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Coexistence of limit cycles and invariant curves for Extended Kukles systems

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Abstract. We work with a certain class of extended Kukles system of arbitrary degree n with at least three invariant straight lines. We show that for a certain values of the parameters, the system has an lower bound of limit cycles. By writing the system as a perturbation of a Hamiltonian system, we show that the first Poincaré-Melnikov integral of the system is a polynomial whose coefficients are the Lyapunov quantities. The maximum number of simples zero of this polynomial gives the maximum number of the global limit cycles; the multiplicity of the origin as a root the polynomial gives the maximum weakness that the weak focus at the origin. On the other hand, we also work with a certain extended Kukles system of order four with a invariant circumference. We show that for certain values of the parameters the system has an lower bound of limit cycles at the origin.

1. Introduction

Let us consider the following differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y) \tag{1.1}$$

in which P and Q are real polynomials of degree at most n. One of the main problems in the qualitative theory of real planar differential systems is the determination of limit cycles.

Suppose that the origin of (1.1) is a critical point of focus type, i.e, the divergence of the linear part of the system at the origin is zero. If the origin is not a center, it is said to be a fine focus. We are interested in answer two closely related questions.

- The first one is the number of limit cycles that bifurcate from a critical point,
- The second one is the derivation of necessary and sufficient conditions for an equilibrium point to be a center.

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A curve C said to be invariant with respect to (1.1) if there exist a polynomial K such that $\dot{C} = CK$. Here $\dot{C} = C_x P + C_y Q$ is the rate of change of C along orbits.

Let us assume that the origin is a critical point of (1.1) and transform the system to canonical form

$$\dot{x} = -y + kxy, \qquad \dot{y} = x + \lambda y + g(x, y) \tag{1.2}$$

where g is a polynomial without linear terms. The system (1.2) is called extended Kukles system. For the origin to be a center we must have $\lambda = 0$. If $\lambda = 0$ and the origin is not a center, it is a fine focus.
The necessary condition for a center are obtained by computing the focal values. These are polynomial in the coefficients arising in g. A function V, analytic in a neighborhood of the origin, such that the rate of change along orbits, $\dot{V}$ is of the form $\eta_2 r^2 + \eta_4 r^4 + \cdots$, where $r^2 = x^2 + y^2$. The focal values are the η_{2k} and the origin is a center if and only if they are all zero. Using the Mathematica software [**2**] we can to calculate the first focal values. These are then reduced in the sense that each is computed modulo the ideal generated by the previous ones, that is, the relation $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ are used to eliminate some of the variables in η_{2k+2}. The reduced focal values η_{2k+2}, with strictly positive factors removed, is known as Lyapunov quantity $L(k)$. Common factors of the reduced focal values are removed and the computation precedes until it can be show the remaining expressions can not be zero simultaneously. The origin of the system (1.2) is fine focus of order k if $L(i) = 0$ for $i = 0, 1, 2, \cdots, k-1$ and $L(k) \neq 0$. System (1.2) can have at most k limit cycles bifurcate out of a fine focus of order k; these are called small-amplitude limit cycles.

We can write extended Kukles system (1.1) as the perturbed with small real parameter ϵ for a Hamiltonian system given by

$$\begin{cases} \dot{x} = P(x, y) = -\dfrac{\partial H}{\partial y} + \displaystyle\sum_{k=1}^{\infty} \epsilon^k f_k(x, y) \\ \dot{y} = Q(x, y) = \dfrac{\partial H}{\partial x} + \displaystyle\sum_{k=1}^{\infty} \epsilon^k g_k(x, y) \end{cases} \tag{1.3}$$

where H, f_k and g_k are real analytic polynomials on $\mathbb{R}^2$, ϵ is small parameter. We say that system (1.3) with $\epsilon = 0$ is the unperturbed Hamiltonian system, while system (1.3) with $\epsilon \neq 0$ is the perturbed one. The problem then is to determine the number, position and shape of the families of limit cycles which emerge from the period orbits of the unperturbed system as the parameter ϵ is varied.
If $T(h)$ is the period the period orbit $H(x, y) = h$ of the unperturbed system, then the first Poincaré-Melnikov integral is

$$V_0(h) = \int_0^{T(h)} \left(f_1 \frac{\partial H}{\partial x} + g_1 \frac{\partial H}{\partial y} \right) dt \tag{1.4}$$

of the system associated to the period orbit $H(x, y) = h$ provided $V_0(h) \neq 0$. Due to Poincaré's work it is well known that the maximum number of simple zeros of V_0 gives the maximum number of limit cycles of the perturbed hamiltonian system which bifurcate from the closes orbits $H(x,y)=h$ when ϵ is small enough.

J.M. Hill, N.G. Lloyd and J.M. Pearson in [**4**] considered the extended Kukles cubic polynomial system and show that under certain condition for their parameters there are centers and small-amplitude limit cycles which coexist with a single invariant straight line $x=\frac{1}{k}$. We will work with an extended Kukles system of arbitrary degree n as given below.

Let us consider the extended Kukles system of degree n given by

$$X_\mu : \begin{cases} \dot{x} = -y + kxy \\ \dot{y} = ky^2 + \left(x + \sum_{k=1}^{n-2} a_k y^k\right) \cdot f(x,y) \end{cases} \tag{1.5}$$

where $f(x,y) = \left(-1 + kx + \frac{m}{2}y\right)\left(-1 + kx - \frac{m}{2}y\right)$ and $\mu = (k, m, a_1, a_2, \ldots, a_{n-2}) \in \mathbb{R}^n$. This system has at least three invariant straight lines. We prove the necessary conditions for center at the origin and coexistence of limit cycles for a certain values of the coefficients of the system (1.5).

Let us consider the extended Kukles system of degree 4 given by

$$Y_\mu : \begin{cases} \dot{x} = P(x,y) = -y + kxy \\ \dot{y} = Q(x,y) = x(1-kx) + (-1 + k^2x^2 + k^2y^2)(\lambda(1-kx)y - \\ \qquad\qquad \frac{1}{2}(a_1x^2 + a_2xy + a_3y^2)) \end{cases} \tag{1.6}$$

where $\mu = (k, \lambda, a_1, a_2, a_3) \in \mathbb{R}^5$ with $k \neq 0$. This system has a invariant circumference, we prove the necessary conditions for center at the origin and coexistence at least of two limit cycles of small-amplitude at the origin for a certain conditions of the coefficients of the system (1.6).

2. Main Results for system X_μ

THEOREM 2.1. *For all $k \in \mathbb{R}^+$ and for all $m \in \mathbb{R}$, $f(x,y) = \left(-1 + kx + \frac{m}{2}y\right)\left(-1 + kx - \frac{m}{2}y\right) = 0$ is an invariant curve for (*1.5*).*

PROOF. For all $\mu \in \mathbb{R}^n$ we have that

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = \frac{y}{2}\left(4k - m^2\sum_{k=1}^{n-2} a_k y^k - m^2 x\right) \cdot f(x,y)$$

This proves that f is an invariant curve for (1.5) . □

COROLLARY 2.2. *For all $k \in \mathbb{R}^+$ and for all $m \in \mathbb{R}$ we have that $f_1(x,y) = -1 + kx + \frac{m}{2}y$, $f_2(x,y) = -1 + kx - \frac{m}{2}y$ and $f_3(x,y) = -1 + kx$ are invariant straight lines for (1.5).*

PROOF. For all $\mu = (k, m, a_1, a_2, \cdots, a_{n-2}) \in \mathbb{R}^n$ we have

$$\begin{aligned} \dot{f_i} &= \frac{\partial f_i}{\partial x}P + \frac{\partial f_i}{\partial y}Q = k_i(x,y)\cdot f_i(x,y), \qquad i=1,2 \\ \dot{f_3} &= \frac{\partial f_3}{\partial x}P + \frac{\partial f_3}{\partial y}Q = k_3(x,y)\cdot f_3(x,y) \end{aligned}$$

where $k_1(x,y) = kx + \frac{m}{2}\left(x + \sum_{k=1}^{n-2} a_k y^k\right)$, $k_2(x,y) = kx - \frac{m}{2}\left(x - \sum_{k=1}^{n-2} a_k y^k\right)$ and $k_3(x,y) = ky$ are the cofactor of f_1, f_2 and f_3 respectively. □

THEOREM 2.3. *If either $n = 2l$ or $n = 2l-1$ for $l \geqslant 2$ with $a_1 = a_3 = \cdots = a_{2l-5} = 0$ and $a_{2l-3} \neq 0$, then system (1.5) has a fine focus of order $l-2$ at the origin.*

PROOF. As $P(x,y) = -y + kxy$ and $Q(x,y) = ky^2 + \left(x + \sum_{k=1}^{n-2} a_k y^k\right)\cdot f(x,y)$, we have that

$$\frac{\partial P(0,0)}{\partial x} + \frac{\partial Q(0,0)}{\partial y} = a_1.$$

For $a_1 = 0$, $L(0) = 0$ and the origin is a center or a focus.
Using Mathematica software, we are able to compute the Lyapunov quantities $L(1) = \frac{3}{8}a_3$. Then

$$\begin{aligned} &\text{if } a_3 = 0 \quad \text{then} \quad L(2) = \frac{5}{16}a_5 \\ &\text{if } a_5 = 0 \quad \text{then} \quad L(3) = \frac{35}{128}a_7 \\ &\qquad\vdots \\ &\text{if } a_{2l-5} = 0 \quad \text{then} \quad L(l-2) = \frac{(2l-3)!!}{(2l-2)!!}a_{2l-3} \text{ and } L(j) = R(j)a_{2l-3} \text{ where } R(j) \text{ is} \end{aligned}$$

a constant.

Note that $n!! = \begin{cases} n(n-2)(n-4)\cdots 3\cdot 1, & \text{if } n \text{ is odd number} \\ n(n-2)(n-4)\cdots 4\cdot 2, & \text{if } n \text{ is ever number} \end{cases}$

If $a_{2l-3} \neq 0$, then $L(j) \neq 0$ for all $j \geqslant l-2$; thus we prove the system (1.5) has a fine focus of order $l-2$ at the origin. □

THEOREM 2.4. *If either $n = 2l$ or $n = 2l-1$ for $l \geqslant 2$, then system (1.5) has a center at the origin if and only if $a_1 = a_3 = a_5 = \cdots = a_{2l-3} = 0$.*

PROOF. By Theorem 2.3 the Lyapunov quantities are given by $L(l-2) = \frac{(2l-3)!!}{(2l-2)!!}a_{2l-3}$ for $l \geqslant 2$. Since all of the computed focal values are zero, we obtain the necessary conditions.
If for $l \geqslant 2$, $a_1 = a_3 = a_5 = \cdots = a_{2l-3} = 0$, by symmetry $P(x,-y) = P(x,y)$ and $Q(x,-y) = -Q(x,y)$ are satisfied and this is sufficient for system (1.5) have a center at the origin. □

THEOREM 2.5. *In the parameters space $\mathbb{R}^n$, there exists an open set $\mathcal{N}$, such that for all $(k, m, a_1, a_2, \cdots, a_{n-2}) \in \mathcal{N}$ with $m \neq 0$ and $k \neq 0$, the system (1.5) of degree either $n = 2l$ or $n = 2l - 1$ with $l \geqslant 2$ has at least $l - 2$ small-amplitude limit cycles limited by the invariant straight lines f_1 and f_2.*

PROOF. If $a_1 = a_3 = \cdots = a_{2l-5} = 0$ and $a_{2l-3} \neq 0$, by Theorem 2.3 the system (1.5) has a fine focus of order $l - 2$ at the origin. Perturbing the system in a_{2l-5} such that $L(2l - 3) < 0$ and $L(0) = L(1) = L(2) = \cdots = L(2l - 7) = 0$ the stability of the origin reverse, a limit cycle is created. Following the same process and finally hyperbolizing the origin, $l - 2$ small-amplitude limit cycles are created. □

2.1. Extended Kukles system Hamiltonian as perturbed of Hamiltonian system. Let us consider the system (1.5) and reescaling $m \to \epsilon m$, $k \to \epsilon k$ and $a_i \to \epsilon a_i$ for all $i = 1, 2, \ldots, n - 2$, where $\epsilon > 0$ is a small parameter. We have the extended Kukles system as perturbed of Hamiltonian system given by

$$X_\epsilon : \begin{cases} \dot{x} = -\frac{\partial H}{\partial y} + \epsilon\, Q_1(x, y) \\ \dot{y} = \frac{\partial H}{\partial x} + \epsilon\, Q_2(x, y) \end{cases} \tag{2.1}$$

where the Hamiltonian function is $H(x, y) = \frac{x^2+y^2}{2}$, the polynomials $Q_1(x, y)$ and $Q_2(x, y)$ in $\mathbb{R}[x, y]$ are given by

$$\begin{aligned} Q_1(x, y) &= kxy \\ Q_2(x, y) &= ky^2 - 2kx^2y^2 - \frac{m^2}{4}xy + \sum_{k=1}^{n-2} a_k y^k \end{aligned}$$

THEOREM 2.6. *If either $n = 2l$ or $n = 2l - 1$ for $l \geqslant 3$, then system (2.1) has at most $l - 2$ global limit cycles bifurcated from the unperturbed Hamiltonian center.*

PROOF. From (1.4), the first Poincaré-Melnikov integral of system (2.1) has the form

$$V_0(h) = \int_0^{2\pi} (y(t)Q_2(x(t), y(t)) + x(t)Q_1(x(t), y(t)))dt$$

where $T = 2\pi$ is the period of the unperturbed orbits of the isochronous center

$$(x(t), y(t)) = (\sqrt{2h}\cos(t), -\sqrt{2h}\sin(t))$$

parameterized with $h > 0$. We note that

i) $$\int_0^{\frac{\pi}{2}} \cos^n(t)dt = \int_0^{\frac{\pi}{2}} \sin^n(t)dt = \begin{cases} \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n}\dfrac{\pi}{2} & \text{if } n \text{ is even} \\ \dfrac{2 \cdot 4 \cdot 6 \cdots n}{1 \cdot 3 \cdot 5 \cdots (n-1)} & \text{if } n \text{ is odd} \end{cases}$$

ii) For all odd number n: $\int_0^{2\pi} (\sin(t))^n dt = 0$

iii) For all ever number n: $\int_0^{\frac{\pi}{2}} (\sin(t))^n\, dt = \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)\pi}{2 \cdot 4 \cdots \cdot n}$

Then we have

$$V_0(h) = 4\pi h^2 \sum_{j=1}^{l-1} 2^j\, h^{j-1}\, a_{2j-1} \cdot \frac{(2j-1)!!}{(2j)!!}$$

This polynomial can have at most $l-2$ real roots different of zero. Then, from Melnikov Theory [**5**], system (2.1) can bifurcate at most $l-2$ global limit cycles from the unperturbed Hamiltonian system. ☐

THEOREM 2.7. *If either $n = 2l$ or $n = 2l-1$ for $l \geqslant 3$ with $a_1 = a_3 = a_5 = \cdots = a_{2l-5} = 0$ and $a_{2l-3} \neq 0$, then system (2.1) has a fine focus of order $l-2$ at the origin.*

PROOF. By Theorem 2.6, the first Poincaré-Melnikov integral is given by

$$\begin{aligned}
V_0(h) &= 4\pi h^2 \sum_{j=1}^{l-1} 2^j\, h^{j-1}\, a_{2j-1} \cdot \frac{(2j-1)!!}{(2j)!!} \\
&= 4\pi h^2 \left(a_1 + \frac{3}{2} a_3 h + \cdots + 2^{l-2} h^{l-3} a_{2l-5} \frac{(2l-5)!!}{(2l-4)!!} + 2^{l-1} h^{l-2} a_{2l-3} \frac{(2l-3)!!}{(2l-2)!!} \right) \\
&= 4\pi h^2 \sum_{j=2}^{l} 2^{j-1} L(j-2) h^{j-2} \qquad \text{where } L(j-2) = \tfrac{(2j-3)!!}{(2j-2)!!} a_{2j-3} \ \text{ for all } j \geqslant 2
\end{aligned}$$

This proves that the first Poincaré-Melnikov of the coefficients correspond to Lyapunov quantities. This polynomial is of degree $l-2$ and it can have at most $l-2$ non-zero, real roots, thus the system (2.1) can have at most $l-2$ global limit cycles from the unperturbed Hamiltonian system.
As $a_1 = a_3 = a_5 = \cdots = a_{2l-5} = 0$ and $a_{2l-3} \neq 0$, we have

$$V_0(h) \quad = \quad 2^{l-1} h^{l-2} a_{2l-3} \frac{(2l-3)!!}{(2l-2)!!}$$

Thus, we have proved that system (2.1) has a fine focus of order $l-2$ at the origin. ☐

Example 2.8. We consider the extended Kukles system of degree 5 and $m = 0$.

$$Z : \begin{cases} \dot{x} = -y + kxy \\ \dot{y} = ky^2 + (x + a_1 y + a_2 y^2 + a_3 y^3)(kx-1)^2 \end{cases} \tag{2.2}$$

Using Mathematica software [**3**], we are able to compute the Lyapunov quantities $L(k)$. Then, we have

$$L(0) = \mathrm{Div} Z(0,0) = a_1.$$

If $a_1 = 0$, $L(0) = 0$ and the origin is a center or focus.
Then $L(1) = \frac{3}{8} a_3$. If $a_3 = 0$ then $L(1) = 0$ and $L(k) = 0$ for all $k \geqslant 2$. By Theorem 2.4, the system (2.2) has a center at the origin. If $a_3 \neq 0$, by Theorem 2.3 the system (1.5) has at most one limit cycle of order one.
Furthermore, the first Poincaré-Melnikov integral by (2.2) is $V_0(h) = 4\pi h^2 \left(a_1 + \frac{3}{2} a_3\right)$. By Theorem 2.3, system (2.2) has at most one limit cycle.

We show a numerical simulation of system (2.2) with $a_1 = -0.1$, $a_2 = 0$, $a_3 = 20$ and $k = 1$. It follows that we have $L(1) > 0$ and $L(0) < 0$. See Figure 1.

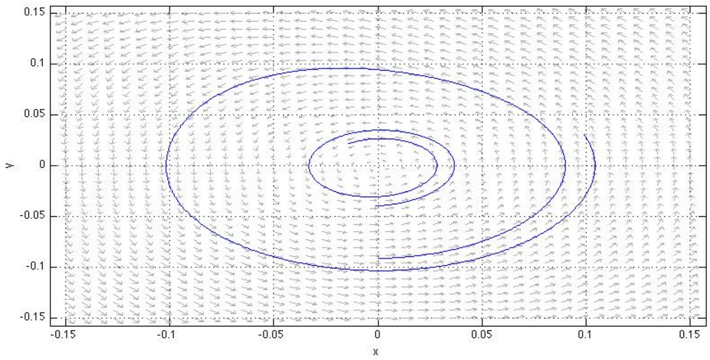

Figure 1: One limit cycle

We note that the outer path moves away of the origin while the inside path attracts of the origin, then the system (2.2) has an unstable limit cycle near the origin. The simulation was obtained using the Pplane7 software with MATLAB [**3**].

3. Main Results for System Y_μ

THEOREM 3.1. *For all $k \in \mathbb{R}-\{0\}$, the circumference $f(x, y) = k^2x^2+k^2y^2-1 = 0$ is invariant of system (1.6).*

PROOF. For all $\mu \in \mathbb{R}^5$ with $k \neq 0$ we have that

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = -k^2y(a_1x^2 + y(a_2x + 2\lambda(-1 + kx) + a_3y))f(x, y)$$

where $k(x, y) = -k^2y(a_1x^2 + y(a_2x + 2\lambda(-1 + kx) + a_3y))$ is the cofactor of $f(x, y)$, this proves that f is an invariant circumference for (1.6). □

THEOREM 3.2. *System (1.6) has a center at the origin if and only if one of the following conditions holds:*

i. $\lambda = a_2 = 0$;
ii. $\lambda = a_2 = 0$, $a_3 = 2k - a_1$.

PROOF. The linear part of the system (1.6) at the origin is given by

$$DY_\mu(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix}$$

For $\lambda = 0$ the origin is a center or focus. Using the Mathematica software, we are able to compute the Lyapunov quantities $L(l)$. We have that $L(0) = 0$ and $L(1) = -\frac{1}{32}a_2(a_3 + a_1 - 2k)$

i. If $a_2 = 0$ then $L(l) = 0$ for all $l \geqslant 1$.
Let $\tilde{\mu} = (k, 0, a_1, 0, a_3) \in \mathbb{R}^5$ we have $\mathrm{Div}Y_{\tilde{\mu}}(0,0) = 0$ and the system

$$\mathrm{Y}_{\tilde{\mu}} : \begin{cases} \dot{x} = -y + kxy \\ \dot{y} = x(1 - kx) - \frac{1}{2}(-1 + k^2x^2 + k^2y^2)(a_3y^2 + a_1x^2) \end{cases}$$

By symmetry

$$P(x, -y) = -P(x, y) \qquad \wedge \qquad Q(x, -y) = Q(x, y)$$

This proves that the origin is a center for (1.6).
In particular, if $a_3 = 2k - a_1$ then the origin is a center for (1.6).

□

THEOREM 3.3. *For system (1.6), we have the following:*

i. *If $\lambda = 0$, $a_2 \neq 0$ and $a_3 \neq 2k - a_1$ then system has a fine focus of order 1 at the origin.*
ii. *If $a_3 = 2k - a_1$, $\lambda = 0$ and $a_2 \neq 0$ then system has a fine focus of order 2 at the origin.*

PROOF. Using the Mathematica Software, we are able to compute the Lyapunov quantities $L(l)$. We have that $\mathrm{Div}Y_\mu(0,0) = L(0) = \lambda$.
If $\lambda = 0$ then $L(0) = 0$,

$$\begin{aligned} L(1) &= -\frac{1}{32}a_2(a_1 + a_3 - 2k), \\ L(2) &= \frac{a_2k^3}{16} \text{ and} \\ L(3) &= -\frac{a_2k^2(8a_1^3 + 164a_1^2k + 2a_1(a_2^2 - 704k^2) + 3k(7a_2^2 + 988k^2))}{6144} \end{aligned}$$

i. If $\lambda = 0$, $a_2 \neq 0$ and $a_3 \neq 2k - a_1$, then system has a fine focus of order 1 at the origin.
ii. If $\lambda = 0$, $a_3 = 2k - a_1$ and $a_2 \neq 0$, then system has a fine focus of order 2 at the origin.

□

THEOREM 3.4. *In the parameters space $\mathbb{R}^5$, there exists an open set $\mathcal{N}$ such that for all $(k, \lambda, a_1, a_2, a_3) \in \mathcal{N}$ with $k \neq 0$, the system (1.6) has at least two small-amplitude limit cycles enclosed by the invariant circumference.*

PROOF. If $a_3 = 2k - a_1$, $\lambda = 0$ and $a_2 > 0$ ($a_2 < 0$ respectively), by the Theorem 3.3 the system (1.6) has a fine focus of order 2 at the origin. Perturbing the system in a_3 such that $L(1) > 0$ and $L(0) = 0$, the stability of the origin reverse, a limit cycle is created. Following same process and finally the system in λ such that $L(0) < 0$ the stability at the origin reverse, 2 small-amplitude limit cycles are created.

If $\lambda = 0$ and $a_2 > 0$ ($a_2 < 0$ respectively), by the Theorem 3.3 the system (1.6) has a fine focus of order one. Perturbing the system (1.6) in λ, such that $L(0) > 0$ ($L(0) < 0$ respectively), the stability at the origin reverse and a small-amplitude limit cycle is created. □

Example 3.5. We consider the system (1.6) with $a_3 = 3$, $a_2 = 1$, $a_1 = 0$, $k = 1$ and $\lambda = 0.1$, then $L(2) > 0$, $L(1) < 0$ and $L(0) > 0$. The system has an unstable limit cycles at the origin. See Figure 2.

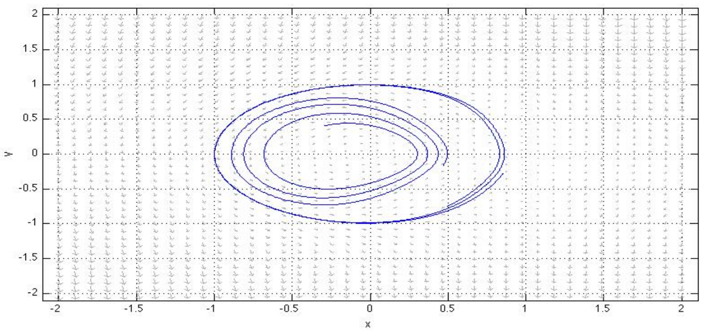

Figure 2: One limit cycle

The simulation was obtained using the Pplane7 software with MATLAB [**3**].

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