin(2n+1)\theta d\theta &= \pi\frac{(-1)^n (2n+1)!}{r^{2n+1}}, \end{cases} \tag{2.11}$$

where $r \in (0, \infty)$.

EXAMPLE 6. Consider $f(z) = \cos\left(\frac{1}{z}\right), z \neq 0$, we have

$$u(r,\theta) = \cos\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right),\ v(r,\theta) = -\sinh\left(\frac{\sin\theta}{r}\right) \cos\left(\frac{\cos\theta}{r}\right) \tag{2.12}$$

and the Laurent series of $f(z)$ is given by $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}}, \mid z \mid > 0$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} \cos\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right) d\theta &= 2\pi, \\ \int_{-\pi}^{\pi} \cos\left(\frac{\cos\theta}{r}\right) \cosh\left(\frac{\sin\theta}{r}\right) \cos(2n\theta) d\theta &= \pi\frac{(-1)^n (2n)!}{r^{2n}}, \\ \int_{-\pi}^{\pi} \sinh\left(\frac{\sin\theta}{r}\right) \cos\left(\frac{\cos\theta}{r}\right) \sin(2n\theta) d\theta &= \pi\frac{(-1)^n (2n)!}{r^{2n}}, \end{cases} \tag{2.13}$$

where $r \in (0, \infty)$.

EXAMPLE 7. Consider $f(z) = e^{z+\frac{1}{z}}, z \neq 0$, we have

$$u(r,\theta) = e^{\frac{r^2+1}{r}\cos\theta} \cos\left(\frac{r^2-1}{r}\sin\theta\right),\ v(r,\theta) = e^{\frac{r^2+1}{r}\cos\theta} \sin\left(\frac{r^2-1}{r}\sin\theta\right) \tag{2.14}$$

and $f(z) = \sum_{n=0}^{\infty} \left[\sum_{q=0}^{\infty} \frac{1}{q!(q+n)!}\right] z^n + \sum_{n=1}^{\infty} \left[\sum_{q=0}^{\infty} \frac{1}{q!(q+n)!}\right] \frac{1}{z^n}, z \neq 0$. Thus

$$\begin{cases} \int_{-\pi}^{\pi} u(r,\theta)\ d\theta &= 2\pi \sum_{q=0}^{\infty} \frac{1}{q! q!}, \\ \int_{-\pi}^{\pi} u(r,\theta) \cos(n\theta)\ d\theta &= \pi(r^n + r^{-n}) \sum_{q=0}^{\infty} \frac{1}{q!(q+n)!}, \\ \int_{-\pi}^{\pi} \sinh\left(\frac{\sin\theta}{r}\right) v(r,\theta) \sin(n\theta)\ d\theta &= \pi(r^n - r^{-n}) \sum_{q=0}^{\infty} \frac{1}{q!(q+n)!}, \end{cases} \tag{2.15}$$

where $r \in (0, \infty)$.

EXAMPLE 8. Consider $f(z) = \sin z \sin(\frac{1}{z}), z \neq 0$, we have

$$u(r,\theta) = \sin^2(r\cos\theta)\cosh^2(r\sin\theta) + \cos^2(r\cos\theta)\sinh^2(r\sin\theta)\,,\ \ v(r,\theta) = 0 \tag{2.16}$$

and $f(z) = \sum_{n=0}^{\infty}(-1)^n \left[\sum_{q=0}^{\infty} \frac{1}{(2q+1)!(2q+2n+1)!}\right] z^{2n} + \sum_{n=1}^{\infty} \left[\sum_{q=0}^{\infty} \frac{1}{(2q+1)!(2q+2n+1)!}\right] \frac{1}{z^{2n}}$.
Thus (2.17)

$$\begin{cases} \int_{-\pi}^{\pi} u(r,\theta)\,d\theta & = & 2\pi \sum_{q=0}^{\infty} \frac{1}{((2q+1)!)^2}, \\ \int_{-\pi}^{\pi} u(r,\theta)\cos(2n\theta)d\theta & = & \pi\left[(-1)^n r^{2n} + r^{-2n}\right] \sum_{q=0}^{\infty} \frac{1}{(2q+1)!(2q+2n+1)!}, \end{cases}$$

where $r \in (0,\infty)$.

3. Conclusion

In [**1**], the authors proposed an interesting method for evaluation of the definite integrals of the forms $\int_{-\pi}^{\pi} u(r,\theta)d\theta$, $\int_{-\pi}^{\pi} u(r,\theta)\cos(n\theta)d\theta$ and $\int_{-\pi}^{\pi} v(r,\theta)\sin(n\theta)d\theta$, where $r > 0$ and $u(r,\theta)$, $v(r,\theta)$ are real and imaginary parts of $f(z)$, respectively; which is represented by the Taylor series as $f(z) = \sum_{n=0}^{\infty} c_n z^n$.
In this note, we have expanded this method to compute the above integrals by developing a useful technique for all analytic complex functions that contain positive and negative powers of z. This technique gives us a process for helping to recognize a whole range of new integrals.

References

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