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Uniqueness Theorem for Hilbert Transform for Boehmians

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Abstract. In this paper a uniqueness theorem is proved for the Hilbert transform for the Boehmians of analytic function by using a relation between the Hilbert transform and the Fourier transform. Hilbert transform of Boehmians of analytic type is also discussed.

1. Introduction

Hilbert transform occurs in many branches of pure and applied mathematics. The transform plays an important role in Schwartz theory of distributions (generalized functions) [**10**]. The distributional Hilbert transformation is defined in terms of analytic representations of distributions by Orton [**9**]. Most classes of generalized functions are constructed analytically, see [**13**]. The most well-known space of generalized functions is the space of distributions [**13**], denoted by $\mathcal{D}^{'}(\mathbb{R})$ (the space of continuous linear functional on $\mathcal{D}(\mathbb{R})$) while $\mathcal{D}(\mathbb{R})$ denotes the set of all complex-valued infinitely differentiable function on $\mathbb{R}$ having compact support. Boehmians (or the generalized quotient spaces) is a generalization of generalized function (Schwartz theory of distributions). The construction of Boehmians was motivated by the concept of regular operators, see [**1**], and given by Mikusiński [**6**]. Karunakaran [**4**] has extended the Hilbert transform to the Boehmian space and studied its properties. The Hilbert transform becomes a continuous linear map from one space of Boehmians into another in contrast to other integral transform of Boehmians wherein the image of these transforms are classical distributions or analytic functions. Nemzer [8] constructed a subspace of Boehmians, called Boehmians of analytic type, which is said to possess a uniqueness property. Singh *et al.* [**12**] developed this theory for the Mellin transform. In the present paper using the technique of [8] and [**12**], respectively a uniqueness theorem for Hilbert transform for Boehmians of analytic functions is obtained and further the Hilbert transform of analytic Boehmians is discussed.

Let the set of all real analytic functions on a given set p is denoted by $C^w(p)$. Then for any open set $U : \Omega \subseteq C$, the set $A(\Omega)$ of all analytic functions $U : \Omega \to C$

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is a Fréchet space with respect to the uniform convergence on compact sets. The boundary-value theory of analytic function in the unit disk has always been a rich area in term of its interesting mathematical problems.

If an analytic function $f(z)$ is bounded in the unit disc $\mathcal{D}$, then it has the uniqueness property that if $\lim_{r\to 1} f(re^{i\theta}) = 0$ on a set of positive measure on the unit circle $S^{'}$, which implies $f(z)$ to be identically zero, where the radial limit $F(re^{i\psi}) = \lim_{h\to 1} f(he^{i\theta}) = 0$, almost everywhere on $S^{'}$. Riesz [**11**] showed that any bounded analytic function in $\mathcal{D}$ has the uniqueness property.

Let $S^{'}$ denote the unit circle, $C(S^{'})$ is the collection of continuous complex valued function of $S^{'}$. By $C^{N}(S^{'})$ we mean collection of sequence of continuous complex valued function on $S^{'}$. The definition for the Hilbert transform of periodic function contains the period 2π, see [**5**]. No distinction is made between a function on $S^{'}$ and a $2\pi-$periodic function on the real line $\mathbb{R}$. The convolution of two functions $f, g \in C(S^{'})$, denoted by $f * g$, is defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt, \qquad x \in [-\pi, \pi]. \tag{1.1}$$

The n^{th} Fourier coefficient of $f \in C(S^{'})$ is defined by [**7**]:

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Let G is linear space, i.e., $G = C^{\infty}(\mathbb{R})$, which is also considered as a quasi-normed space that is equipped with the topology of uniform convergence on compact set $S \in \mathcal{D}(\mathbb{R})$, and Δ be the class of sequence from $\mathcal{D}$ which satisfies the following conditions

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta_n(x)dx = 1$ for all $n \in \mathbb{N}$

(ii) supp$\delta_n \subseteq (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$,

where δ_n is a delta sequence (a sequence of continuous non-negative functions).

A pair of sequence (f_n, δ_n), denoted by $\frac{f_n}{\delta_n}$, is called quotient of sequence, if

$$A = \left\{(\{f_n\}, \{\delta_n\}) : f_k * \delta_n = f_n * \delta_k \text{ for all } n, k \in \mathbb{N}\right\},$$

where $f_n \in C(\mathbb{R})$, $n = 1, 2, ...,$ and $A \subseteq C^{\mathbb{N}}(S^{'}) \times \Delta$.

Two quotients of sequences $\frac{f_n}{\delta_n}$ and $\frac{g_n}{\sigma_n}$ are called equivalent if $f_n * \sigma_m = g_m * \delta_n$ for all $m, n \in \mathbb{N}$, which is said to be an equivalence relation on A.

The equivalence classes are called periodic Boehmians [**7**], defined by

$$\mathcal{B} = \left\{\left[\frac{\{f_n\}}{\{\delta_n\}}\right] : (\{f_n\}, \{\delta_n\}) \in A\right\}. \tag{1.2}$$

The natural addition, multiplication and scalar multiplication on $\mathcal{B}$ imply

$$\frac{f_n}{\delta_n} + \frac{g_n}{\sigma_n} = \frac{(f_n * \sigma_n + g_n * \delta_n)}{\delta_n * \sigma_n} \tag{1.3}$$

$$\frac{f_n}{\delta_n} * \frac{g_n}{\sigma_n} = \frac{(f_n * g_n)}{\delta_n * \sigma_n} \tag{1.4}$$

$$\alpha\left(\frac{f_n}{\delta_n}\right)=\frac{\alpha f_n}{\delta_n} \tag{1.5}$$

where α is a complex number and $\mathcal{B}$ becomes a commutative algebra with identity $\delta=\frac{\delta_n}{\delta_n}$.

2. Uniqueness theorem for Hilbert transform for Boehmians of analytic functions

DEFINITION 2.1. [**10**] Let $f(t)$ be a periodic function with period 2τ which is L^p integrable over the interval $[-\tau,\tau]$. Then the Hilbert transform of $(Hf)(x)$ of $f(t)$ is given by

$$(Hf)(x)=\tilde{f}(x)=\frac{1}{\pi}\lim_{N\to\infty}\int_{-N}^{N}\frac{f(t)}{x-t}dt=\frac{1}{2\tau}(P)\int_{-\tau}^{\tau}f(x-t)\cot(\frac{t\pi}{2\tau})dt \tag{2.1}$$

for almost all $x\in\mathbb{R}$, where P is the principal value of the integral.

The Hilbert transform $\tilde{f}$ of any $f\in P_{2\tau}^{'}$ is defined as a functional by

$$\langle\tilde{f},\theta\rangle=\langle -f,\tilde{\theta}\rangle,\qquad(\theta\in P_{2\tau}). \tag{2.2}$$

NOTE 2.1. Here the testing function space $P_{2\tau}$ is defined as the space of all smooth functions with period 2τ and $P_{2\tau}^{'}$ denotes the dual of $P_{2\tau}$ with its weak* topology.

In [**10**] it is proved that the Hilbert transform of any element in $P_{2\tau}$ exists and is also in $P_{2\tau}$. It is clear that the Hilbert transform is linear on $P_{2\tau}$ and it is also proved that the Hilbert transform of any element in $P_{2\tau}^{'}$ belongs to $P_{2\tau}^{'}$. It is clear that the Hilbert transform in linear on $P_{2\tau}^{'}$. The following Lemmas can easily be proved [**10**].

LEMMA 2.1. *If $\varphi\in P_{2\tau}$ and $x\in\mathbb{R}$ then $(\tau_x\check{\varphi})\tilde{}=-\tau_x(\tilde{\varphi})\check{}$. In particular, $(\check{\varphi})\tilde{}=-(\tilde{\varphi})\check{}$.*

LEMMA 2.2. *If $f\in P_{2\tau}$ and $\varphi\in\mathbb{R}$ then $(f*\varphi)\tilde{}=(\tilde{f})*\varphi=f*(\tilde{\varphi})$ in $P_{2\tau}$.*

LEMMA 2.3. *If $f\in P_{2\tau}^{'}$ and $\varphi\in\mathbb{R}$ then $(f*\varphi)\tilde{}=(\tilde{f})*\varphi=f*(\tilde{\varphi})$ in $P_{2\tau}$.*

The Hilbert transform of compactly supported distribution u (an element of the dual $\varepsilon^{'}$ of $\varepsilon=C^\infty(\mathbb{R})$) is usually defined as an analytic function as follows. Let z belongs to the complement of the support of u and let $\alpha(t)\in C^\infty(\mathbb{R})$ be such that in neighbourhood of the support of u, $\alpha(t)=1$ and $\alpha(t)=0$ for sufficiently large t. Then the Hilbert transform of u is defined by

$$(\tilde{f})(x)=\left\langle f(t),\frac{\alpha(t)}{t-x}\right\rangle.$$

It is defined [**2, 3**] that this is an analytic function of $z\in C\setminus\operatorname{supp}u$.

The classical relation between Fourier transform and the Hilbert transform [**10**, pp. 125-130], is given by

$$[\hat{\tilde{\phi}}](x)=i\,\pi sgn(x)[\hat{\phi}](x),\qquad\forall\phi\in S, \tag{2.3}$$

where sgn is the signum function:

$$sgn\ x = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0. \end{cases}$$

Further, if f is a periodic function, then the k^{th} coefficient $\tilde{f}(k)$ of f is given by,

$$c_k(Hf) = c_k(\tilde{f}) = i\pi\ sgn(k)(\hat{f}).$$

LEMMA 2.4. *Let $F = [\frac{f_n}{\varphi_n}] \in \mathcal{B}$. Then for each k, the sequence $\{c_k(\tilde{f}_n)\}_{n=1}^{\infty}$ converges.*

PROOF. Let $k \in \mathbb{Z}$. Since $\{\varphi_w\}_{w=1}^{\infty}$ is a delta sequence, there exists a $w \in \mathbb{N}$ such that $\tilde{\varphi}_w(k) \neq 0$. Now

$$\begin{aligned} c_k(\tilde{f}_n) &= c_k(\tilde{f}_n)\frac{\hat{\varphi}_w(k)}{\hat{\varphi}_w(k)} \\ &= i\pi sgn(k)\frac{(f_n * \varphi_w)\hat{}(k)}{\hat{\varphi}_w(k)} \\ &= i\pi sgn(k)\frac{(f_w * \varphi_n)\hat{}(k)}{\hat{\varphi}_w(k)} \end{aligned}$$

i.e.

$$\begin{aligned} &= i\pi sgn(k)\frac{(\hat{f}_w(k))}{\hat{\varphi}_w(k)}.\hat{\varphi}_n(k) \\ &\longrightarrow i\pi sgn(k)\frac{(\hat{f}_w(k))}{\hat{\varphi}_w(k)}, \text{ as } n \to \infty. \end{aligned}$$

Thus the lemma is proved. □

DEFINITION 2.2. Let $F = [\frac{f_n}{\varphi_n}] \in \mathcal{B}$. Then the k^{th} Hilbert coefficient of f is defined by

$$c_k(\tilde{f}) = \lim_{n\to\infty} c_k(\tilde{f}_n). \tag{2.4}$$

DEFINITION 2.3. A Boehmian F is said to be zero on an open set Ω, denote by $F = 0$ on Ω, if there exist a delta sequence $\{\delta_n\}$ such that $F * \delta_n \in C(S')$ for all $n \in \mathbb{N}$ and $F * \delta_n \to 0$ uniformly on compact subset of Ω as $n \to \infty$.

DEFINITION 2.4. A Boehmian $F = [\frac{f_n}{\varphi_n}] \in \mathcal{B}$ is said to be of analytic type if $\tilde{F}(k) = 0$ for $k = -1, -2, ...$

THEOREM 2.2. *If $F = [\frac{f_n}{\varphi_n}] \in \mathcal{B}$ be a Boehmian of analytic type and $\tilde{F}(k)$ denotes its Hilbert transform such that $F = 0$ on some open arc Ω, then $F \equiv 0$.*

PROOF. Let $F \in \mathcal{B}$ be a analytic type such that $F = 0$ on Ω. Since $\tilde{F}(n) = 0$ for $n = -1, -2, ...$, while for each n

$$\tilde{f}_n(k) = \tilde{F}(k)\delta_n(k) = 0 \text{ for } k = -1, -2, ... \tag{2.5}$$

Invoking the Definition of Boehmians, $f_n * \varphi_\omega = f_\omega * \varphi_n$ for all $n, \omega \in \mathbb{N}$, we have

$$f_n = f_n - (f_n * \delta_\omega) + (f_n * \delta_\omega), \text{ for all } n, \omega \in \mathbb{N}. \tag{2.6}$$

Since $\{\delta_n\}$ is a delta sequence, for each ω, $(f_n * \delta_\omega) \to f_n$ uniformly on T as $\omega \to \infty$. Let J be any closed subinterval on Ω. Then there exist a closed interval I, for an $\alpha > 0, J \subset I \subset \Omega$, and $(-\alpha, \alpha) + J \subseteq I$. Also there exists an $n_0 \in \mathbb{N}$ such that supp $\delta_n \subseteq (-\alpha, \alpha)$, for all $n \geqslant n_0$. Let n_o be any fixed integer, $n > n_0$. Then for all $\omega \geqslant \omega_0$, let $\varepsilon > 0$. Since $f_\omega \to 0$ uniformly on I as $\omega \to \infty$, there exist a $\omega_0 \in \mathbb{N}$ such that for all $\omega \geqslant \omega_0$, $|f_\omega(x)| < \varepsilon$ for all $x \in I$. Then

$$\begin{aligned} |(f_n * \delta_\omega)(x)| &= |(f_\omega * \delta_n)(x)| \\ &\leqslant \frac{1}{2\pi} \int_{-\alpha}^{\alpha} |f_\omega(x-t)| \delta_n(t) dt \\ &\leqslant \frac{\varepsilon}{2\pi} \int_{-\alpha}^{\alpha} \delta_n(t) dt = \varepsilon, \quad \text{for all } x \in J. \end{aligned} \tag{2.7}$$

Therefore, $(f_n * \delta_\omega) \to 0$ uniformly on J as $\omega \to \infty$, for each $n \geqslant n_0$. By combining (2.5), (2.6) and (2.7), we see that for each $n \geqslant n_0$, f_n vanishes on J. This completes the proof of the theorem. □

3. Analytic Boehmian and Hilbert transform

A Boehmian space consists of analytic functions in the open unit disc $\mathcal{D}$ of $\mathbb{C}$, called analytic Boehmians, defined by

$$A = \left\{ f(z) = \sum_{k=0}^{\infty} a_k z^k : |a_k| \leqslant Ce^{-\lambda\omega(k)} \;\; \forall k \in \mathbb{N}, \text{and for each } \lambda > 0 \right\}. \tag{3.1}$$

We also take the multiplication in G as the Hadamard product is defined by

$$\sum_{k=0}^{\infty} a_k z^k \star \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} (a_k . b_k) z^k.$$

The Hilbert transform of an periodic function $f(t)$ which is the uniform limit of the sequence of trigonometric polynomial

$$f_n(t) = \sum_{m=1}^{n} a_m e^{\lambda_m t i}$$

may be defined by

$$(\tilde{f})(x) = \lim_{n\to\infty} (\tilde{f_n}(t)) = \sum_{m=1}^{\infty} -a_m i \operatorname{sgn}(\lambda_m) e^{\lambda_m x i} \tag{3.2}$$

provided the limit exists.

In [**15**] it has been shown that if f is periodic function with period 2π and of bounded variation then its Fourier series and the corresponding conjugate Fourier series are related by Hilbert transform, i.e., if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$g(x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)$$

then

$$g(x) = \frac{1}{2\pi}(P)\int_{-\pi}^{\pi} f(t)\cot\frac{x-t}{2}dt = \frac{1}{2\pi}(P)\int_{-\pi}^{\pi} f(x-t)\cot\frac{t}{2}dt. \tag{3.3}$$

In fact it is proved in [**15**] that the Fourier series of $f(x)$ converges to $f(x)$ iff the integral in (3.3) converges. It is true that if f_n is the partial sum of the Fourier series on $f(x)$ and $g_n(x)$ is the partial sum of the corresponding conjugate Fourier series then $\tilde{f}_n(x) = g_n(x)$ and that

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \tilde{f}_n(x) \Rightarrow g(x) = (\tilde{f})(x).$$

Thus, for this class of periodic function we have

$$D^k\tilde{f} = (D^k f)\tilde{}$$

and also

$$\langle \tilde{f}, \varphi\rangle = \langle f, -\tilde{\varphi}\rangle, \qquad \forall \varphi \in S.$$

Definition 3.1. Let $X = (\{f_k\}, \{\phi_k\}) \in \mathcal{B}$. The Hilbert coefficient of X, is defined by $c_k(\tilde{X}) = \lim_{k\to\infty} i\pi sgn(k)(\hat{f}_k)$.

Definition 3.2. A sequence $\{X_n\}$ in $\mathcal{B}$ is said to δ-converge to some $X \in \mathcal{B}$, denoted by $\{X_n\} \xrightarrow{\delta} X$ as $n \to \infty$ if there exists a sequence $\{\phi_k\} \in \Delta$ such that $X_n * \phi_k, X * \phi_k \in L^1(S')(n, k \in \mathbb{N})$ and $X_n * \phi_k \to X * \phi_k \in L^1(S')$ as $n \to \infty$ for each fixed k.

Theorem 3.1. *Let T be a Schwartz distribution exist trigonometric series $\sum_{n=1}^m a_n e^{\lambda_n ti}$ which converges in the weak distribution sense to T and that the k^{th} order distribution T is given by $D^kT = \sum_{n=1}^m a_n(\lambda_n i)^k e^{\lambda_n ti}$. Also, let $f \in \zeta$ (class of all analytic functions defined in the open unit disc $\mathcal{D}$). Then there exists a sequence $\{S_m\}$ in A and $X \in \mathcal{B}$ such that $S_m \to f$ in ζ and $D^kT \xrightarrow{\delta} X$ in $\mathcal{B}$ as $m \to \infty$.*

Proof. Let $f(z) = \sum_{n=0}^m a_n z^n \in \zeta$ with $\sum_{n=0}^m |a_n| < \infty$. For each $m \in \mathbb{N}$, define

$$S_m(z) = \sum_{n=0}^{m} a_n z^n, \;\; z \in \mathcal{D}.$$

Then obviously $S_m \in A$ and $S_m \to f$ in ζ as $m \to \infty$. Also $T = \sum_{n=1}^{m} a_n e^{\lambda_n t i}$. By convolution of two functions we have

$$
\begin{aligned}
(T * f_k)(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} T(t-\eta) f_k(\eta) d\eta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{m} a_n e^{i\lambda_n (t-\eta)} f_k(\eta) d\eta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=0}^{m} a_n e^{i\lambda_n t} e^{-i\lambda_n(\eta)} f_k(\eta) d\eta \\
&= \sum_{n=0}^{m} a_n e^{i\lambda_n t} \hat{f}_k(\lambda_n) \\
&= \frac{i}{sgn(\lambda_n)} \lim_{n\to\infty} (\tilde{f}_n(t)) \hat{f}_k(\lambda_n). \qquad \text{(by (3.2))}
\end{aligned}
$$

Define $g_k(t) = \sum_{n=0}^{m} a_n e^{i\lambda_n(t)} \hat{f}_k(\lambda_n) \in L^1(S')$. By choosing

$$
\hat{f}_k(\lambda_n) = \begin{cases} \left(1 + \dfrac{|\lambda_n|}{k+1}\right), \text{ for } |\lambda_n| \leqslant k \\ \\ 0, \quad \text{otherwise,} \end{cases}
$$

we can easily prove that for each $k, T * f_k \to \sum_{n=0}^{m} a_n e^{i\lambda_n(t)} \in L^1(S')$ as $m \to \infty$ and $(\{g_k\}, \{f_k\}) \in A$. Therefore, if we take $X = (\{g_k\}, \{f_k\}) \in \mathcal{B}$, by definition $S_m \xrightarrow{\delta} X$ in $\mathcal{B}$ as $m \to \infty$.

This completes the proof of the Theorem. □

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