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Certain Inequalities Related to the Chebyshev's Functional Involving Erdélyi-Kober Operators

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Abstract. The object of the present investigation is to obtain certain Chebyshev type integral inequalities involving the Erdélyi-Kober fractional integral operators. Some consequent results and special cases of the main results are also pointed out.

1. Introduction

In the present paper, our work is based on a celebrated functional introduced by Chebyshev ([**5**]), which is defined by

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \quad (1.1)$$

where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$\{(f(x)-f(y))\,(g(x)-g(y))\} \geqslant 0, \quad (1.2)$$

for any $x, y \in [a,b]$, then the Chebyshev inequality is given by $T(f,g) \leqslant 0$.
The functional (1.1) has applications in numerical quadrature, transform theory, probability, study of existence of solutions of differential equations and in statistical problems. Therefore, in the literature several generalizations of the Chebyshev type integral inequality ([**5**]) are considered by many authors; for instance, Belarbi and Dahmani [**4**], Dahmani *et al.* [**7**], Dahmani and Tabharit [**8**] and Sulaiman [**18**];and they derived certain Chebyshev type integral inequalities involving Riemann-Liouville fractional integral operators. Recently, Purohit and Raina [**14**], and Purohit *et al.* [**15**] investigated certain integral inequalities that are associated with Chebyshev functional, involving the Saigo fractional integral operators ([**16**]), and also established the q-extensions of the main results. Further, Baleanu *et al.* [**2**]-[**3**] established certain generalized integral inequalities for synchronous functions that are related to the Chebyshev functional using the fractional hypergeometric operator, introduced by Curiel and Galué

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[6]. Recently, Anastassiou [1] has considered the most general fractional representation formulae for a function in terms of the most general fractional integral operators due to Kalla [10]-[11] and derived general fractional Ostrowski type inequalities.
In this paper, we add one more dimension to this study by introducing certain new integral inequalities for synchronous functions, involving the Erdélyi-Kober fractional integral operators. Further, we develop certain known and new integral inequalities for the fractional integrals by suitably choosing the special cases of our main results.

First we give some necessary definitions and mathematical preliminaries of fractional calculus operators which are used in our analysis.

Definition 1. Let $\alpha > 0$, $\beta > 0$ and $\eta \in \mathbb{R}$, then the Erdélyi-Kober fractional integral operators $I_{\beta}^{\eta,\alpha}$ of order α for a real-valued continuous function $f(t)$ is defined by (see [12, p. 14, eqn. (1.1.17)]):

$$\begin{aligned} I_{\beta}^{\eta,\alpha}\{f(t)\} &= \frac{t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\beta\eta}(t^{\beta}-\tau^{\beta})^{\alpha-1} f(\tau) d(\tau^{\beta}) \\ &= \frac{\beta\, t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\beta(\eta+1)-1}(t^{\beta}-\tau^{\beta})^{\alpha-1} f(\tau) d\tau. \end{aligned} \tag{1.3}$$

Our results in this paper are based on the following preliminary assertions giving composition formula of fractional integral (1.3) with a power function (see also as special case of image formula [12, p. 29, eqn. (1.2.26)]).

$$I_{\beta}^{\eta,\alpha}\left\{t^{\lambda}\right\} = \frac{\Gamma(1+\eta+\frac{\lambda}{\beta})}{\Gamma(1+\alpha+\eta+\frac{\lambda}{\beta})}\, t^{\lambda}\ (\lambda > -\beta(\eta+1)). \tag{1.4}$$

2. Main results

In this section, we obtain certain fractional integral inequalities for synchronous functions which are related to the Chebyshev functional ([5]) by using the Erdélyi-Kober fractional integral operators (1.3) (defined above).

Theorem 1. *Let f and g be two synchronous functions on $[0,\infty)$ then*

$$I_{\beta}^{\eta,\alpha}\{f(t)g(t)\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\ I_{\beta}^{\eta,\alpha}\{f(t)\}\ I_{\beta}^{\eta,\alpha}\{g(t)\}, \tag{2.1}$$

for all $t > 0,\ \alpha > 0, \beta > 0$ and $\eta \in \mathbb{R}$ such that $\eta > -1$.

Proof: Let f and g be two synchronous functions, then for all $\tau,\ \rho \in (0,t),\ t \geqslant 0$, we have

$$\{(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\} \geqslant 0,$$

which implies that

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geqslant f(\tau)g(\rho) + f(\rho)g(\tau). \tag{2.2}$$

Consider

$$\mathcal{F}(t,\tau) = \frac{\beta\ t^{-\beta(\eta+\alpha)}\tau^{\beta(\eta+1)-1}}{\Gamma(\alpha)}(t^{\beta}-\tau^{\beta})^{\alpha-1}. \tag{2.3}$$

We observe that $\alpha, \beta > 0$ before, and hence each factor of the above function is positive in view of the conditions stated with Theorem 1, and hence, the function $\mathcal{F}(t,\tau)$ remains positive, for all $\tau \in (0,t)$ $(t > 0)$.

Multiplying both sides of (2.2) by $\mathcal{F}(t,\tau)$ (where $\mathcal{F}(t,\tau)$ is given by (2.3)) and integrating with respect to τ from 0 to t, and using operator (1.3), we get

$$I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\} + f(\rho)g(\rho)\ I_{\beta}^{\eta,\alpha}\left\{1\right\} \geqslant g(\rho)\ I_{\beta}^{\eta,\alpha}\left\{f(t)\right\} + f(\rho)\ I_{\beta}^{\eta,\alpha}\left\{g(t)\right\}. \tag{2.4}$$

Next, multiplying both sides of (2.4) by $\mathcal{F}(t,\rho)$ $(\rho \in (0,t)\,,\ t > 0)$, where $\mathcal{F}(t,\rho)$ is given by (2.3), and integrating with respect to ρ from 0 to t, and using formula (1.4) (for $\lambda = 0$), we arrive at the desired result (2.1).

The following result gives a generalization of Theorem 1.

Theorem 2 *Let f and g be two synchronous functions on $[0,\infty)$, then*

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)\right\} + \frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)} I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\} \geqslant I_{\beta}^{\eta,\alpha}\left\{f(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)\right\} + I_{\delta}^{\zeta,\gamma}\left\{f(t)\right\}\ I_{\beta}^{\eta,\alpha}\left\{g(t)\right\}, \tag{2.5}$$

for all $t > 0$, $\alpha > 0, \beta > 0$, $\gamma > 0, \delta > 0$ and $\eta, \zeta \in \mathbb{R}$ such that $\eta > -1$ and $\zeta > -1$.

Proof: To prove the above theorem, we use the inequality (2.4). Multiplying both sides of (2.4) by

$$\mathcal{H}(t,\rho) = \frac{\delta\ t^{-\delta(\zeta+\gamma)}\rho^{\delta(\zeta+1)-1}}{\Gamma(\gamma)}(t^{\delta}-\rho^{\delta})^{\gamma-1}, \tag{2.6}$$

which remains positive in view of the conditions stated with (2.5) then integrating with respect to ρ from 0 to t, we get

$$I_{\delta}^{\zeta,\gamma}\left\{1\right\} I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\} + I_{\beta}^{\eta,\alpha}\left\{1\right\} I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)\right\} \geqslant$$
$$I_{\beta}^{\eta,\alpha}\left\{f(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)\right\} + I_{\delta}^{\zeta,\gamma}\left\{f(t)\right\}\ I_{\beta}^{\eta,\alpha}\left\{g(t)\right\},$$

which on using (1.4) yields the desired result (2.5).

Theorem 3. *Let $(f_i)_{i=1,\cdots,n}$ be n positive increasing functions on $[0,\infty)$, then*

$$I_{\beta}^{\eta,\alpha}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \left[\frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\right]^{n-1}\ \prod_{i=1}^{n} I_{\beta}^{\eta,\alpha}\left\{f_i(t)\right\}\ , \tag{2.7}$$

for all $t > 0$, $\alpha > 0, \beta > 0$ and $\eta \in \mathbb{R}$ such that $\eta > -1$.

Proof: We prove this theorem by mathematical induction. Clearly, for $n = 1$ in (2.7), we have

$$I_{\beta}^{\eta,\alpha}\left\{f_1(t)\right\} \geqslant I_{\beta}^{\eta,\alpha}\left\{f_1(t)\right\}\ (t > 0, \alpha > 0).$$

Next, for $n = 2$, in (2.7), we get

$$I_\beta^{\eta,\alpha}\left\{f_1(t)f_2(t)\right\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\left\{f_1(t)\right\}\; I_\beta^{\eta,\alpha}\left\{f_2(t)\right\}\;\; (t>0,\alpha>0),$$

which holds in view of (2.1) of Theorem 1.
By the induction principle, we suppose that the inequality

$$I_\beta^{\eta,\alpha}\left\{\prod_{i=1}^{n-1} f_i(t)\right\} \geqslant \left[\frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\right]^{n-2} \prod_{i=1}^{n-1} I_\beta^{\eta,\alpha}\left\{f_i(t)\right\}\;, \tag{2.8}$$

holds true for some positive integer $n \geqslant 2$.
Now $(f_i)_{i=1,\cdots,n}$ are increasing functions which imply that the function $\prod_{i=1}^{n-1} f_i(t)$ is also an increasing function. Therefore, we can apply inequality (2.1) of Theorem 1 to the functions $\prod_{i=1}^{n-1} f_i(t) = g$ and $f_n = f$ to get

$$I_\beta^{\eta,\alpha}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\left\{\prod_{i=1}^{n-1} f_i(t)\right\}\; I_\beta^{\eta,\alpha}\left\{f_n(t)\right\},$$

provided that $t>0,\ \alpha>0, \beta>0,\ \eta>-1$.
Making use of (2.8) now, this last inequality above leads to the result (2.7), which proves Theorem 3.

Theorem 4. *Let f and g be two synchronous functions on $[0,\infty)$, $h>0$, then for all $t>0,\ \alpha>0, \beta>0, \eta>-1$*

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}\; I_\beta^{\eta,\alpha}\left\{f(t)g(t)h(t)\right\} \geqslant I_\beta^{\eta,\alpha}\left\{f(t)\right\}\; I_\beta^{\eta,\alpha}\left\{g(t)h(t)\right\} + I_\beta^{\eta,\alpha}\left\{g(t)\right\}\; I_\beta^{\eta,\alpha}\left\{f(t)h(t)\right\} - I_\beta^{\eta,\alpha}\left\{h(t)\right\}\; I_\beta^{\eta,\alpha}\left\{f(t)g(t)\right\}. \tag{2.9}$$

Proof: Using (1.2) and $h>0$, for all $\tau,\ \rho \geqslant 0$, we have

$$\left\{\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right)\left(h(\tau)+h(\rho)\right)\right\} \geqslant 0, \tag{2.10}$$

which implies that

$$f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) \geqslant f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + g(\tau)f(\rho)h(\rho) + g(\rho)f(\tau)h(\tau) - h(\tau)f(\rho)g(\rho) - h(\rho)f(\tau)g(\tau). \tag{2.11}$$

Multiplying both sides of (2.11) by $\mathcal{F}(t,\tau)$ (defined above by (2.3)) and integrating with respect to τ from 0 to t, and using (1.3), we get

$$I_\beta^{\eta,\alpha}\left\{f(t)g(t)h(t)\right\} + f(\rho)g(\rho)h(\rho)\; I_\beta^{\eta,\alpha}\left\{1\right\} \geqslant g(\rho)h(\rho)\; I_\beta^{\eta,\alpha}\left\{f(t)\right\} + f(\rho)\; I_\beta^{\eta,\alpha}\left\{g(t)h(t)\right\} + f(\rho)h(\rho)\; I_\beta^{\eta,\alpha}\left\{g(t)\right\} + g(\rho)\; I_\beta^{\eta,\alpha}\left\{f(t)h(t)\right\} - f(\rho)g(\rho)\; I_\beta^{\eta,\alpha}\left\{h(t)\right\} - h(\rho)\; I_\beta^{\eta,\alpha}\left\{f(t)g(t)\right\}. \tag{2.12}$$

Next, multiplying both sides of (2.12) by $F(t,\rho)$ $(\rho \in (0,t)\,,\ t>0)$, where $\mathcal{F}(t,\rho)$ is given by (2.3), and integrating with respect to ρ from 0 to t, and using formula (1.4), we arrive at the desired result (2.9).

Theorem 5. *Let f and g be two synchronous functions on $[0,\infty)$, and $h>0$, then*

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)h(t)\right\}+\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}\ I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)h(t)\right\}\geqslant I_{\beta}^{\eta,\alpha}\left\{f(t)\right\}$$
$$I_{\delta}^{\zeta,\gamma}\left\{g(t)h(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{g(t)h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{g(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)h(t)\right\}$$
$$+I_{\beta}^{\eta,\alpha}\left\{f(t)h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{h(t)\right\}, \tag{2.13}$$

for all $t>0,\ \alpha>0,\beta>0,\ \gamma>0,\delta>0$ and $\eta,\zeta\in\mathbb{R}$ such that $\eta>-1$ and $\zeta>-1$.

Proof: To prove the above theorem, we start with the inequality (2.12). On multiplying both sides of (2.12) by $\mathcal{H}(t,\rho)$ (defined above by (2.6)) and integrating with respect to ρ from 0 to t, we get

$$I_{\delta}^{\zeta,\gamma}\left\{1\right\}I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)h(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{1\right\}I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)h(t)\right\}\geqslant I_{\beta}^{\eta,\alpha}\left\{f(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)h(t)\right\}+$$
$$I_{\beta}^{\eta,\alpha}\left\{g(t)h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{g(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)h(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{f(t)h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)\right\}-$$
$$I_{\beta}^{\eta,\alpha}\left\{h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{h(t)\right\},$$

which on using (1.4) yields the desired result (2.13).

Theorem 6. *Let f, g and h be three monotonic functions on $[0,\infty)$ satisfying the inequality*

$$\left\{\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right)\left(h(\tau)-h(\rho)\right)\right\}\geqslant 0, \tag{2.14}$$

then for all $t>0,\ \alpha>0,\beta>0,\ \gamma>0,\delta>0$ and $\eta,\zeta\in\mathbb{R}$ such that $\eta>-1$ and $\zeta>-1$,

$$\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)h(t)\right\}-\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}\ I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)h(t)\right\}\geqslant I_{\beta}^{\eta,\alpha}\left\{g(t)h(t)\right\}$$
$$I_{\delta}^{\zeta,\gamma}\left\{f(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{f(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)h(t)\right\}+$$
$$I_{\beta}^{\eta,\alpha}\left\{f(t)h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{g(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{g(t)\right\}$$
$$I_{\delta}^{\zeta,\gamma}\left\{f(t)h(t)\right\}-I_{\beta}^{\eta,\alpha}\left\{h(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{f(t)g(t)\right\}+I_{\beta}^{\eta,\alpha}\left\{f(t)g(t)\right\}\ I_{\delta}^{\zeta,\gamma}\left\{h(t)\right\}. \tag{2.15}$$

Proof: By applying the similar procedure as of Theorem 2 or 5, one can easily establish the above theorem. Therefore, we omit the details of the proof of this theorem.

Remark 1. It may be noted that the inequalities (2.1), (2.5), (2.9) and (2.13) are reversed if the functions are asynchronous on $[0,\infty)$, i.e.

$$\{(f(x)-f(y))\,(g(x)-g(y))\} \leqslant 0, \tag{2.16}$$

for any $x, y \in [0,\infty)$.

Remark 2. For $\gamma=\alpha, \delta=\beta, \zeta=\eta$, Theorems 2 and 5 immediately reduce to the Theorems 1 and 4, respectively.

Remark 3. If we consider the function $h(t)$ as a constant (>0), the Theorems 4 and 5 immediately reduce to the Theorems 1 and 2, respectively.

Now, we consider some other variations of the fractional integral inequality of Theorem 1:

Theorem 7. *Let f and g be two functions defined on $[0,\infty)$, such that f is increasing, g is differentiable and there exists a real number $m = inf_{t\geqslant 0} g'(t)$, then*

$$I_\beta^{\eta,\alpha}\{f(t)g(t)\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; I_\beta^{\eta,\alpha}\{g(t)\}$$
$$-\frac{m\;\Gamma(1+\alpha+\eta)\Gamma(1+\eta+\frac{1}{\beta})}{\Gamma(1+\alpha+\eta+\frac{1}{\beta})\Gamma(1+\eta)} I_\beta^{\eta,\alpha}\{f(t)\} + m\; I_\beta^{\eta,\alpha}\{tf(t)\}, \tag{2.17}$$

for all $t>0$, $\alpha>0, \beta>0$ and $\eta\in\mathbb{R}$ such that $\eta>-1$.

Proof: Consider the function $h(t) = g(t) - mt$, such that h and f are synchronous. It is clear that h is differentiable and it is increasing on $[0,\infty)$, therefore, by using Theorem 1, we get

$$I_\beta^{\eta,\alpha}\{f(t)(g(t)-mt)\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; I_\beta^{\eta,\alpha}\{g(t)-mt\}$$

$$\geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; I_\beta^{\eta,\alpha}\{g(t)\} - \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; m\; I_\beta^{\eta,\alpha}\{t\}$$

$$\Rightarrow\; I_\beta^{\eta,\alpha}\{f(t)g(t)\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; I_\beta^{\eta,\alpha}\{g(t)\}$$
$$-\frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; m\; I_\beta^{\eta,\alpha}\{t\} + m\; I_\beta^{\eta,\alpha}\{f(t)t)\}$$

Now, on making use of the formula (1.4) (for $\lambda = 1$), we are lead to the result (2.17) after some simplifications.

Theorem 8. *Let f and g be two functions defined on $[0,\infty)$, such that f is decreasing, g is differentiable and there exists a real number $M = Sup_{t\geqslant 0} g'(t)$, then*

$$I_\beta^{\eta,\alpha}\{f(t)g(t)\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\; I_\beta^{\eta,\alpha}\{f(t)\}\; I_\beta^{\eta,\alpha}\{g(t)\}$$

$$-\frac{M\ \Gamma(1+\alpha+\eta)\Gamma(1+\eta+\frac{1}{\beta})}{\Gamma(1+\alpha+\eta+\frac{1}{\beta})\Gamma(1+\eta)} I_{\beta}^{\eta,\alpha}\left\{f(t)\right\} + M\ I_{\beta}^{\eta,\alpha}\left\{tf(t)\right\}, \tag{2.18}$$

for all $t > 0,\ \alpha > 0, \beta > 0$ *and* $\eta \in \mathbb{R}$ *such that* $\eta > -1$.

Proof: Consider the function $h(t) = g(t) - Mt$, such that h and f are synchronous. Then by applying the similar procedure as of Theorem 7, one can easily establish the above theorem. Therefore, we omit the details of the proof of this theorem.

3. Special Cases

Following Kiryakova [**12**], a number of generalized integration and differentiation operators introduced and used by various authors are included as special cases of the operator (1.3). Some important special cases of the integral operator $I_{\beta}^{\eta,\alpha}$ are mentioned below :
(1) If we set $\eta = 0$, $\alpha = n$ ($integer > 0$) and $\beta = 1$, then the operator (1.3) yields the following ordinary n-fold integrations:

$$l^n\left\{f(t)\right\} = t^n I_1^{0,n}\left\{f(t)\right\} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{3.1}$$

(2) For $\eta = 0$ and $\beta = 1$, (1.3) contain the Riemann-Liouville fractional integral operators, by means of the following relationships:

$$R^{\alpha}\left\{f(t)\right\} = t^{\alpha} I_1^{0,\alpha}\left\{f(t)\right\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau. \tag{3.2}$$

(3) Again, for $\eta = 0$, $\alpha = 1$ and $\beta = 1$, the operator (1.3) leads to the Hardy-Littlewood (Cesaro) integration operator:

$$L_{1,0}\left\{f(t)\right\} = I_1^{0,1}\left\{f(t)\right\} = \frac{1}{t}\int_0^t f(\tau)d\tau, \tag{3.3}$$

and as its generalization for integers $m, n; n > m-1$ (when $\eta = n$, $\alpha = 1$ and $\beta = 1$), we have

$$L_{m,n}\left\{f(t)\right\} = t^{n-m+1} I_1^{n,1}\left\{f(t)\right\} = t^{-m}\int_0^t \tau^n f(\tau)d\tau. \tag{3.4}$$

(4) When $\beta = 1$, operator (1.3) reduces to the fractional integral operator, which originally considered by Kober [**13**] and Erdélyi [**9**]:

$$I^{\alpha,\eta}\left\{f(t)\right\} = I_1^{\eta,\alpha}\left\{f(t)\right\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\ \tau^{\eta} f(\tau)d\tau\ (\alpha > 0, \eta \in \mathbb{R}). \tag{3.5}$$

(5) Also for $\beta = 2$, the operator (1.3) yields the Erdélyi-Kober fractional integral operator $I_{\eta,\alpha}$ (introduced by Sneddon [**17**]):

$$I_{\eta,\alpha} = I_2^{\eta,\alpha}\left\{f(t)\right\} = \frac{2\ t^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{2\eta+1}(t^2-\tau^2)^{\alpha-1} f(\tau)d\tau. \tag{3.6}$$

(6) Further, if we set $\eta = -\frac{1}{2}$, $\beta = 2$ and α replaced by $\alpha + \frac{1}{2}$, the Uspensky integral transform ([**19**]), can easily be obtained as under:

$$P^{\alpha}\{f(t)\} = \frac{1}{2} I_2^{-\frac{1}{2},\alpha+\frac{1}{2}}\{f(t)\} = \frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-\tau^2)^{\alpha-\frac{1}{2}} f(t\tau)d\tau. \tag{3.7}$$

For a detailed information about fractional integral operator (1.3) and its more special cases one may refer the book [**12**, pp. 15-17].
We now, briefly consider some consequences of the results derived in the previous section. To this end, if we consider $\beta = 1$ (and $\delta = 1$ additionally for Theorem 2, 5 & 6), and make use of the relation (3.5), the Theorems 1 to 6 provide, respectively, the known fractional integral inequalities due to Purohit and Raina [**14**] and Purohit *et al.* [**15**]. Again, for $\beta = 1$ the Theorems 7 & 8 provide, respectively, the known integral inequalities involving the Erdélyi-Kober operators due to Baleanu *et al.* [**3**, p.4, Corollary 14 & 15].

Further, if we take $\eta = 0$, $\beta = 1$ ($\zeta = 0$, $\delta = 1$ additionally for Theorem 2, 5 & 6) and make use of (3.2), then the Theorems 1 to 3, 7 & 8 yields the known results due to Belarbi and Dahmani [**4**], whereas the Theorems 4 to 6 provides the results due to Sulaiman [**18**].
Indeed, by suitably specializing the values of parameters η, α and β the results presented in this paper may generate some more known and possibly new inequalities involving the various type of integral operator, on taking relations (3.1) to (3.7) into account. Additionally, by suitably choosing the function $h(t)$, one can obtain further inequalities involving the fractional integral operators and various type of special functions from our results Theorem 4, 5 & 6.

The integral operator $I_{\beta}^{\eta,\alpha}$, has number of applications in the generalized axially symmetric potential theory and other physical problems like in electrostatics, elasticity, etc (se Sneddon [**17**]). The results derived in this paper are general in character and give some contributions to the theory integral inequalities and fractional calculus. Therefore, we conclude this paper with the remark that, the results derived here are expected to find some applications in generalized axially symmetric potential theory and for establishing uniqueness of solutions in fractional boundary value problems. Moreover, one can further easily obtain additional integral inequalities involving the various type of integral operators as special cases of our main results Theorems 1 to 8.

References

[1] Anastassiou, G.A. Most general fractional representation formula for functions and implications, Serdica Math. J., 40(1) (2014), 89-98.

[2] Baleanu, D. and Purohit, S.D. Chebyshev type integral inequalities involving the fractional hypergeometric operators, Abstr. Appl. Anal. 2014 (2014), Article ID 609160, 10(pp.).

[3] Baleanu, D., Purohit, S.D. and Agarwal, P. On fractional integral inequalities involving hypergeometric operators, Chinese Journal of Mathematics, 2014, Article ID 609476, 5(pp.).

[4] Belarbi, S. and Dahmani, Z.th., 10(3)(2009), Art. 86, 5 (pp)(electronic).

[5] Chebyshev, P.L. Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites, Proc. Math. Soc. Charkov, 2(1882), 93-98.
[6] Curiel, Luis and Galué, Leda A generalization of the integral operators involving the Gauss' hypergeometric function, Rev. Téc. Ing. Unlv. Zulla., 19(1) (1996), 17-22.
[7] Dahmani, Z., Mechouar, O. and Brahami, S. Certain inequalities related to the Chebyshev's functional involving a Riemann-Liouville operator, Bull. Math. Anal. Appl., 3(4) (2011), 38-44.
[8] Dahmani, Z. and Tabharit, L. Certain inequalities involving fractional integrals, Journal of Advanced Research in Scientific Computing, 2(1) (2010), 55-60.
[9] Erdélyi, A. On fractional integration and its applications to the theory of Hankel transforms, Quart. J. Math. Oxford, 11 (1940), 293-303.
[10] Kalla, S.L. On operators of fractional integration, I, Mat. Notae 22 (1970), 89-93.
[11] Kalla, S.L. On operators of fractional integration, II, Mat. Notae 25 (1976), 29-35.
[12] Kiryakova, V. Generalized Fractional Calculus and Applications (Pitman Res. Notes Math. Ser. 301), Longman Scientific & Technical, Harlow, 1994.
[13] Kober, H. On a fractional integral and derivative, Quart. J. Math. Oxford, 11 (1940), 193-211.
[14] Purohit, S.D. and Raina, R.K. Chebyshev type inequalities for the saigo fractional integrals and their q-analogues, J. Math. Inequal., 7(2) (2013), 239-249.
[15] Purohit, S.D., Uçar, F. and Yadav, R.K. On fractional integral inequalities and their q-analogues, Revista Tecnocientifica URU, 6 (2014), 53-66.
[16] Saigo, M. A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ., 11 (1978) 135-143.
[17] Sneddon, I.N. Mixed Boundary Value Problems in Potential Theory, North-Holand Publ. Co., Amsterdam, 1966.
[18] Sulaiman, W.T. Some new fractional integral inequalities, J. Math. Anal., 2(2) (2011), 23-28.
[19] Uspensky, J.V. On the development of arbitrary functions in series of Hermite's and Laguerre's polynomials, Ann. Math., (2) 28 (1927), 593-619.

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