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An arithmetic conjecture on an arctangent sum

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Abstract. A sequence x_n, defined in terms of a sum of arctangent values, satisfies the nonlinear recurrence $x_n = (n + x_{n-1})/(1 - nx_{n-1})$, with $x_1 = 1$, which has been conjectured not to be an integer for $n \geqslant 5$. This problem is analyzed here in terms of divisibility questions of an associated sequence. Properties of this new sequence are employed to prove that the subsequences $\{x_{19n+5} : n \in \mathbb{N}\}$ and $\{x_{19n+13} : n \in \mathbb{N}\}$ contain no integer values.

1. Introduction

The evaluation of arctangent sums of the form

$$\sum_{k=1}^{\infty} \tan^{-1} h(k) \tag{1.1}$$

for a rational function h reappear in the literature from time to time. The reader will find in [**3**] a survey of a variety of methods employed to obtain results such as

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4} \tag{1.2}$$

as well as

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{1}{k^2} = \tan^{-1} \frac{\tan(\pi/\sqrt{2}) - \tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2}) + \tanh(\pi/\sqrt{2})}. \tag{1.3}$$

An example of the corresponding finite sum

$$\sum_{k=1}^{n} \tan^{-1} h(k) \tag{1.4}$$

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was discussed at the end of [**3**] in the form

$$x_n = \tan \sum_{k=1}^{n} \tan^{-1} k \tag{1.5}$$

that satisfies the nonlinear recurrence

$$x_n = \frac{x_{n-1} + n}{1 - n x_{n-1}} \tag{1.6}$$

and the initial condition $x_1 = 1$. The paper above observes that $x_3 = 0$ and ends with the question of whether x_n ever vanishes again.

This problem was addressed in [**1**] on the basis of the computation of the 2-adic valuation of x_n. Recall that if p is a prime and $0 \neq x \in \mathbb{Z}$, the p-adic valuation of x is the highest power of p that divides x. This is denoted by $\nu_p(x)$. This notion is extended to $\mathbb{Q}$ via $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ and the special value $\nu_p(0) = +\infty$. In particular, if $\nu_p(x) < \infty$ for some prime p, then $x \neq 0$. The result

$$\nu_2(x_n) = \begin{cases} \nu_2(2n(n+1)) & \text{if } n \equiv 0,\, 3 \bmod 4 \\ 0 & \text{if } n \equiv 1,\, 2 \bmod 4, \end{cases} \tag{1.7}$$

valid for $n \geqslant 5$, shows that $x_n = 0$ only when $n = 3$.

The question addresed here is whether $x_n \in \mathbb{Z}$ when $n \geqslant 4$. The authors of [**1**] stated the following conjecture.

Conjecture 1.1. The number x_n is not an integer when $n \geqslant 5$.

This conjecture remains open and some evidence pointing towards its validity are stated in [**1**]. For example, with

$$\omega_n = \prod_{j=1}^{n} (1 + j^2) \tag{1.8}$$

the authors established the following criterion:

Theorem 1.2. Assume that for $n \geqslant 5$, the term ω_n is a square. Then x_n is not and integer.

The usefulness of this statement was very short-lived, since J. Cilleruelo [**5**] proved the next result.

Theorem 1.3. The product ω_n is a square only for $n = 3$.

The conjecture is equivalent to the fact that the graph of the fractional part of x_n, shown in Figure 1, does not intersect the x-axis. For $5 \leqslant n \leqslant 50000$, the minimum height is 2.39245×10^{-6}.

Note 1.4. The graph shown in Figure 1 is reminiscent of the plot of

$$y_i(k) = \frac{i \bmod k}{k}, \qquad \text{for } 1 \leqslant k \leqslant i \tag{1.9}$$

analyzed in Chapter 5 of [**6**]. The result is that, when $i \to \infty$, the rescaled arithmetic random variables $y_i(k)$, where k is taken uniformly on $[1, i]$, converge in law to the uniform distribution on $[0, 1]$. Figure 2 shows the function $y_i(k)$ for $i = 5000$.

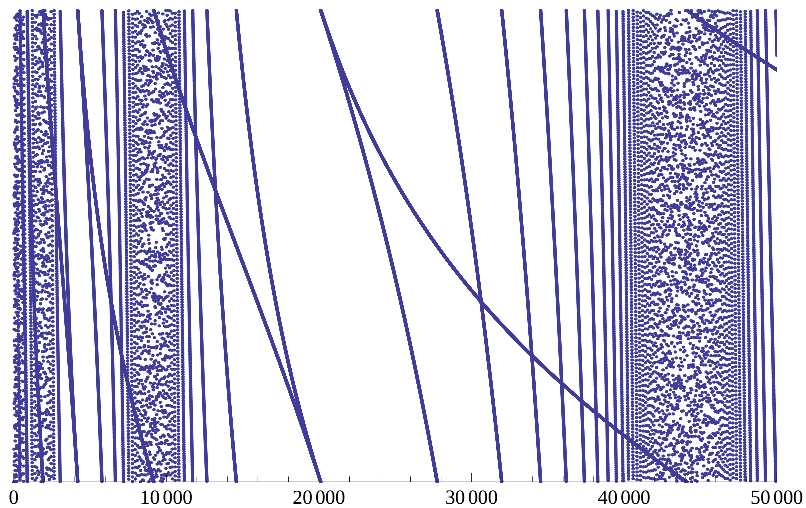

FIGURE 1. The fractional part of x_n for $1 \leqslant n \leqslant 50000$

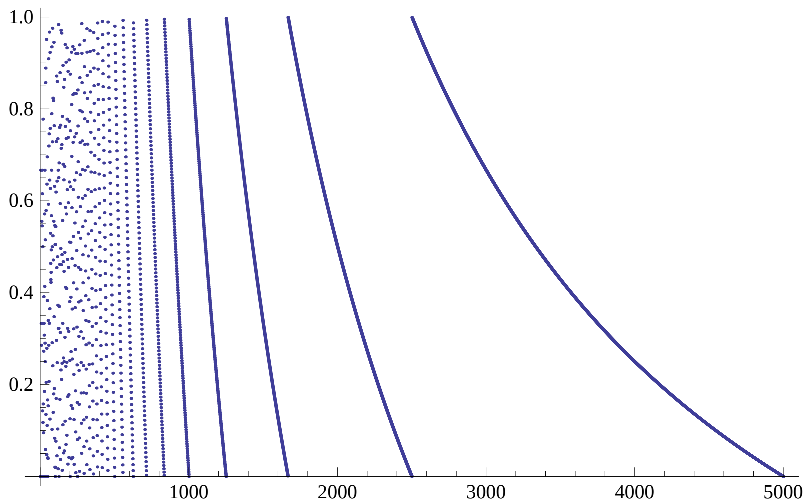

FIGURE 2. The function $y_{5000}(k)$ for $1 \leqslant k \leqslant 5000$

Note 1.5. The relation $\tan^{-1} k + \tan^{-1}(1/k) = \frac{\pi}{2}$ is used in comparing the sequence x_n against the sequence

$$a_n := \sum_{k=1}^{n} \tan^{-1} \frac{1}{k}. \tag{1.10}$$

A simple calculation shows that $b_n = \tan a_n$ satsifies

$$x_n = \tan\left(\frac{\pi n}{2} - a_n\right) = \begin{cases} -b_n & \text{for } n \text{ even} \\ 1/b_n & \text{for } n \text{ odd.} \end{cases} \tag{1.11}$$

Now

$$a_n = \sum_{k=1}^{n} \frac{1}{k} + O(1), \tag{1.12}$$

with the error term given by

$$\begin{aligned} \sum_{k=1}^{n} \left(\frac{1}{k} - \tan^{-1} \frac{1}{k} \right) &= \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j+1)k^{2j+1}} \\ &= \frac{1}{3} \sum_{k=1}^{n} \frac{1}{k^3} - \frac{1}{5} \sum_{k=1}^{n} \frac{1}{k^5} + \frac{1}{7} \sum_{k=1}^{n} \frac{1}{k^7} - \cdots \end{aligned}$$

and this is bounded by $\zeta(3)/3 < 0.41$. The harmonic sum in (1.12) can be replaced by $\log n$ with an error term

$$\sum_{k=1}^{n} \frac{1}{k} - \log n < \gamma < 0.58, \tag{1.13}$$

where γ is Euler's constant. It follows that the dynamics of b_n is comparable to $c_n = \tan \log n$. This example represents a caricature of the original sequence x_n and it will analyzed in a future publication.

Introduce the sequence f_n implicitly by

$$x_n = \frac{f_{n+1} + f_n}{(n+1) f_n} \tag{1.14}$$

with $f_1 = 1$. The fact $f_n \in \mathbb{Z}$ is based on the closed-form expression (1.15). The following arithmetic criterion is established:

if f_{n-1} does not divide f_n, then x_n is not an integer.

This criteria is used to construct subsequences of x_n which do not contain integer values. Still, the main conjecture stating that $x_n \notin \mathbb{Z}$ remains open.

The sequence f_n is given explicitly by

$$f_n = (-1)^{n+1} \operatorname{Re} \prod_{k=0}^{n} (1 + ik). \tag{1.15}$$

and it satisfies the recurrence

$$n f_{n+1} = -(2n+1) f_n - (n+1)(n^2+1) f_{n-1} \tag{1.16}$$

with initial conditions $f_1 = f_2 = 1$. Section 2 discusses a family of matrices $B_{n,j}$, with entries that are polynomials in n, such that

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = B_{n,j} \begin{bmatrix} f_{n-j} \\ f_{n-j-1} \end{bmatrix}. \tag{1.17}$$

Section 3 gives the bound $|f_n| \leqslant Cn!$ for some constant C, with the optimal constant $C_* = \sqrt{\sinh \pi/\pi}$. An interesting modulo 4 phenomena for the function $q_n = f_n/n!$ is also discussed in this section.

The arithmetic criterion stated above motivated the search of primes p which divide f_{n-1} but not f_n. This is explained in Section 4. The data presented there indicates that it is unlikely that the present method will produce a proof of the main conjecture discussed in this paper.

The valuations of f_n are discussed in Section 5, for instance the formulas

$$\nu_2(f_n) = \left\lfloor \frac{n+1}{4} \right\rfloor \text{ and } \nu_3(f_n) = 0 \tag{1.18}$$

Obviously, $\nu_3(f_n) = 0$ means that 3 never divides f_n. The set of primes is divided into three types: *i*) primes p which never divide an element of the sequence f_n; *ii*) primes p for which $\nu_p(f_n)$ is asymptotically linear; *iii*) those primes for which $\nu_p(f_n)$ displays an oscillatory behavior. *A precise description of this concept is missing.*

It is conjectured that the class of primes *iii*) produces subsequences of $\{x_n\}$ that are guaranteed not to contain any integer values. Section 6 contains all the details for $p = 19$, the first prime of this class. The analysis exploits the periodicity of the sequence $\text{Mod}(f_n, 19)$ and the matrices in (1.17) modulo 19. This periodicity is not a direct fact since the recurrence satisfied by f_n has non-constant coefficients. The main result of Section 7 is:

Theorem 1.6. The subsequences x_{19n+5} and x_{19n+13} contain no integer values.

An analytic formula for $\nu_p(f_n)$, similar to the classical formula of Legendre for $\nu_p(n!)$, seems to be possible for primes in the class *ii*). Details of an experimental attempt to find this formula are provided in Section 8 for the prime $p = 13$. An exact formula for $\nu_{13}(f_n)$ remains an open problem, but simple expressions that match this valuation for almost all values of n are described.

2. An associated sequence

The recurrence

$$x_n = \frac{n + x_{n-1}}{1 - nx_{n-1}}$$

yields

$$\begin{aligned} x_n &= \frac{1}{n}\frac{n^2 + nx_{n-1}}{1 - nx_{n-1}} \\ &= \frac{1}{n}\frac{(nx_{n-1} - 1) + n^2 + 1}{1 - nx_{n-1}} \\ &= -\frac{1}{n} + \frac{n + n^{-1}}{1 - nx_{n-1}}. \end{aligned}$$

Multiply through by $n + 1$ and simplify to get

$$(n+1)x_n - 1 = -2 - \frac{1}{n} + \frac{(n+1)(n+n^{-1})}{1 - nx_{n-1}}. \tag{2.1}$$

This motivates the introduction of

$$u_n = nx_{n-1} - 1. \tag{2.2}$$

LEMMA 2.1. *The sequence u_n satisfies the recurrence*

$$u_{n+1} + \frac{(n+1)(n+n^{-1})}{u_n} + \frac{2n+1}{n} = 0. \tag{2.3}$$

The first few values are

$$u_1 = 1,\ u_2 = -10,\ u_3 = -1,\ u_4 = 19,\ u_5 = -\frac{73}{19},\ u_6 = \frac{662}{73}.$$

A new sequence $\{f_n\}$ is introduced as follows: $f_1 = 1$ and recursively $f_n = u_n f_{n-1}$.

Note 2.1. The relation to the original sequence is given by

$$x_n = \frac{f_{n+1} + f_n}{(n+1)f_n}. \tag{2.4}$$

A recurrence for f_n is described next.

PROPOSITION 2.1. *The sequence f_n satisfies*

$$f_{n+1} + \frac{2n+1}{n} f_n + (n+1)(n+n^{-1})f_{n-1} = 0. \tag{2.5}$$

Equivalently

$$(n-1)f_n = -\left[(2n-1)f_{n-1} + n(n^2-2n+2)f_{n-2}\right], \qquad \text{for } n \geqslant 3, \tag{2.6}$$

with initial conditions $f_1 = f_2 = 1$.

PROOF. Replace in (2.3). □

The first few values of f_n are

$$f_1 = 1,\ f_2 = 1,\ f_3 = -10,\ f_4 = 10,\ f_5 = 190,\ f_6 = -730,\ f_7 = -6620,\ f_8 = 55900.$$

It is a remarkable fact that the numbers f_n are integers.

Theorem 2.2. The numbers f_n are given by

$$f_n = (-1)^{n+1} \operatorname{Re} \prod_{k=0}^{n} (1+ik). \tag{2.7}$$

In particular $f_n \in \mathbb{Z}$.

PROOF. It will be shown that the right hand side of (2.7) satisfies the recurrence (2.5) and that the initial conditions match. Define

$$R_n = (-1)^{n+1} \operatorname{Re} \prod_{k=0}^{n} (1+ik) \text{ and } I_n = (-1)^{n+1} \operatorname{Im} \prod_{k=0}^{n} (1+ik).$$

Then $R_0=-1$, $R_1=1$, $I_0=0$, $I_1=1$ and

$$\begin{aligned} R_n &= (-1)^{n+1}\operatorname{Re}\left((1+in)\times\prod_{k=0}^{n-1}(1+ik)\right) \\ &= (-1)^{n+1}\operatorname{Re}\left(\prod_{k=0}^{n-1}(1+ik)\right)\cdot 1-(-1)^{n+1}n\cdot\operatorname{Im}\left(\prod_{k=0}^{n-1}(1+ik)\right) \\ &= -R_{n-1}+nI_{n-1}. \end{aligned}$$

Similarly

$$\begin{aligned} I_n &= (-1)^{n+1}\operatorname{Im}\left((1+in)\times\prod_{k=0}^{n-1}(1+ik)\right) \\ &= (-1)^{n+1}\operatorname{Re}\left(\prod_{k=0}^{n-1}(1+ik)\right)\cdot n+(-1)^{n+1}\cdot\operatorname{Im}\left(\prod_{k=0}^{n-1}(1+ik)\right) \\ &= -nR_{n-1}-I_{n-1}. \end{aligned}$$

Now

$$\begin{aligned} R_n &= -R_{n-1}+nI_{n-1} \\ &= -R_{n-1}+n\times(-(n-1)R_{n-2}-I_{n-2}) \\ &= -R_{n-1}-n(n-1)R_{n-2}-n\times\left(\frac{R_{n-1}+R_{n-2}}{n-1}\right). \end{aligned}$$

This yields

$$\begin{aligned} (n-1)R_n &= -(n-1)R_{n-1}-n(n-1)^2R_{n-2}-nR_{n-1}-nR_{n-2} \\ &= -(2n-1)R_{n-1}-n(n^2-2n+2)R_{n-2} \end{aligned}$$

and hence

$$(n-1)R_n+(2n-1)R_{n-1}+n(n^2-2n+2)R_{n-2}=0. \tag{2.8}$$

This is the recurrence satisfied by f_n. The initial conditions match: $f_0=R_0=-1$ and $f_1=R_1=1$. The proof is complete. □

Note 2.3. The recurrence (2.6) implies

$$\begin{aligned} (n-2)(n-1)f_n &= (n-2)\left[-(2n-1)f_{n-1}-n(n^2-2n+2)f_{n-2}\right] \\ (n-2)f_{n-1} &= -(2n-3)f_{n-2}-(n-1)(n^2-4n+5)f_{n-3}. \end{aligned}$$

Replace the second equation into the first one to obtain

$$f_n=-\frac{(n+1)(n-1)(n-3)}{n-2}f_{n-2}+\frac{(2n-1)(n^2-4n+5)}{n-2}f_{n-3}. \tag{2.9}$$

The recurrence (2.6) can be written as

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}=A_n\begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} \tag{2.10}$$

with

$$A_n = \begin{bmatrix} -(2n-1)/(n-1) & -n(n^2-2n+2)/(n-1) \\ 1 & 0 \end{bmatrix}. \tag{2.11}$$

Then (2.9) is simply

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = A_n \cdot A_{n-1} \begin{bmatrix} f_{n-2} \\ f_{n-3} \end{bmatrix}. \tag{2.12}$$

Define the following product of matrices

$$B_{n,j} = A_n \cdot A_{n-1} \cdot \cdots \cdot A_{n-j+1}. \tag{2.13}$$

Then

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = B_{n,j} \begin{bmatrix} f_{n-j} \\ f_{n-j-1} \end{bmatrix}. \tag{2.14}$$

The matrices $B_{n,j}$ have some special form that is described next. The first few examples are

$$\begin{aligned} B_{n,1} &= \frac{1}{n-1} \begin{bmatrix} 2n-1 & -n(n^2-2n+2) \\ n-1 & 0 \end{bmatrix} \\ B_{n,2} &= \frac{1}{n-2} \begin{bmatrix} -(n-1)(n+1)(n-3) & (2n-1)(n^2-4n+5) \\ (2n-3) & -(n-1)(n^2-4n+5) \end{bmatrix} \\ B_{n,3} &= \frac{1}{n-3} \begin{bmatrix} 2n(n-3)(2n-3) & (n-3)(n-1)(n+1)(n^2-6n+10) \\ -n(n-2)(n-4) & (2n-3)(n^2-6n+10) \end{bmatrix}. \end{aligned}$$

These examples suggest to write

$$B_{n,j} = \frac{1}{n-j} \begin{bmatrix} \alpha(n,j) & \beta(n,j) \\ \gamma(n,j) & \delta(n,j) \end{bmatrix}. \tag{2.15}$$

The definition (2.13) gives

$$B_{n,j} = B_{n,j-1} \cdot A_{n-j+1} \tag{2.16}$$

and the recurrences

$$\begin{aligned} \alpha_{n,j} &= \frac{1}{n-j+1} \left[-(2n-2j+1)\alpha_{n,j-1} + (n-j)\beta_{n,j-1} \right] \\ \beta_{n,j} &= -\left[(n-j)^2 + 1 \right] \alpha_{n,j-1} \\ \gamma_{n,j} &= \frac{1}{n-j+1} \left[-(2n-2j+1)\gamma_{n,j-1} + (n-j)\delta_{n,j-1} \right] \\ \delta_{n,j} &= -\left[(n-j)^2 + 1 \right] \gamma_{n,j-1}, \end{aligned} \tag{2.17}$$

having initial conditions

$$\alpha_{n,1} = -(2n-1),\ \beta_{n,1} = -n(n^2-2n+2),\ \gamma_{n,1} = n-1,\ \delta_{n,1} = 0. \tag{2.18}$$

The next step is showing that $\alpha_{n,j}$, $\beta_{n,j}$, $\gamma_{n,j}$, $\delta_{n,j}$ are polynomials in n. The proof is by induction.

Observe first that the recurrence for $\alpha_{n,j}$ may be written as

$$\alpha_{n,j} = -2\alpha_{n,j-1} + \beta_{n,j-1} + \frac{\alpha_{n,j-1} - \beta_{n,j-1}}{n-j+1} \tag{2.19}$$

and assume that $\alpha_{n,j-1}$ and $\beta_{n,j-1}$ are polynomials. Then, replace $n = j$ to obtain

$$\alpha_{j,j} = -\alpha_{j,j-1}. \tag{2.20}$$

Similarly, the recurrence for $\beta_{n,j}$ gives

$$\beta_{j,j} = -\alpha_{j,j-1}. \tag{2.21}$$

It follows that $\alpha_{j,j} = \beta_{j,j}$ for any $j \in \mathbb{N}$. The induction hypothesis states that $\alpha_{n,j-1} - \beta_{n,j-1}$ is a polyomial in n. The previous identity shows that it vanishes at $n = j-1$ proving that the last term in (2.19) is a polynomial in n. Therefore $\alpha_{n,j}$ is a polynomial. A similar argument for $\beta_{n,j}$, $\gamma_{n,j}$ and $\delta_{n,j}$ provides a complete proof of the next result.

Theorem 2.4. The functions $\alpha_{n,j}$, $\beta_{n,j}$, $\gamma_{n,j}$, $\delta_{n,j}$, defined by (2.17), are polynomials in n.

Note 2.5. The recurrence (2.1) verifies that the generating function $F(x) = f_1x + f_2x^2 + \cdots$ of the sequence f_n satisfies the third order linear differential equation

$$x^5F^{(3)}(x) + 7x^4F^{(2)}(x) + x(11x^2 + 2x + 1)F^{(1)}(x) + (4x^2 + x - 1)F(x) = 4x^2.$$

3. Bounds on f_n

The sequence $\{f_n\}$ will determine arithmetic properties of the original sequence $\{x_n\}$. These issues will be discussed in Section 4. The goal of the present section is to establish bounds on the growth of f_n.

Theorem 3.1. There is a constant C, such that

$$|f_n| \leqslant Cn! \tag{3.1}$$

for all $n \in \mathbb{N}$. The best constant in (3.1) is

$$C_* = \sqrt{\frac{\sinh \pi}{\pi}} \sim 1.91731007\ldots \tag{3.2}$$

Proof. The first step is to produce a bound of the form (3.1) for some constant C. The optimal bound is constructed next. *The fact that this is the optimal constant remains an open question.*

The identity

$$f_n = (-1)^{n+1}\,\mathrm{Re}\prod_{k=0}^{n}(1+ik) \tag{3.3}$$

yields

$$|f_n| \leqslant \prod_{k=1}^{n}(1+k^2)^{1/2} = n! \times \prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right)^{1/2}. \tag{3.4}$$

To bound this product employ the arithmetic-mean inequality

$$x_1x_2\cdots x_m \leqslant \left(\frac{x_1+x_2+\cdots+x_m}{m}\right)^m, \tag{3.5}$$

with $x_k = 1 + 1/k^2$ to obtain

$$\prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right) \leqslant \left(1+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^2}\right)^n \leqslant \left(1+\frac{\zeta(2)}{n}\right)^n \leqslant e^{\zeta(2)}.$$

Then (3.4) yields

$$\prod_{k=1}^{n}(1+k^2)^{1/2} \leqslant e^{\frac{1}{2}\zeta(2)}n! \tag{3.6}$$

and the result holds.

The optimal constant C_* is computed next. The bound (3.4) on f_n gives

$$\begin{aligned} |f_n| &\leqslant \prod_{k=1}^{n}(1+k^2)^{1/2} = n! \times \prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right)^{1/2} \\ &\leqslant n! \times \prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)^{1/2} \\ &= n!\sqrt{\frac{\sinh\pi}{\pi}}. \end{aligned}$$

The last evaluation follows directly from the infinite product representation for $\sin z$

$$\frac{\sin \pi z}{\pi z} = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \tag{3.7}$$

evaluated at $z = i$. This product may also be found on page 753 of [**7**], formula 6.2.1.6. □

Definition 3.1. Introduce the notation

$$q_n = \frac{f_n}{n!}, \tag{3.8}$$

so that Theorem 3.1 states that $|q_n| \leqslant C_*$.

The recurrence for f_n in (2.6) produces one for q_n.

Lemma 3.1. *The sequence q_n satisfies the recurrence*

$$q_n = -\frac{2n-1}{n(n-1)}q_{n-1} - \left[1+\frac{1}{(n-1)^2}\right]q_{n-2} \tag{3.9}$$

with initial conditions $q_1 = 1$ and $q_2 = 1/2$.

Problem 3.1. *Four different colors are employed on the graph on the left of Figure 3. Each of the subsequences q_{4n}, q_{4n+1}, q_{4n+2} and q_{4n+3} are painted with a different color. The picture on the right contains the subsequences q_{3n}, q_{3n+1} and q_{3n+2}, each painted with its own color. The fact that colors distinguish branches seems to occur only for subsequences modulo 4. In all other cases examined, there is a mixing of the colors involved.* There is no available explanation for this phenomenon.

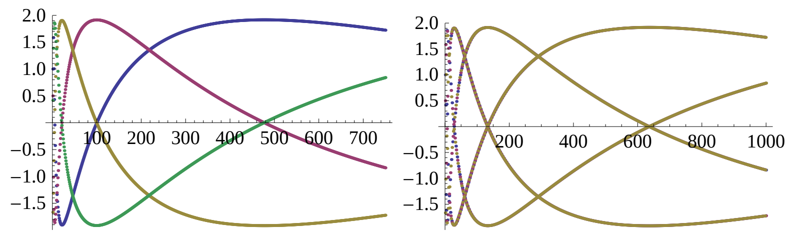

FIGURE 3. The function q_n with n painted modulo 4 (on the left) and modulo 3 (on the right). Each graph contains 3000 points

4. A sequence of special primes.

This section considers divisibility properties of the sequence f_n. In particular, certain prime divisors of this sequence are responsible in establishing the non-integrality of the original sequence x_n.

LEMMA 4.1. *Assume u_n is not an integer. Then x_{n-1} is not an integer.*

PROOF. This follows directly from the relation

$$u_n = nx_{n-1} - 1. \tag{4.1}$$

□

Theorem 4.1. Suppose a prime p divides f_{n-1} and not f_n. Then x_{n-1} is not an integer.

PROOF. The assumptions implies that $u_n = f_n/f_{n-1}$ is not an integer. The result now follows from Lemma 4.1. □

EXAMPLE 4.1. The prime $p = 19$ divides $f_5 = 190 = 2 \cdot 5 \cdot 19$ and it does not divide $f_6 = -730 = -2 \cdot 5 \cdot 73$. This confirms $x_5 = -9/19$ is not an integer. Similarly, the prime $p = 83$ divides $f_{11} = -28269800 = -2^3 \cdot 5^2 \cdot 13 \cdot 83 \cdot 131$ and it does not divide $f_{12} = 839594600 = 2^3 \cdot 5^2 \cdot 13 \cdot 322921$, confirming that $x_{11} = -26004/10873$ is not an integer.

DEFINITION 4.1. A prime p is called a *non-integrality certificate* for x_{n-1} if it satisfies the condition of Theorem 4.1. For $n \in \mathbb{N}$, let p_n be smallest prime with this property. If there is no such prime, set $p_n = \infty$.

EXAMPLE 4.2. The behavior of the primes p_n appears difficult to figure out. The table below show such primes for $6 \leqslant n \leqslant 35$.

These primes grow in unexpected manner. For instance,

$$p_{40} = 9681381484475904765200453. \tag{4.2}$$

It is unlikely that this method will yield a proof of Conjecture 1.1.

6	7	8	9	10	11	12	13	14	15
19	73	331	43	281	13	83	322921	19	17
16	17	18	19	20	21	22	23	24	25
13	1087	1185403	5	17	5323	5	8629	71	19
26	27	28	29	30	31	32	33	34	35
5	269	163	5	1367	199	5	19	41	43

TABLE 1. Non-integrality certificates

Note 4.2. The result of Theorem 4.1 suggests the factorization of f_n in the form

$$f_{n-1} = \text{sign}(f_{n-1})\ \gcd(f_n, f_{n-1}) \times \prod p^{\nu_p(f_{n-1})} \tag{4.3}$$

where the product runs over all primes that divide f_{n-1} but not f_n. A property of the first factor in (4.3) is described next.

PROPOSITION 4.1. *The* $\gcd(f_n, f_{n-1})$ *divides* $n \prod_{k=1}^{n-1} (k^2+1)$.

PROOF. Consider two sequences h_n and g_n which satisfy the recurrence

$$x_n + b_n x_{n-1} + c_n x_{n-2} = 0, \tag{4.4}$$

with initial conditions h_0, h_1 and g_0, g_1, respectively. The coefficients b_n and c_n are given. For any sequence γ_n, it follows that

$$\begin{aligned} \gamma_n \left(h_n g_{n-1} - h_{n-1} g_n \right) &= \gamma_n \left[g_{n-1}(-b_n h_{n-1} - c_n h_{n-2}) - h_{n-1}(-b_n g_{n-1} - c_n g_{n-2}) \right] \\ &= \gamma_n c_n \left(h_{n-1} g_{n-2} - h_{n-2} g_{n-1} \right). \end{aligned}$$

This is valid for arbitrary γ_n. Now assume γ_n is defined by

$$\gamma_{n-1} = \gamma_n c_n, \quad \text{for} \quad n \geqslant 2, \tag{4.5}$$

and initial condition $\gamma_1 = 1$. Then the previous computation gives

$$\gamma_n \left(h_n g_{n-1} - h_{n-1} g_n \right) = \gamma_{n-1} \left(h_{n-1} g_{n-2} - h_{n-2} g_{n-1} \right). \tag{4.6}$$

Repeated iteration shows that

$$\gamma_n \left(h_n g_{n-1} - h_{n-1} g_n \right) = \gamma_1 (h_1 g_0 - h_0 g_1). \tag{4.7}$$

This is now employed to evaluate γ_n. Rewrite (4.5) in the form

$$\frac{\gamma_{k-1}}{\gamma_k} = c_k \tag{4.8}$$

and multiply from $k = 2$ to n. Now use $\gamma_1 = 1$ to arrive at

$$\gamma_n = \prod_{k=2}^{n} 1/c_k. \tag{4.9}$$

The sequence $\{f_n\}$ defined in Section 2 satisfies

$$f_n + \frac{2n-1}{n-1} f_{n-1} + \frac{n((n-1)^2+1)}{n-1} f_{n-2} = 0, \tag{4.10}$$

with $f_1 = f_2 = 1$. The value $f_0 = -1$ is chosen in order make this definition consistent. This is of the type (4.4) with

$$c_n = \frac{n(\,(n-1)^2+1\,)}{n-1}. \tag{4.11}$$

Then (4.9) gives

$$\gamma_n = \prod_{k=2}^{n} \frac{k-1}{k((k-1)^2+1)} = \prod_{k=1}^{n-1} \frac{k}{k+1}\frac{1}{(k^2+1)}. \tag{4.12}$$

The factors $k/(k+1)$ telescope to the value $1/n$ and thus

$$\gamma_n = \frac{1}{n}\prod_{k=1}^{n-1}\frac{1}{k^2+1}. \tag{4.13}$$

The sequence f_n is an example of h_n in the discussion above. Now choose the companion sequence g_n as the solution of (4.10) with the initial conditions $g_0 = 1$ and $g_1 = 0$. As before, it can be checked that $g_n \in \mathbb{Z}$. The relation (4.7) generates

$$f_n g_{n-1} - f_{n-1} g_n = n \prod_{k=1}^{n-1}(k^2+1). \tag{4.14}$$

Observe that $\gcd(f_n, f_{n-1})$ divides the left-hand side of (4.14). The proof is complete. □

Note 4.3. The statement in Proposition 4.1 motivated the computation of the largest prime factor of $\gcd(f_n, f_{n-1})$. Empirical evidence suggests that this prime is bounded by $2n$.

5. The valuations of f_n.

The relation $u_n = f_n/f_{n-1}$ shows that u_n is not an integer if there is a prime p, such that

$$\nu_p(f_{n-1}) > \nu_p(f_n). \tag{5.1}$$

This is a slight generalization of Theorem 4.1, where $\nu_p(f_{n-1}) > 0 = \nu_p(f_n)$. In this section, we analyze the graph of the valuation $\nu_p(f_n)$. The goal is to look for places where this graph is decreasing.

The prime $p = 2$. The graph shown in Figure 4 depicts the 2-adic valuation of f_n.
In this case it is possible to obtain an exact expression for $\nu_2(f_n)$.

Theorem 5.1. The 2-adic valuation of f_n is given by

$$\nu_2(f_n) = \left\lfloor \frac{n+1}{4} \right\rfloor.$$

In particular, the graph is non-decreasing.

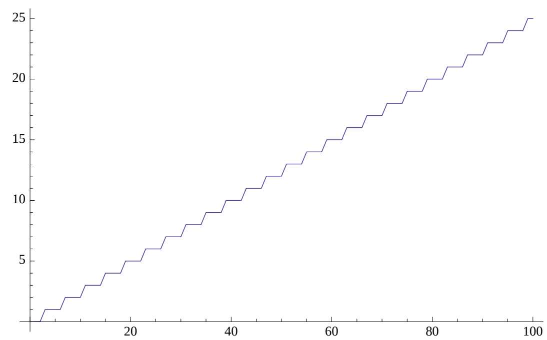

FIGURE 4. Power of 2 that divides f_n

PROOF. The proof is based on the recurrence

$$(n-1)f_n = -(2n-1)f_{n-1} - n(n^2-2n+2)f_{n-2}. \tag{5.2}$$

Write

$$f_n = 2^{\lfloor (n+1)/4 \rfloor} f_n^* \tag{5.3}$$

and the result is equivalent to showing f_n^* is odd.

Case 1. Assume $n \equiv 0 \bmod 4$ and write $n = 4t$. Then (5.2) gives

$$(4t-1)f_{4t} = -2^t \left[(8t-1)f_{4t-1}^* + 4t\left((4t)(2t-1)+1\right)\, f_{4t-2}^* \right]. \tag{5.4}$$

The right-hand side is of the form $2^t\times$ an odd number. It follows that

$$\nu_2(f_{4t}) = t = \left\lfloor \frac{4t+1}{4} \right\rfloor, \tag{5.5}$$

as claimed.

Case 2. Assume $n \equiv 2 \bmod 4$ and write $n = 4t+2$. Then (5.2) gives

$$(4t+1)f_{4t+2} = -2^t \left[(8t+3)f_{4t+1}^* + 2(2t+1)\left((4t)(4t+2)+2\right)\, f_{4t}^* \right]. \tag{5.6}$$

The right-hand side is of the form $2^t\times$ an odd number. It follows that

$$\nu_2(f_{4t+2}) = t = \left\lfloor \frac{4t+2+1}{4} \right\rfloor, \tag{5.7}$$

as claimed.

Case 3. Assume $n \equiv 3 \bmod 4$. Use (5.2) with $n = 4t+3$ to obtain

$$(4t+2)f_{4t+3} = -(8t+5)f_{4t+2} - (4t+3)(16t^2+16t+5)f_{4t+1} \tag{5.8}$$

and with $n = 4t+2$ to produce

$$(4t+1)f_{4t+2} = -(8t+3)f_{4t+1} - 4(2t+1)(8t^2+4t+1)f_{4t}. \tag{5.9}$$

Multiply (5.8) by $4t+1$ and replace the value from (5.9) to obtain

$$\begin{aligned} 2(4t+1)(2t+1)f_{4t+3} &= -64t(t+1)(2t+1)^2 f_{4t+1} + \\ &+ 4(8t+5)(2t+1)(8t^2+4t+1)f_{4t}. \end{aligned} \tag{5.10}$$

Now use $n = 4t+1$ in (5.2) to obtain

$$4t f_{4t+1} = -(8t+1)f_{4t} - (4t+1)(16t^2+1)f_{4t-1} \tag{5.11}$$

and replacing in the first term of (5.10) transforms this expression into

$$\begin{aligned} 2(2t+1)(4t+1)f_{4t+3} &= 4(2t+1)(4t+1)(4t+3)(8t+3)f_{4t} \\ &+ 16(t+1)(2t+1)^2(4t+1)(16t^2+1)f_{4t-1}. \end{aligned} \tag{5.12}$$

The result is now established by induction. The first term on the right-hand side of (5.12) has 2-adic valuation $2+t$ and the second one, using the inductive hypothesis, has valuation $4+t$. Therefore, the left-hand side has valuation $t+2$. This proves $\nu_2(f_{4t+3}) = t+1$, completing the inductive argument.

Case 4. Assume $n \equiv 1 \bmod 4$. The identity (5.8) is

$$2(2t+1)f_{4t+3} + (8t+5)f_{4t+2} = -(4t+3)(16t^2+16t+5)f_{4t+1}. \tag{5.13}$$

The 2-adic valuation of the first term on the left-hand side is $1+t+1 = t+2$ and for the second term t. Therefore, the right-hand side has valuation t, as claimed.

This completes the proof. □

The prime $p = 3$. In this case, the analysis is simpler.

Theorem 5.2. The number f_n is not divisible by 3. The patterns modulo 3 are given by

$$f_n \equiv \begin{cases} 1 \bmod 3 & \text{if } n \equiv 1, 2 \mod 3 \\ 2 \bmod 3 & \text{if } n \equiv 0 \mod 3. \end{cases}$$

Proof. Write $n = 3t+j$ and then (5.2) gives

$$(j-1)f_{3t+j} \equiv -(2j-1)f_{3t+j-1} - j(j^2-2j+2)f_{3t+j-2} \bmod 3. \tag{5.14}$$

Proceed by induction.

In the case $j=0$, the identity (5.14) becomes $-f_{3t} \equiv f_{3t-1} \bmod 3$. The induction hypothesis gives $f_{3t} \equiv 2 \bmod 3$.

If $j=2$, then (5.14) gives $f_{3t+2} \equiv -f_{3t} \mod 3$ and it produces $f_{3t+2} \equiv 1 \bmod 3$, as claimed.

To prove the remaining case, start with the identities

$$\begin{aligned} 3t f_{3t+1} &= -(6t+1)f_{3t} - (3t+1)(9t^2+1)f_{3t-1} \\ (3t-1)f_{3t} &= -(6t-1)f_{3t-1} - 3t(9t^2-6t+2)f_{3t-2} \end{aligned}$$

obtained from (5.2). Add these two equations and divide by $3t$ to get

$$f_{3t+1} + 3t f_{3t} = -3t(3t^2+t+1)f_{3t-1} - (9t^2-6t+2)f_{3t-2}. \tag{5.15}$$

Reducing modulo 3 gives $f_{3t+1} \equiv f_{3t-2} \bmod 3$. It follows that $f_{3t+1} \equiv 1 \bmod 3$. □

The prime $p = 5$. In this case, the function $\nu_5(f_n)$ decreases in some intervals (see Figure 5). It is evident that a delicate analysis of this function will be required to capture these decreasing segments.

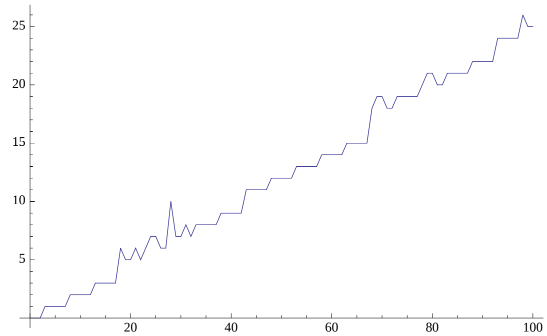

FIGURE 5. Power of 5 that divides f_n

The prime $p = 7$ is similar to $p = 3$.

Theorem 5.3. The number f_n is not divisible by 7. In fact, it is periodic modulo 42, with

$$f_n \equiv \begin{cases} 1 & \text{if } n \equiv 1, 2, 5, 11, 21, 31, 41 \mod 42 \\ 2 & \text{if } n \equiv 7, 17, 27, 29, 30, 33, 39 \mod 42 \\ 3 & \text{if } n \equiv 4, 14, 24, 34, 36, 37, 40 \mod 42 \\ 4 & \text{if } n \equiv 3, 13, 15, 16, 19, 25, 35 \mod 42 \\ 5 & \text{if } n \equiv 6, 8, 9, 12, 18, 28, 38 \mod 42 \\ 6 & \text{if } n \equiv 10, 20, 22, 23, 26, 32, 42 \mod 42 \end{cases} \tag{5.16}$$

A proof in the style similar to the case $p = 3$ is left to the reader.

Note 5.4. The experiments conducted with the valuations of f_n suggest that there are three types of primes:

Type 1. The prime p *does not* divide any element of the sequence f_n. The first few examples are $\{3, 7, 11, 23, 31, 47, 59\}$.

Type 2. The valuation $\nu_p(f_n)$ has *asymptotically linear* behavior. The first few examples are $\{2, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97\}$. Figure 6 shows the graph of $\nu_{13}(f_n)$. The deviation from its linear asymptote is also shown in Figure 6.

Conjecture 5.5. Assume p is a prime of type 2. Then

$$\nu_p(f_n) \sim \frac{n}{p-1}, \quad \text{as } n \to \infty. \tag{5.17}$$

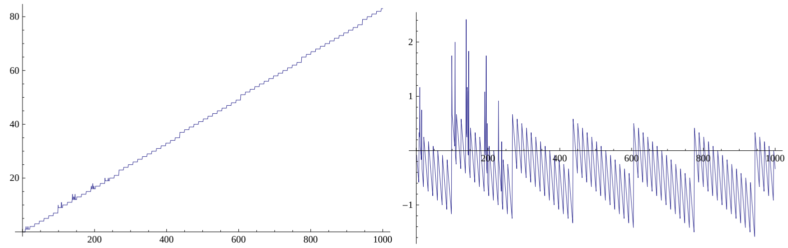

FIGURE 6. The 13-adic valuation of f_n and its deviation from asymptotic behavior.

Type 3. These are primes p for which $\nu_p(f_n)$ exhibits a well-defined oscillation. Figure 7 shows the examples $p = 19$ and $p = 43$. These primes play an important role in the integrality question of the original sequence $\{x_n\}$. The first few cases are

19	43	71	79	83	131	163	191	199	211
223	227	263	311	331	347	379	431	463	467
491	499	563	659	727	811	839	863	883	971

TABLE 2. Oscillating primes

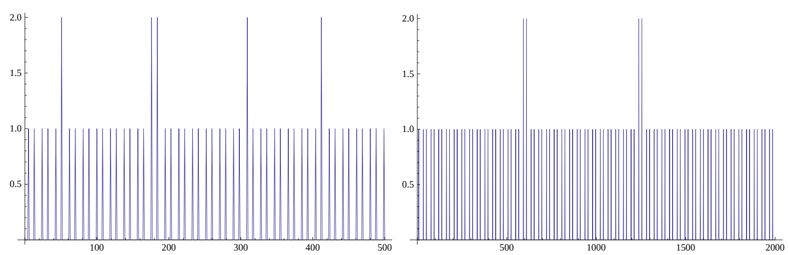

FIGURE 7. The valuation $\nu_{19}(f_n)$ and $\nu_{43}(f_n)$

Note 5.6. This sequence of primes does not appear in The On-Line Encyclopedia of Integer Sequences (OEIS).

The next section presents an argument geared towards the existence of subsequences of $\{x_n : n \in \mathbb{N}\}$ which are non-integers. It is expected that any oscillating prime will produce such subsequences.

6. A periodic example

In the case of a sequence satisfying a recurrence with constant coefficients, it is clear that the residues modulo a prime p form a periodic sequence. For example, for the Fibonacci numbers F_n given by $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. To verify this fact define $h_{n,p} := \text{Mod}(F_n, p)$ and observe that the pigeon-hole principle shows that the list $\{h_{n,p} : n \in \mathbb{N}\}$ contains indices $n_0 < n_1$ with

$$(h_{n_0,p}, h_{n_0+1,p}) = (h_{n_1,p}, h_{n_1+1,p}). \tag{6.1}$$

The recurrence for the Fibonacci numbers shows that the string

$$(h_{n_0,p},\, h_{n_0+1,p},\, h_{n_0+2,p}, \cdots, h_{n_1-1,p}) \tag{6.2}$$

is a period for $\{h_{n,p} : n \in \mathbb{N}\}$.

The recurrence satisfied by the sequence $\{f_n : n \in \mathbb{N}\}$

$$nf_{n+1} = -(2n+1)f_n - (n+1)(n^2+1)f_{n-1}, \tag{6.3}$$

from (2.5), has non-constant coefficients. Therefore the previous periodicity argument is not applicable for this situation. Nevertheless, there are some primes for which the residues do form a periodic sequence. The case $p = 19$ is discussed in detail here as it has arithmetical consequences for the original sequence $\{x_n\}$.

A direct computation of the residues of $\{f_n : n \in \mathbb{N}\}$ gives evidence that the numbers $f_n \bmod 19$ form a periodic sequence of period $171 = 9 \cdot 19$. This is the content of the next result.

Theorem 6.1. The sequence $\{f_n \bmod 19 : n \in \mathbb{N}\}$ is a periodic sequence of minimal period 171.

The idea of the proof is to expand the index n in base 19 in the form

$$n = n_0 + 19n_1 + 19^2 n_2 + 19^3 n_3 + \cdots, \tag{6.4}$$

and then determine conditions on the digits n_j for a possible exception to the theorem. Lemma 6.1 shows that any such exception must have $n_0 = 0$. Lemma 6.2 shows that $n_1 = 14$ and Lemma 6.3 gives the contradictory statement that $n_1 = 9$. This proves the theorem.

The recurrence for f_n is repeated here

$$(n-1)f_n = -\left[(2n-1)f_{n-1} + n(n^2-2n+2)f_{n-2}\right], \quad \text{for } n \geqslant 3, \tag{6.5}$$

for the convenience of the reader.

Lemma 6.1. *Assume $n \not\equiv 0 \bmod 19$; that is $n_0 \neq 0$. Then $f_n \equiv f_{n-171} \bmod 19$.*

Proof. The first row of identity (2.14) with $j = 171$ becomes

$$(n-171)f_n = \alpha_{n,171} f_{n-171} + \beta_{n,171} f_{n-172}. \tag{6.6}$$

The polynomial $\alpha_{n,171}$ is of degree 171 and its first few terms are

$$\begin{aligned}\alpha_{n,171} &= 172n^{171} - 2514726n^{170} + 18238895910n^{169} \\ &\quad -87492422433780n^{168} + 312275766371812152n^{167} - \cdots.\end{aligned}$$

The coefficients of $\alpha_{n,171}$ and $\beta_{n,171}$ grow very rapidly.

The relation (6.6) is considered now modulo 19 and written as

$$nf_n \equiv z_1(n) f_{n-171} + z_2(n) f_{n-172} \bmod 19 \tag{6.7}$$

with $z_1(n)$ the polynomial $\alpha_{n,171}$ with coefficients reduced modulo 19 and $z_2(n)$ the corresponding one for $\beta_{n,171}$. A direct symbolic calculation produces

$$\begin{aligned} z_1(n) \quad := \quad & 15n + 9n^3 + 13n^5 + 18n^7 + 5n^{19} + 2n^{21} + 7n^{23} + 14n^{25} + n^{27} + 7n^{39} + 13n^{41} \\ & 13n^{43} + +11n^{45} + n^{57} + 17n^{59} + 7n^{61} + 9n^{63} + 5n^{77} + 15n^{79} + n^{81} + 2n^{95} \\ & 12n^{97} + 13n^{99} + 8n^{115} + n^{117} + 8n^{133} + 9n^{135} + 11n^{153} + n^{171}, \end{aligned}$$

and

$$\begin{aligned} z_2(n) \quad = \quad & 14n + 13n^3 + 16n^5 + 9n^7 + 10n^9 + 18n^{11} + 5n^{19} + 8n^{21} + 13n^{23} + 9n^{25} \\ & +8n^{27} + 9n^{29} + 16n^{39} + 15n^{41} + 2n^{43} + 5n^{45} + 2n^{47} + n^{57} + 2n^{59} \\ & +15n^{61} + 3n^{63} + 8n^{65} + 9n^{77} + 14n^{79} + 12n^{81} + 7n^{83} + 2n^{95} + 18n^{97} \\ & +9n^{99} + 12n^{101} + n^{115} + 12n^{117} + 11n^{119} + 8n^{133} + 6n^{135} + 17n^{137} \\ & +10n^{153} + 10n^{155} + n^{171} + n^{173}. \end{aligned}$$

The polynomials z_1, z_2 are further reduced using Fermat's little theorem $n^a \equiv n^r \bmod 19$, where $a = 18t + r$ and $0 \leqslant r \leqslant 17$. This gives

$$z_1(n) \equiv n \bmod 19 \text{ and } z_2(n) \equiv 0 \bmod 19. \tag{6.8}$$

Therefore (6.7) is simply

$$nf_n \equiv nf_{n-171} \bmod 19. \tag{6.9}$$

The proof is complete. □

Note 6.2. The previous lemma shows that any exception to Theorem 6.1 forces $n_0 = 0$; that is, n has an expansion of the form

$$n = 19n_1 + 19^2 n_2 + 19^3 n_3 + \cdots \tag{6.10}$$

LEMMA 6.2. *Assume* $n_0 = 0$ *and* $n_1 \neq 14$. *Then* $f_n \equiv f_{n-171} \bmod 19$.

PROOF. Let $n = 19m$. Then (6.6) yields

$$(19m - 171) f_{19m} = \alpha_{19m,171} f_{19m-171} + \beta_{19m,171} f_{19m-172}. \tag{6.11}$$

A symbolic computation reveals that $\alpha_{19m,171}$ and $\beta_{19m,171}$ have all their coefficients divisible by 19. Define

$$\alpha^*_{19m,171} = \frac{1}{19}\alpha_{19m,171} \text{ and } \beta^*_{19m,171} = \frac{1}{19}\beta_{19m,171}. \tag{6.12}$$

Then (6.11) takes the form

$$(m - 9) f_{19m} = \alpha^*_{19m,171} f_{19m-171} + \beta^*_{19m,171} f_{19m-172}. \tag{6.13}$$

A computation of (6.13) modulo 19 produces

$$(m - 9) f_{19m} = (15m + 18) f_{19m-171} + (14m + 8) f_{19m-172} \bmod 19. \tag{6.14}$$

The recurrence (6.5) is

$$(n-1)f_n = -(2n-1)f_{n-1} - n(n^2-2n+2)f_{n-2} \tag{6.15}$$

and replacing n by $19m$ gives

$$(19m-1)f_{19m} = -(38m-1)f_{19m-1} - 19m(361m^2-38m+2)f_{19m-2}. \tag{6.16}$$

Computing modulo 19 implies

$$f_{19m} \equiv -f_{19m-1} \bmod 19. \tag{6.17}$$

Lemma 6.1 shows that

$$f_{19m-172} \equiv f_{19m-1} \bmod 19 \tag{6.18}$$

since $19m-172 \not\equiv 0 \bmod 19$. Then (6.14) gives

$$\begin{aligned}(m-9)f_{19m} &\equiv (15m+18)f_{19m-171} + (14m+8)f_{19m-172} \bmod 19\\ &\equiv (15m+18)f_{19m-171} + (14m+8)f_{19m-1} \bmod 19\\ &\equiv (15m+18)f_{19m-171} - (14m+8)f_{19m} \bmod 19.\end{aligned}$$

Therefore

$$(15m-1)f_{19m} \equiv (15m-1)f_{19m-171} \bmod 19. \tag{6.19}$$

The congruence $15m-1 \equiv 0 \bmod 19$ is equivalent to $m \equiv 14 \bmod 19$, thus $m \not\equiv 14 \bmod 19$ implies

$$f_{19m} \equiv f_{19m-171} \bmod 19. \tag{6.20}$$

This gives the result. □

Lemma 6.3. *Assume $n_0 = 0$ and $n_1 \neq 9$. Then $f_n \equiv f_{n-171} \bmod 19$.*

Proof. Replacing m by $m-9$ in (6.17) gives

$$f_{19m-171} \equiv -f_{19m-172} \bmod 19. \tag{6.21}$$

Then (6.14) produces

$$\begin{aligned}(m-9)f_{19m} &\equiv (15m+18)f_{19m-171} + (14m+8)f_{19m-172} \bmod 19\\ &\equiv (15m+18)f_{19m-171} - (14m+8)f_{19m-171} \bmod 19\\ &\equiv (m-9)f_{19m-171} \bmod 19.\end{aligned} \tag{6.22}$$

This gives the result. □

Lemmas 6.2 and 6.3 complete the proof of Theorem 6.1.

Note 6.3. Symbolic computations show that for primes $p \equiv 3 \bmod 4$, the sequence $\text{Mod}(f_n, p)$ has minimal period $p(p-1)/2$ if $p \equiv 3 \bmod 8$ and $p(p-1)$ if $p \equiv 7 \bmod 8$.

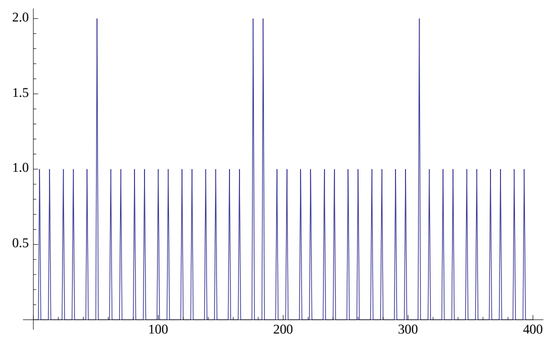

FIGURE 8. Power of 19 that divides f_n

7. Non-integral subsequences of x_n

The existence of non-integral values of x_n can be seen directly from the graph of $\nu_p(f_n)$. Theorem 4.1 states that every decreasing section of this graph corresponds to non-integral x_n. The graph in Figure 8 contains many such decreasing segments. This will be used to verify the existence of two non-integral arithmetic subsequences of x_n.

Theorem 6.1 shows that f_n mod 19 is a periodic sequence, with period 171. Table 3 gives the residues modulo 19, where the columns are indexed modulo 19 and the rows are indexed modulo 9. For instance, the first row states that f_{19n}, with $n \equiv 1 \bmod 9$ satisfies $f_{19n} \equiv 2 \bmod 19$. Also f_{19n}, with $n \equiv 2 \bmod 9$ satisfies $f_{19n} \equiv 15 \bmod 19$; and so on.

The data given in Table 3 is a complete listing, from a direct symbolic evaluation of f_n, for the values in the range $1 \leqslant n \leqslant 171$. It is also possible to verify these residues using the recurrence (2.6). Indeed, replacing n by $19n + a - 1$ in (2.6) gives

(7.1)
$$\begin{aligned}(19n+a-1)f_{19n+a} &= -(38n+2a-1)f_{19n+a-1} \\ &\quad -(19n+a)(361n^2+38an-38n+a^2-2a+2)f_{19n+a-2}\end{aligned}$$

and reducing modulo 19 yields

$$(a-1)f_{19n+a} \equiv -(2a-1)f_{19n+a-1} - a(a^2-2a+2)f_{19n+a-2} \bmod 19. \tag{7.2}$$

This identity is now employed to justify the values given in Table 3, inductively. Recall that the indices n are further computed modulo 9.

Example 1. Take $a = 0$, then (7.2) yields

$$f_{19n} \equiv -f_{19n-1} = -f_{19(n-1)+18} \bmod 19. \tag{7.3}$$

A couple of examples are provided to illustrate the procedure.

If $n \equiv 1 \bmod 9$, then $19n - 1 = 19(n-1) + 18$ and $n - 1 \equiv 0 \bmod 9$. The induction hypothesis shows that $f_{19n-1} = f_{19(n-1)+18} = 17 \bmod 9$. This shows that $f_{19n} \equiv -17 \equiv 2 \bmod 19$ as claimed.

f_{19n}	$\equiv$	2	15	8	3	13	12	14	10	18
f_{19n+1}	$\equiv$	17	4	11	16	6	7	5	9	1
f_{19n+2}	$\equiv$	17	4	11	16	6	7	5	9	1
f_{19n+3}	$\equiv$	1	17	4	11	16	6	7	5	9
f_{19n+4}	$\equiv$	18	2	15	8	3	13	12	14	10
f_{19n+5}	$\equiv$	0	0	0	0	0	0	0	0	0
f_{19n+6}	$\equiv$	16	6	7	5	9	1	17	4	11
f_{19n+7}	$\equiv$	16	6	7	5	9	1	17	4	11
f_{19n+8}	$\equiv$	15	8	3	13	12	14	10	18	2
f_{19n+9}	$\equiv$	7	5	9	1	17	4	11	16	6
f_{19n+10}	$\equiv$	14	10	18	2	15	8	3	13	12
f_{19n+11}	$\equiv$	8	3	13	12	14	10	18	2	15
f_{19n+12}	$\equiv$	1	17	4	11	16	6	7	5	9
f_{19n+13}	$\equiv$	0	0	0	0	0	0	0	0	0
f_{19n+14}	$\equiv$	4	11	16	6	7	5	9	1	17
f_{19n+15}	$\equiv$	8	3	13	12	14	10	18	2	15
f_{19n+16}	$\equiv$	11	16	6	7	5	9	1	17	4
f_{19n+17}	$\equiv$	17	4	11	16	6	7	5	9	1
f_{19n+18}	$\equiv$	4	11	16	6	7	5	9	1	17

TABLE 3. Values modulo 19

If $n \equiv 2 \bmod 9$, then $19n-1 = 19(n-1)+18$ and $n-1 \equiv 1 \bmod 9$. The induction hypothesis shows that $f_{19n-1} = f_{19(n-1)+18} = 4 \bmod 9$. This shows that $f_{19n} \equiv -4 \equiv 15 \bmod 19$ as stated.

Example 2. The only special case of equation (7.2) is $a = 1$, in which instance

$$19nf_{19n+1} = -(38n+1)f_{19n} - (19n+1)(361n^2+1)f_{19n-1}. \tag{7.4}$$

Use $a = 0$ in (7.1) to obtain

$$(19n-1)f_{19n} = -(38n-1)f_{19n-1} - 19n(361n^2-38n+2)f_{19n-2}. \tag{7.5}$$

Multiply (7.4) by $19n-1$ and replace in (7.5) to get

$$(19n-1)f_{19n+1} = -19n(19n-2)(19n+2)f_{19n-1} + (38n+1)(361n^2-38n+2)f_{19n-2},$$

then modulo 19 it becomes

$$f_{19n+1} \equiv 17 f_{19n-2} \bmod 19. \tag{7.6}$$

The data in Table 3 shows that this must be consistent with

$$\{17, 4, 11, 16, 6, 7, 5, 9, 1\} \equiv 17 \times \{1, 17, 4, 11, 16, 6, 7, 5, 9\} \bmod 19. \tag{7.7}$$

This is indeed true.

The argument above is summarized in the following statement.

PROPOSITION 7.1. *The prime* 19 *divides* f_{19n+5} *and it does not divide* f_{19n+6}*. Therefore* f_{19n+5} *does not divide* f_{19n+6}*. Similarly,* f_{19n+13} *does not divide* f_{19n+14}.

Theorem 4.1 now gives the next statement.

Corollary 7.1. *The numbers* $\{x_{19n+5} : n \in \mathbb{N}\}$ *and* $\{x_{19n+13} : n \in \mathbb{N}\}$ *are not integers.*

Note 7.1. The reader will verify, along the same lines as described above, that $\{x_{43n+8} : n \in \mathbb{N}\}$ and $\{x_{43n+34} : n \in \mathbb{N}\}$ are not integers. The proof should start by checking that f_n mod 43 is a periodic sequence with minimal period $301 = 43 \cdot 7$. Then verify that 43 divides f_{43n+8} and f_{43n+34} but it divides neither f_{43n+9} nor f_{43n+35}.

8. Case study $p = 13$: asymptotic linear growth

This section reports on some experimental observations for the valuation $\nu_{13}(f_n)$. The goal is to present a formula analogous to the classical formula of Legendre for valuations of factorials:

$$\nu_p(n!) = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor. \tag{8.1}$$

The formula (8.1) gives the p-adic valuation of n as

$$\nu_p(n) = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{n-1}{p^j} \right\rfloor \right). \tag{8.2}$$

The summand in (8.2) is a periodic function of period p^j.

This approach has been applied in [**2**] in synthesising the p-adic valuation of ASM-numbers. An *alternating sign matrix* (ASM) is an array of 0, 1 and -1, such that the entries of each row and column add up to 1 and the non-zero entries of a given row/column alternate. After a fascinating sequence of events, D. Zeilberger [**8**] proved that the cardinality of such matrices is enumerated by

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{8.3}$$

In particular, the product in (8.3) is an integer: not an obvious fact. The story behind this formula and its many combinatorial interpretations are given in D. Bressoud's book [**4**].

The main result of [**2**] is a formula for the p-adic valuation of A_n similar to (8.2).

Theorem 8.1. Let $n \in \mathbb{N}$ and $p \geqslant 5$ be a prime. Define

$$\mathrm{Per}_{j,p}(n) = \begin{cases} 0 & \text{if} \quad 0 \leqslant n \leqslant \left\lfloor \frac{p^j+1}{3} \right\rfloor \\ n - \left\lfloor \frac{p^j+1}{3} \right\rfloor & \text{if} \quad \left\lfloor \frac{p^j+1}{3} \right\rfloor + 1 \leqslant n \leqslant \frac{p^j-1}{2} \\ \left\lfloor \frac{2p^j+1}{3} \right\rfloor - n & \text{if} \quad \frac{p^j+1}{2} \leqslant n \leqslant \left\lfloor \frac{2p^j+1}{3} \right\rfloor \\ 0 & \text{if} \quad \left\lfloor \frac{2p^j+1}{3} \right\rfloor + 1 \leqslant n \leqslant p^j - 1. \end{cases} \tag{8.4}$$

Then

$$\nu_p(A_n) = \sum_{j=1}^{\infty} \mathrm{Per}_{j,p}\left(n \mod p^j\right). \tag{8.5}$$

The description of $\nu_{13}(f_n)$ given below is an initial step in establishing a theorem similar to Theorem 8.1 for the p-adic valuation of the sequence f_n. It is important to recall that the expressions in (8.4) and (8.5) were discovered experimentally. The process of obtaining the correct formula for $\nu_p(A_n)$ was the hardest part of the proof of Theorem 8.1. The graphs presented below represent the initial guess for a possible analytic expression of $\nu_{13}(f_n)$.

Step 1. Figure 9 shows the valuation $\nu_{13}(f_n)$ in the range $1 \leqslant n \leqslant 300$. This graph shows the asymptotic behavior $\nu_{13}(f_n) \sim \frac{n}{13}$ as well as some peculiar small oscillations in the range $1 \leqslant n \leqslant 267$. This disappears for values $n \geqslant 267$ as shown in the figure on the right with range $300 \leqslant n \leqslant 600$.

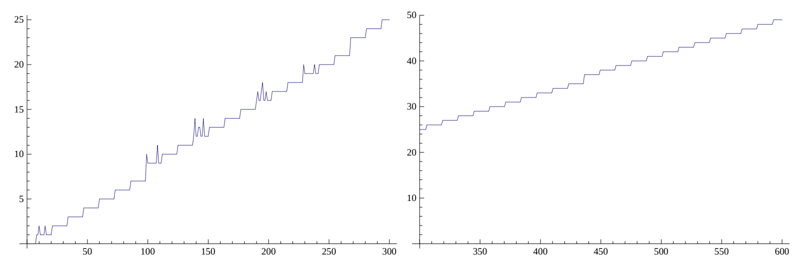

FIGURE 9. $\nu_{13}(f_n)$ for $1 \leqslant n \leqslant 300$ and $301 \leqslant n \leqslant 600$

The graph in Figure 10 shows this valuation in the range $1 \leqslant n \leqslant 1000$, this pointing to a clear linear asymptotic behavior. The figure on the right shows the deviation from the asymptote. The oscillations at the beginning of the graph correspond to the range $1 \leqslant n \leqslant 267$.

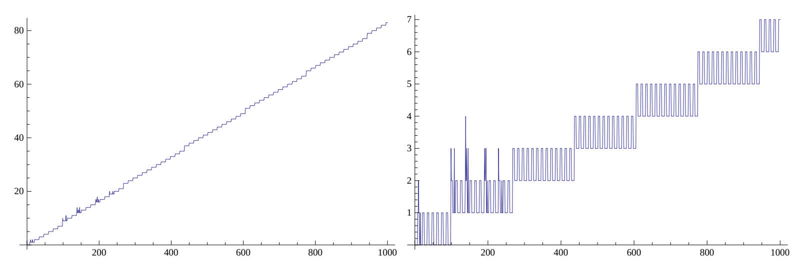

FIGURE 10. $\nu_{13}(f_n)$ and deviation from asymptotes

Step 2. Define the function

$$T_1(n) = \nu_{13}(f_n) - \left\lfloor \frac{n}{13} \right\rfloor \tag{8.6}$$

measuring the error of $\nu_{13}(f_n)$ against its asymptote.

In order to ignore the initial oscillation, it is convenient to define the function

$$T_2(n) = T_1(n + 267) \tag{8.7}$$

and the first error term

$$E_1(n) = T_2(n) - 2. \tag{8.8}$$

is shown in Figure 11.

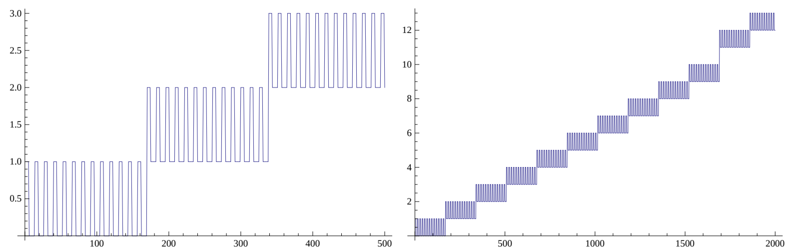

FIGURE 11. The error term $E_1(n)$ for $1 \leqslant n \leqslant 500$ and $1 \leqslant n \leqslant 2000$

Note 8.2. The valuation has been expressed as

$$\nu_{13}(f_{n+267}) = \left\lfloor \frac{n+7}{13} \right\rfloor + 22 + E_1(n) \tag{8.9}$$

where the bounds for the error $E_1(n)$ are shown in Table 6.

Step 3. The first correction to the error $E_1(n)$ is based on the graph seen in Figure 12 showing $E_1(n)$ for $1 \leqslant n \leqslant 52 = 4 \cdot 13$. The periodicity shown here is described by the function

$$x_1(n) = \begin{cases} 1 & \text{if } 1 \leqslant n \leqslant 5 \\ 0 & \text{otherwise.} \end{cases} \tag{8.10}$$

Figure 13 presents the error term

$$E_2(n) = E_1(n) - x_1(\text{mod}(n, 13)) \tag{8.11}$$

for the same range of values shown in Figure 11.

Figure 14 shows the error term $E_2(n)$ for $1 \leqslant n \leqslant 10000$.

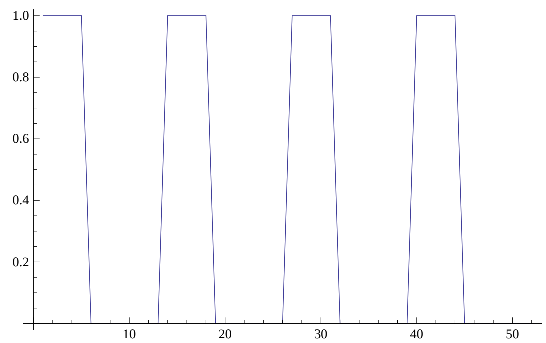

FIGURE 12. The correction term $x_1(n)$ for $1 \leqslant n \leqslant 52$

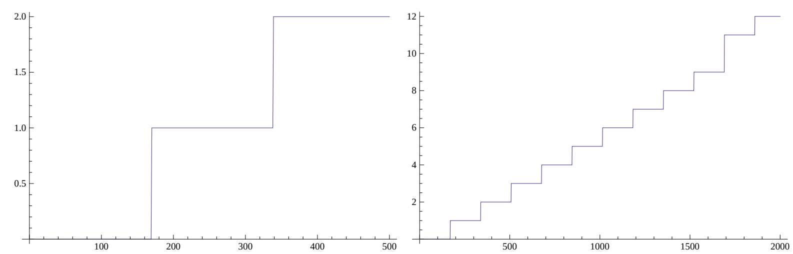

FIGURE 13. The error term $E_2(n)$ for $1 \leqslant n \leqslant 500$ and $1 \leqslant n \leqslant 2000$

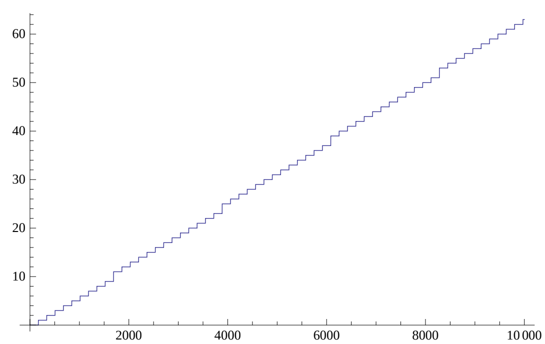

FIGURE 14. The error term $E_2(n)$ for $1 \leqslant n \leqslant 10000$

Note 8.3. The expression for $\nu_{13}(f_{n+267})$ in Note 8.2 has been replaced by

$$\nu_{13}(f_{n+267}) = \left\lfloor \frac{n+7}{13} \right\rfloor + 22 + E_2(n) + x_1(\mathrm{Mod}(n, 13)). \tag{8.12}$$

The identity

$$x_1(\mathrm{Mod}(n,13)) + \left\lfloor \frac{n+7}{13} \right\rfloor = \left\lceil \frac{n}{13} \right\rceil \tag{8.13}$$

implies

$$\nu_{13}(f_{n+267}) = \left\lceil \frac{n}{13} \right\rceil + 22 + E_2(n). \tag{8.14}$$

Step 4. The linear asymptotic growth of $E_2(n)$ depicted in Figure 14 motivates the definition of the next correction for the error. The graph in Figure 15 shows the possible corrections $E_2(n) - \left\lfloor \frac{n}{13^2} \right\rfloor + 1$ and $E_2(n) - \left\lceil \frac{n}{13^2} \right\rceil + 1$, in the range $1 \leqslant n \leqslant 5000$.

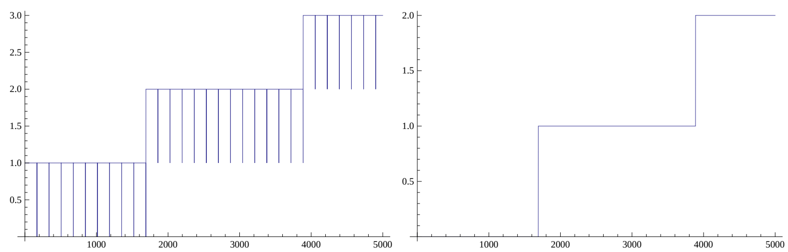

FIGURE 15. Possible corrections to the error term $E_2(n)$

The graphs in Figure 15 motivate the definition

$$E_3(n) = E_2(n) - \left\lceil \frac{n}{13^2} \right\rceil + 1. \tag{8.15}$$

This function is shown in Figure 16 in the range $1 \leqslant n \leqslant 10000$ and $1 \leqslant n \leqslant 50000$.

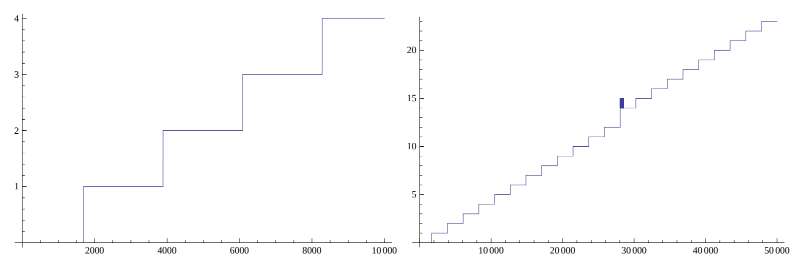

FIGURE 16. The error term $E_3(n)$ for $1 \leqslant n \leqslant 10000$ and $1 \leqslant n \leqslant 50000$

Note 8.4. The valuation is now expressed as

$$\nu_{13}(f_{n+267}) = \left\lceil \frac{n}{13} \right\rceil + \left\lceil \frac{n}{13^2} \right\rceil + 21 + E_3(n), \tag{8.16}$$

and the bounds for the error $E_3(n)$ are shown in Table 6.

Step 5. The functions

$$E_4(n) = E_3(n) - \left\lfloor \frac{n}{13^3} \right\rfloor \tag{8.17}$$

and

$$E_5(n) = E_4(n) - x_2\left(\text{Mod}(n, 13^3)\right) \tag{8.18}$$

with

$$x_2(n) = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant 1690 \\ 1 & \text{otherwise,} \end{cases} \tag{8.19}$$

form the next two components of this approximation process. Figure 17 and Table 6 shows these errors.

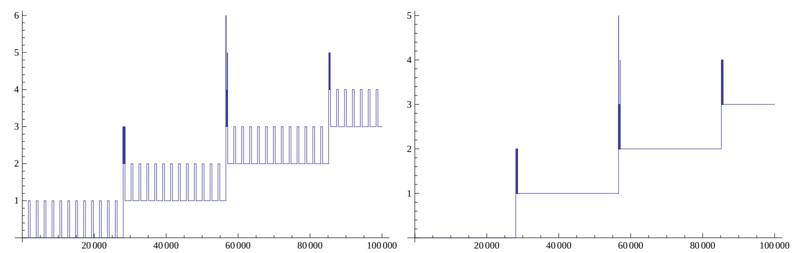

FIGURE 17. The error terms $E_4(n)$ and $E_5(n)$ for $1 \leqslant n \leqslant 100000$

For example, $\nu_{13}(f_{n+267})$ and the function

$$21 + \left\lceil \frac{n}{13} \right\rceil + \left\lceil \frac{n}{13^2} \right\rceil + \left\lfloor \frac{n}{13^3} \right\rfloor + x_2\left(\text{Mod}(n, 13^3)\right) \tag{8.20}$$

differ by at most 7 in the range $1 \leqslant n \leqslant 200000$. The next table shows the distribution of these values.

0	1	2	3	4	5	6	7
28054	28535	28559	28571	28558	28540	28601	582

TABLE 4. Value distribution of the error term E_5

Step 6. The last correction term is defined by

$$E_6(n) = E_5(n) - \left\lceil \frac{n}{13^4} \right\rceil + 1 \tag{8.21}$$

and the data shows that $|E_6(n)| \leqslant 4$ for $1 \leqslant n \leqslant 200000$. The table shows the distribution of the values taken by E_6:

0	1	2	3	4
196451	3419	124	5	1

TABLE 5. Value distribution of the error term E_6

Note 8.5. The goal of this section was to obtain an analytic expression for the p-adic valuations of f_n, for those primes p where $\nu_p(f_n)$ grows linearly. The empirical functions described above, show that the functions $\nu_{13}(f_{n+267})$ and

$$h_6(n) := 19 + \left\lceil \frac{n}{13} \right\rceil + \left\lceil \frac{n}{13^2} \right\rceil + \left\lceil \frac{n}{13^3} \right\rceil + \left\lceil \frac{n}{13^4} \right\rceil + x_2\left(\text{Mod}\left(n, 13^3\right)\right) \tag{8.22}$$

agree in 196451 out of the first 200000 values of n (this is 98.22% of the cases). Moreover in 99.93% of the cases, these two functions differ by at most 1. The data for the errors is summarized in Table 6.

Max n	Max $\nu_{13}(f_{n+267})$	Max E_1	Max E_2	Max E_3	Max E_4	Max E_5	Max E_6
10000	832	64	63	4	1	0	0
50000	4165	319	318	23	3	2	2
100000	8332	640	639	48	6	5	4
150000	12498	961	960	73	8	7	4
200000	16666	1282	1281	98	8	7	4
250000	20832	1603	1602	123	13	12	5
300000	24999	1923	1922	147	14	13	5

TABLE 6. The errors in the approximations to $\nu_{13}(f_{n+267})$

Conclusion. An analytic formula for $\nu_{13}(f_n)$ has not been obtained. The search for this formula has produced a simple analytic expression that matches this valuation at almost all integer values.

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