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On the analytic evaluation of a certain class of trigonometric sums

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Abstract. We present an analytic evaluation of the solution to a problem by A. de Moivre. Our approach is based on simple combinatorial arguments. Some identities involving a class of trigonometric sums and multinomial coefficients are also exhibited.

1. Introduction

Dice problems have a long history in probability. For example, the three dice problem goes back to the 13th century (see, e.g. [**2**]). In the present paper we focus on the solution of a variant of a De Moivre's problem: Consider a fair die with m numbered sides (from 1 to m) and let p be a prime number less than m, such that p does not divide m. Thus, there exist positive integers a and b such that

$$m = a\,p + b$$

with $a \in \{1, 2, 3, \cdots\}$ while $b \in \{1, 2, 3, \cdots, p-1\}$, the divisor and the remainder respectively. Next, we roll the die n times independently and add the resulting numbers. Let $X_j = 1, 2, 3, \cdots, p$ be the outcome of the die $\pmod p$, at the j-th roll. Set

$$S_n = X_1 + X_2 + \cdots + X_n \pmod p.$$

The goal is to calculate the probability that S_n is divisible by p, i.e.

$$P\left\{S_n \equiv 0 \pmod p\right\}.$$

One can solve the problem, as De Moivre, using generating functions; (for an alternative approach, see [**3**]). It follows easily that

$$P\left\{S_n \equiv 0 \pmod p\right\} = \frac{1}{p}\left\{1 + \frac{1}{m^n}\sum_{k=1}^{p-1}\left(\sum_{j=1}^{b}\omega^{kj}\right)^n\right\}, \tag{1.1}$$

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and in general

$$P\{S_n \equiv j \pmod p\} = \frac{1}{p}\left\{1 + \frac{1}{m^n}\sum_{k=1}^{p-1}\omega^{p-kj}\left(\sum_{i=1}^{b}\omega^{ki}\right)^n\right\} \tag{1.2}$$

for $j = 1, 2, \cdots, p-1, p$. Here, and in what follows, $\omega^p = 1$, $\omega \neq 1$. As already mention, we restrict ourselves in the case where p be a prime number; however the results will be easily generalized for any number.

2. Analytic evaluation of the solution for $S_n \equiv 0 \pmod p$

Let us now discuss the evaluation of the quantity appearing in (1.1)

$$S_0(b) := \sum_{k=1}^{p-1}\left(\sum_{j=1}^{b}\omega^{kj}\right)^n = \left(\sum_{k=1}^{b}\omega^{k}\right)^n + \left(\sum_{k=1}^{b}\omega^{2k}\right)^n + \cdots + \left(\sum_{k=1}^{b}\omega^{(p-1)k}\right)^n, \tag{2.1}$$

where, p is a prime number and b is the remainder of the division of m by p $(b \neq 0, m > p)$. First, we evaluate $S_0(b)$ for the extreme values of b. For $b = 1$ (2.1) yields

$$S_0(1) = \omega^n + \omega^{2n} + \cdots + \omega^{(p-1)n} = \sum_{k=1}^{p-1}\omega^{kn} = \begin{cases} p-1, & n \equiv 0 \pmod p; \\ -1, & \text{elsewhere.} \end{cases} \tag{2.2}$$

For $b = p-1$ we have

$$S_0(p-1) = \sum_{k=1}^{p-1}\left(\sum_{j=1}^{p-1}\omega^{kj}\right)^n = (-1)^n(p-1) = \begin{cases} p-1, & \text{n is even;} \\ 1-p, & \text{n is odd.} \end{cases} \tag{2.3}$$

For $b = 2$ and by the Binomial Theorem we have

$$\begin{aligned} S_0(2) &= \left(\omega + \omega^2\right)^n + \left(\omega^2 + \omega^4\right)^n + \cdots + \left(\omega^{p-1} + \omega^{2(p-1)}\right)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\omega^{n+k} + \sum_{k=0}^{n}\binom{n}{k}\omega^{2(n+k)} + \cdots + \sum_{k=0}^{n}\binom{n}{k}\omega^{(p-1)(n+k)} \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\omega^{n+k} + \omega^{2(n+k)} + \cdots + \omega^{(p-1)(n+k)}\right). \end{aligned} \tag{2.4}$$

To continue we have to talk about n.

Lemma 2.1. *Let $n \equiv d \pmod p$, i.e. $n = cp + d$. Then, (2.4) yields*

$$S_0(2) = \begin{cases} \displaystyle\sum_{j=0}^{c}\binom{n}{jp}p - 2^n & d = 0, \\ \displaystyle\sum_{j=0}^{c-1}\binom{n}{(p-d)+jp}p - 2^n & 0 < d \leqslant \left\lfloor \frac{p}{2} \right\rfloor, \\ \displaystyle\sum_{j=0}^{c}\binom{n}{(p-d)+jp}p - 2^n & d > \left\lfloor \frac{p}{2} \right\rfloor. \end{cases} \tag{2.5}$$

Proof. If $d = 0$, there are $c+1$ multiples of p in the interval $[n, 2n]$. In view of (2.2), one has

$$S_0(2) = \sum_{j=0}^{c} \binom{n}{jp}(p-1) - \left[2^n - \binom{n}{0} - \binom{n}{p} - \binom{n}{2p} - \cdots - \binom{n}{(c-1)p} - \binom{n}{n}\right]$$

$$= \sum_{j=0}^{c} \binom{n}{jp} p - 2^n; \tag{2.6}$$

(we have used the fact that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$).
For $0 < d \leqslant \lfloor \frac{p}{2} \rfloor$, there exist c multiples of p in the interval $[n, 2n]$. We have

$$\begin{aligned} S_0(2) &= \sum_{j=0}^{c-1} \binom{n}{(p-d)+jp}(p-1) \\ &- \left[2^n - \binom{n}{p-d} - \binom{n}{(p-d)+p} - \cdots - \binom{n}{(p-d)+(c-1)p}\right] \\ &= \sum_{j=0}^{c-1} \binom{n}{(p-d)+jp} p - 2^n. \end{aligned} \tag{2.7}$$

Finally, if $d > \lfloor \frac{p}{2} \rfloor$, we see that there are $c+1$ multiples of p in the interval $[n, 2n]$. In a similar way we have

$$\begin{aligned} S_0(2) &= \sum_{j=0}^{c} \binom{n}{(p-d)+jp}(p-1) \\ &- \left[2^n - \binom{n}{p-d} - \binom{n}{(p-d)+p} - \cdots - \binom{n}{(p-d)+cp}\right] \\ &= \sum_{j=0}^{c} \binom{n}{(p-d)+jp} p - 2^n. \end{aligned} \tag{2.8}$$

$\square$

Consider now the case of a general b and have

$$\begin{aligned} S_0(b) = & \sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} \omega^{k_1}\omega^{2k_2}\cdots\omega^{bk_b} \\ + & \sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} \omega^{2k_1}\omega^{4k_2}\cdots\omega^{2bk_b} \\ + & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + & \sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} \omega^{(p-1)k_1}\omega^{2(p-1)k_2}\cdots\omega^{b(p-1)k_b}, \end{aligned}$$

where,

$$\binom{n}{k_1, k_2, \cdots, k_b} = \frac{n!}{k_1!k_2!\cdots k_b!},$$

is the multinomial coefficient, and $k_1, k_2, \cdots, k_b$ are nonnegative integers, such that $\sum_{i=1}^{b} k_i = n$. It follows that

$$\begin{aligned} S_0(b) = & \sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} \Big[\omega^{k_1+2k_2+\cdots+bk_b} + \omega^{2(k_1+2k_2+\cdots+bk_b)} \\ & \qquad + \omega^{(p-1)(k_1+2k_2+\cdots+bk_b)}\Big] \\ = & \sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} \Big[\omega^{n+(k_2+\cdots+(b-1)k_b)} + \omega^{2[n+(k_2+\cdots+(b-1)k_b)]} \\ & \qquad + \omega^{(p-1)[n+(k_2+\cdots+(b-1)k_b)]}\Big]. \end{aligned} \tag{2.9}$$

Next, notice that the quantity

$$A := k_2 + 2k_3 + 3k_4 + \cdots + (b-1)k_b \tag{2.10}$$

attains its minimum value (namely $A = 0$), for

$$(k_1, k_2, k_3, \cdots k_b) = (n, 0, 0, \cdots, 0)$$

and its maximum value (namely $A = (b-1)\,n$), for

$$(k_1, k_2, \cdots, k_{b-1} k_b) = (0, 0, 0, \cdots, 0, n)\,.$$

Hence, the quantity

$$n + A = n + k_2 + 2k_3 + 3k_4 + \cdots + (b-1)k_b$$

takes values in the interval, $[n, bn]$. As in Lemma 2.1 we assume that $n \equiv d \pmod p$. If $d = 0$, there are $cb - (c-1) = c(b-1) + 1$ multiples of p in the interval $[n, bn]$. In view of (2.2), we have

$$\begin{aligned} S_0(b) &= (p-1)\left[\binom{n}{\vec{a}_1} + \binom{n}{\vec{a}_2} + \cdots + \binom{n}{\vec{a}_{cb-c+1}}\right] \\ &\quad - \left[b^n - \binom{n}{\vec{a}_1} + \binom{n}{\vec{a}_2} + \cdots + \binom{n}{\vec{a}_{cb-c+1}}\right] \\ &= p\left[\binom{n}{\vec{a}_1} + \binom{n}{\vec{a}_2} + \cdots + \binom{n}{\vec{a}_{cb-c+1}}\right] - b^n, \end{aligned} \tag{2.11}$$

where we have used the fact that

$$\sum_{k_1+k_2+\cdots+k_b=n} \binom{n}{k_1, k_2, \cdots, k_b} = b^n.$$

We only have to explain how the vectors $\vec{a}_j$, (which in general are not unique) can be found. Notice that, for each $j = 1, 2, \cdots, (cb - c + 1)$ we have a different system of linear Diophantine equations (mostly with multiple solutions). $\vec{a}_1$ is the solution(s) of

$$\left\{ A = 0, \qquad \sum_{i=1}^{b} k_i = n, \qquad k_i \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}, \quad i = 1, 2, \cdots, b \right\}. \tag{2.12}$$

$\vec{a}_2$ is the solution(s) of

$$\left\{A = p, \qquad \sum_{i=1}^{b} k_i = n, \qquad k_i \in \mathbb{Z}_+ = \{0,1,2,\cdots\}, \quad i = 1,2,\cdots,b\right\}. \tag{2.13}$$

and so on. Finally $\vec{a}_{cb-c+1}$ is the solution(s) of

$$\left\{A = (cb-c)\,p, \qquad \sum_{i=1}^{b} k_i = n, \qquad k_i \in \mathbb{Z}_+ = \{0,1,2,\cdots\}, \quad i = 1,2,\cdots,b\right\}. \tag{2.14}$$

Clearly, $\vec{a}_1 = (n,0,0,\cdots,0)$ and $\vec{a}_{cb-c+1} = (0,0,\cdots,0,n)$.
In case $d > \lfloor \frac{p}{2} \rfloor$, there exist $(b-1)c + \lfloor \frac{bd}{p} \rfloor$ multiples of p in the interval $[n, bn]$. Thus,

$$S_0(b) = p\left[\binom{n}{\vec{a}_1} + \binom{n}{\vec{a}_2} + \cdots + \binom{n}{\vec{a}_{cb-c+\lfloor \frac{bd}{p} \rfloor}}\right] - b^n, \tag{2.15}$$

where $\vec{a}_1$ is the solution(s) of the system of linear Diophantine equations

$$\left\{n + k_2 + 2k_3 + \cdots + (b-1)k_b = (c+1)\,p \Leftrightarrow d + A = p, \ \sum_{i=1}^{b} k_i = n, \ k_i \in \mathbb{Z}_+, \ i = 1,2,\cdots,b\right\}. \tag{2.16}$$

$\vec{a}_2$ is the solution(s) of the system

$$\left\{n + k_2 + 2k_3 + \cdots + (b-1)k_b = (c+2)\,p \Leftrightarrow d + A = 2p, \ \sum_{i=1}^{b} k_i = n, \ k_i \in \mathbb{Z}_+, \ i = 1,2,\cdots,b\right\}, \tag{2.17}$$

and so on. Finally, $\vec{a}_{cb-c+\lfloor \frac{bd}{p} \rfloor}$ is the solution(s) of the system

$$\left\{d + A = \left((b-1)\,c + \lfloor \frac{bd}{p} \rfloor\right)p, \ \sum_{i=1}^{b} k_i = n, \ k_i \in \mathbb{Z}_+, \ i = 1,2,\cdots,b\right\}. \tag{2.18}$$

If $0 < d \leqslant \lfloor \frac{p}{2} \rfloor$, we see that there are $(b-1)c + \lfloor \frac{bd}{p} \rfloor$ multiples of p in the interval $[n, bn]$. Actually, the number of multiples in this case is one minus the numer of multiples in case $d > \lfloor \frac{p}{2} \rfloor$. Hence, equation (2.15) and systems (2.16), (2.17),$\cdots$, (2.18) also hold in this case. We have just one system less.
Notice that the case $b = 2$ is in accordance with the general case.

Remark 1. Systems like (2.16) have multiple solutions, but are easy to solve. All one needs is patience and paper. It is easy to see that k_1 attains its minimum value namely, $(c-1)p + 2d = n - (p-d)$, when $(k_2, k_3, k_4, \cdots, k_{b-1}) = (n - k_1, 0, 0, \cdots, 0)$ and its maximum value, $n-1$, when $k_j = 1$, for $j = p - d + 1$, and $k_i = 0$, for $i \in \{2, \cdots, b\} \setminus \{j\}$. The maximum range for k_b is $\left\{0, 1, \cdots, \left\lfloor \frac{p-d}{b-1} \right\rfloor\right\}$, and in general, the maximum range for k_j is $\left\{0, 1, \cdots, \left\lfloor \frac{p-d}{j-1} \right\rfloor\right\}$, for $j = 2, 3, \cdots, b$.

Remark 2. It is notable that the above results indicate that $S_0(b)$ is always an *integer* for all $b \in \{1, 2, 3, \cdots, p-1\}$.

3. Further results

In this section we present some identities arising from the results of Section 2. From (2.1) we have

$$S_0(b) = \sum_{k=1}^{p-1} \left(\sum_{j=1}^{b} \omega^{kj} \right)^n = S_1^n + S_2^n + \cdots + S_{p-1}^n,$$

where, $S_j = \sum_{k=1}^{b} \omega^{kj}, \quad j = 1, 2, \cdots, p-1$. Since $S_{p-j} = \overline{S_j}$ are complex conjugate numbers and p is any odd prime, one has

$$S_0(b) = 2 \sum_{k=1}^{\frac{p-1}{2}} \Re\left(S_k^n\right). \tag{3.1}$$

Now,

$$\begin{aligned} S_k^n &= \left(\omega^k + \omega^{2k} + \cdots + \omega^{bk}\right)^n = \left(e^{\frac{2k\pi i}{p}} + e^{\frac{4k\pi i}{p}} + \cdots + e^{\frac{2bk\pi i}{p}}\right)^n \\ &= \left(\sum_{j=1}^{b} \cos\left(\frac{2k\pi j}{p}\right) + i \sum_{j=1}^{b} \sin\left(\frac{2k\pi j}{p}\right) \right)^n. \end{aligned}$$

Using the identities (see, e.g., [**1**])

$$\sum_{j=1}^{n} \cos(j\vartheta) = \frac{\sin\left(\frac{n\vartheta}{2}\right)\cos\left(\frac{(n+1)\vartheta}{2}\right)}{\sin\left(\frac{\vartheta}{2}\right)}$$

and

$$\sum_{j=1}^{n} \sin(j\vartheta) = \frac{\sin\left(\frac{n\vartheta}{2}\right)\sin\left(\frac{(n+1)\vartheta}{2}\right)}{\sin\left(\frac{\vartheta}{2}\right)},$$

we arrive at

$$S_k^n = \left(\frac{\sin\frac{\pi k b}{p}}{\sin\frac{\pi k}{p}} \right)^n \left[\cos\left(\frac{(b+1)\, nk\pi}{p}\right) + i\ \sin\left(\frac{(b+1)\, nk\pi}{p}\right) \right]. \tag{3.2}$$

In view of (3.2), (3.1) yields

$$S_0(b) = 2 \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{\sin\left(\frac{\pi k b}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)} \right)^n \cos\left(\frac{(b+1)\, nk\pi}{p}\right) \tag{3.3}$$

which is an *integer* for all prime numbers p and for all $b \in \{1, 2, 3, \cdots, p-1\}$, as noticed in Remark 2. For all prime numbers p and for all $n \in \mathbb{N}$ we are able to evaluate sums of type (3.3) by using Lemma 2.1. For example, if $b = p-1$, (3.3) yields

$$S_0(p-1) = 2\sum_{k=1}^{\frac{p-1}{2}} \left(\frac{\sin\left(\frac{k\pi(p-1)}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)} \right)^n \cos(nk\pi) \overset{(2.3)}{=} \begin{cases} p-1, & \text{n is even;} \\ 1-p, & \text{n is odd.} \end{cases} \tag{3.4}$$

For $b = 2$ (3.3) yields
(3.5)

$$S_0(2) = 2\sum_{k=1}^{\frac{p-1}{2}} \left(\frac{\sin\left(\frac{2k\pi}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)} \right)^n \cos\left(\frac{3nk\pi}{p}\right) = 2^{n+1} \sum_{k=1}^{\frac{p-1}{2}} \cos\left(\frac{k\pi}{p}\right)^n \cos\left(\frac{3nk\pi}{p}\right).$$

Now, if $n = cp$, where c is a positive integer and p any odd prime number, (3.5) can be evaluated from (2.6) and have

$$2^{n+1} \sum_{k=1}^{\frac{p-1}{2}} \cos\left(\frac{k\pi}{p}\right)^n \cos\left(\frac{3nk\pi}{p}\right) = \sum_{j=0}^{c} \binom{cp}{jp} p - 2^{cp}. \tag{3.6}$$

For $n = cp + 1$ we get

$$2^{n+1} \sum_{k=1}^{\frac{p-1}{2}} \cos\left(\frac{k\pi}{p}\right)^n \cos\left(\frac{3nk\pi}{p}\right) = \sum_{j=0}^{c-1} \binom{cp+1}{(p-1)+jp} p - 2^{cp+1}, \tag{3.7}$$

and if, for example, $n = cp + (p-1)$ one has

$$2^{n+1} \sum_{k=1}^{\frac{p-1}{2}} \cos\left(\frac{k\pi}{p}\right)^n \cos\left(\frac{3nk\pi}{p}\right) = \sum_{j=0}^{c} \binom{cp+p-1}{jp+1} p - 2^{cp+p-1}. \tag{3.8}$$

Thus, in order to evaluate trigonometric sums of Section 2, one has only to calculate a few binomial coefficients (al least one of them is elementary).

4. Analytic evaluation of the solution for $S_n \equiv j \pmod p$

The quantity

$$S_j(b) = \sum_{k=1}^{p-1} \omega^{p-kj} \lambda_k^n, \qquad j = 1, 2, \cdots, p, \tag{4.1}$$

appearing in (1.2) has similar properties to $S_0(b)$ which has been studied in Sections 2–3. In fact $S_p(b) = S_0(b)$, of (2.1). Also, for $j = 1, 2, \cdots, p-1$, $S_j(b)$ is always an *integer*. As in Section 3 we get

$$S_j(b) = 2\sum_{k=1}^{\frac{p-1}{2}} \left(\frac{\sin\left(\frac{\pi k b}{p}\right)}{\sin\left(\frac{\pi k}{p}\right)} \right)^n \cos\left(\frac{(b+1)\,nk\pi - 2jk\pi}{p}\right) \tag{4.2}$$

for any prime number p, $n \in \mathbb{N}$, and $b = 1, 2, \cdots, p-1$. In particular, for $b = 1$ we have

$$S_j(1) = \omega^{-j}\omega^n + \omega^{-2j}\omega^{2n} + \cdots + \omega^{-(p-1)j}\omega^{(p-1)n},$$

hence

$$S_j(1) = \begin{cases} p-1, & n-j \equiv 0 \pmod p; \\ \\ -1, & \text{elsewhere.} \end{cases} \tag{4.3}$$

For $b = p-1$ we have

$$S_j(p-1) = (-1)^n \sum_{k=1}^{p-1} \omega^{-kj} = (-1)^n (p-1) = \begin{cases} p-1, & \text{n is even;} \\ \\ -1, & \text{n is odd.} \end{cases} \tag{4.4}$$

For $b = 2$, if $n - j \equiv d \pmod p$, i.e. $n - j = cp + d$, we have

$$S_j(2) = \sum_{k=0}^{n} \binom{n}{k} \left(\omega^{n-j+k} + \omega^{2(n-j+k)} + \cdots + \omega^{(p-1)(n-j+k)} \right).$$

In a similar way as in Lemma 2.1. we get

$$S_j(2) = \begin{cases} \displaystyle\sum_{i=0}^{c} \binom{n-j}{ip} p - 2^n & d = 0, \\ \displaystyle\sum_{i=0}^{c-1} \binom{n-j}{(p-d)+ip} p - 2^n & 0 < d \leqslant \left\lfloor \frac{p}{2} \right\rfloor, \\ \displaystyle\sum_{i=0}^{c} \binom{n-j}{(p-d)+ip} p - 2^n & d > \left\lfloor \frac{p}{2} \right\rfloor. \end{cases} \tag{4.5}$$

Likewise, for general b we arrive in *similar* formulas as in the case $j = 0$, by replacing n with $n - j$.

Remark 3. As in Section 3, sums of type (4.2) may be evaluated via formulas like, (4.5).

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