

A family of palindromic polynomials

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ABSTRACT. The roots of a polynomial coming from a nonselfadjoint boundary value problem are described. The polynomial has palindromic coefficients and it depends on a single parameter. The structure of the roots as a function of this parameter is given.

1. Introduction

The relation between the roots of a polynomial $P(z)$ and its coefficients was one of the driving forces in the development of Algebra. The desire to obtain a closed-form expression for the roots of $P(z) = 0$ ended with a negative solution at the hands of N. Abel and E. Galois at the beginning of the 19th century: a generic polynomial equation of degree 5 or more cannot be solved by radicals. The reader should be aware of the existence of analytic expressions for the roots of a polynomial. Naturally these involve non-algebraic functions: for the quintic equation this is done using *elliptic functions*, as explained in the beautiful text [3], and for higher degree the formulas involve the so-called *theta functions*.

In spite of this set-back, the study of roots of polynomials has continued throughout the centuries. Classical results include Newton's statement that if P has only negative real roots then the coefficients a_j of P form a *logconcave sequence*, that is, $a_j^2 - a_{j-1}a_{j+1} \geq 0$. For starters, the reader is referred to [4]. Logconcave sequences arise in many combinatorial contexts, the simplest of which are the binomial coefficients $\{\binom{n}{k} : 0 \leq k \leq n\}$. Of course, there are artificial ways to construct logconcave polynomials. Here is a simple-minded example

$$(1.1) \quad A(z) = \sum_{r=0}^m r(m-r)z^r.$$

The reader will see this polynomial appearing later in this paper, in a natural way.

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It is well-known that if a polynomial $P(z) = \sum_{k=0}^n a_k z^k$ has only *unimodular* roots (i.e. $|z| = 1$) then it is either *palindromic* (i.e. $a_k = a_{n-k}$) or *anti-palindromic* (i.e. $a_k = -a_{n-k}$). The converse is not true however; for example $P(z) = z^2 - 3z + 1$. In light of this, the polynomials $F_{\lambda,m}(z)$ defined below, being anti-palindromic, are perfect candidates for a study on unimodular zeros.

The main task in this work aims at locating the roots of the polynomial

$$(1.2) \quad \begin{aligned} F_{\lambda,m}(z) &= (z^m + 1)(z - 1) - \lambda(z^m - 1)(z + 1) \\ &= (1 - \lambda)z^{m+1} - (1 + \lambda)z^m + (1 + \lambda)z - (1 - \lambda), \end{aligned}$$

as a function of the parameter $\lambda \in \mathbb{R}$. Observe that, for $\lambda = 0$, this polynomial reduces to $(z^m + 1)(z - 1)$ and all its roots are on the unit circle $|z| = 1$.

The critical case $\lambda = 1/m$, where the equation $F_{\lambda,m}(z) = 0$ turns out to be

$$(1.3) \quad (m - 1)z^{m+1} - (m + 1)z^m + (m + 1)z - (m - 1) = 0,$$

appeared in the context of the so-called *interior transmission problem*. This is a nonselfadjoint boundary-value problem which has attracted a lot of attention in the recent years. The problem relates the refractive index of a medium to a sequence of complex numbers called the transmission eigenvalues. In the spherically symmetric case, there is a natural special subset of those eigenvalues which coincides with the set of zeros of an entire function $D(\lambda)$. If the refractive index of the spherical medium (of radius b) is identically equal to an integer $m \geq 2$, then $D(\lambda)$ becomes $D(\lambda) = g(m\sqrt{\lambda})/2m\sqrt{\lambda}$, where

$$\begin{aligned} g(\tau) &= 2m \sin(\tau) \cos(m\tau) - 2 \cos(\tau) \sin(m\tau) \\ &= (m - 1) \sin((m + 1)\tau) - (m + 1) \sin((m - 1)\tau). \end{aligned}$$

The reader finds further details in [1].

Setting $z = e^{i\tau}$, one obtains the polynomial of present concern:

$$4i(1/m)z^{m+1}g(\tau) = F_{1/m,m}(z).$$

The fact that the zeros of $F_{1/m,m}(z)$ lie on the unit circle implies immediately that the zeros of $g(\tau)$ are real, and hence those of $D(\lambda)$, i.e. the special transmission eigenvalues, are real too (which is, somehow, surprising). Distribution of the roots of $F_{\lambda,m}$ as a function of the parameter λ is presented in the next theorem. The proof is discussed in the upcoming sections.

THEOREM 1.1. *The polynomial $F_{\lambda,m}(z)$ always has $z = 1$ as a root. The remaining roots solve the equation $H_{\lambda,m}(z) = 0$, where $H_{\lambda,m}(z) = F_{\lambda,m}(z)/(z - 1)$. Moreover, $H_{\lambda,m}(z)$ has*

- (a) *at least $m - 2$ unimodular roots with (possibly) two exceptions;*
- (b) *the exceptions are reciprocal real roots;*
- (c) *all roots are simple except for a double root at $z = 1$;*
- (d) *if $\lambda \leq 1/m$ then all roots are unimodular;*
- (e) *in the special case $\lambda = 1$, the polynomial $H_{\lambda,m}$ is of degree m having $z = 0$ as a simple root, the rest being unimodular.*

2. Some basic properties

In this section we collect some basic properties of the polynomial $F_{\lambda,m}(z)$. The first lemma establishes part (e) of Theorem 1.1.

LEMMA 2.1. *The value $z = 0$ is a root of $F_{\lambda,m}(z) = 0$ only for $\lambda = 1$. In this case, the other $m - 1$ roots are roots of unity; hence unimodular.*

PROOF. This follows directly from $F_{\lambda,m}(0) = \lambda - 1$ and $F_{1,m}(z) = -2z(z^{m-1} - 1)$. \square

The polynomial $F_{\lambda,m}(z)$ has real coefficients, therefore the complex roots occur as conjugates pairs. The next result shows that the roots can be partitioned into groups each consisting of four elements.

LEMMA 2.2. *The polynomial $F_{\lambda,m}(z)$ satisfies*

$$(2.1) \quad F_{\lambda,m}(1/z) = -\frac{1}{z^{m+1}} F_{\lambda,m}(z).$$

Thus each root $r \in \mathbb{C}$ is part of a quartet $\{r, \bar{r}, 1/r, 1/\bar{r}\}$.

PROOF. The relation (2.1) is straightforward. Naturally, the value $\lambda = 1$ which leads to the root $z = 0$ has to be excluded. The roots in the quartet can coalesce, as in the case of $r = 1$ where they are all equal. \square

LEMMA 2.3. *The value $z = 1$ is always a root of $F_{\lambda,m}$. This root is simple for $\lambda \neq 1/m$. Also, $F_{\lambda,m}(z) = (z - 1)H_{\lambda,m}(z)$ factorizes as*

$$(2.2) \quad H_{\lambda,m}(z) = (1 - \lambda)z^m - 2\lambda \sum_{j=1}^{m-1} z^j + (1 - \lambda).$$

PROOF. Clearly $F_{\lambda,m}(1) = 0$. Similarly $F'_{\lambda,m}(1) = 2(1 - \lambda m)$ gives the simplicity statement. The expression for the quotient $F_{\lambda,m}(z)/(z - 1)$ is obtained by writing $z^k = (z^k - 1) + 1$ in (1.2). \square

LEMMA 2.4. *The value $z = -1$ is a root of $F_{\lambda,m} = 0$ for m odd. This root is simple for m odd and $\lambda \neq m$.*

PROOF. Put $z = -1$ in Lemma 2.2. The value $F'_{\lambda,m}(-1) = 2(\lambda - m)$ gives the simplicity statement. \square

3. The roots of $F_{\lambda,m}(z) = 0$

This section is devoted to a proof of Theorem 1.1. The discussion is divided into cases according to the value of λ .

Case 1: $\lambda \leq 0$.

PROPOSITION 3.1. *If $\lambda \leq 0$, then $F_{\lambda,m}(z)$ possesses only unimodular roots.*

PROOF. The equation $F_{\lambda,m}(z) = 0$ implies $z^m = \frac{(1-\lambda) - (1+\lambda)z}{(1-\lambda)z - (1+\lambda)}$. Denote the moduli by $A = |(1-\lambda) - (1+\lambda)z|$, $B = |(1-\lambda)z - (1+\lambda)|$ with $z = x + iy$. Then, the claim holds for $\lambda = 0$ since the equation becomes $z^m = -1$. For $\lambda < 0$ a direct calculation gives $A^2 = B^2 - 4\lambda(1 - |z|^2)$. This yields

$$(3.1) \quad |z|^{2m} = 1 - \frac{4\lambda}{B^2}(1 - |z|^2).$$

This equation is solvable if and only if $|z| = 1$, hence the assertion follows. \square

Case 2: $0 < \lambda < 1/m$.

The analysis is based on the fact that non-unimodular roots bifurcate from a value λ for which $F_{\lambda,m}$ has multiple unimodular roots. It is shown that this does not occur for $\lambda < 1/m$.

Lemma 2.2 shows that if r is a root of $F_{\lambda,m}(z)$, then so is $1/r$. Assume that, as λ increases from $\lambda = 0$ to some $\lambda^* > 0$, one of the roots of $F_{\lambda^*,m}(z)$ has modulus strictly bigger than 1. It follows that there is a second root with modulus less than 1. The continuity of the number of roots as a function of λ shows that these non-unimodular roots must bifurcate from a multiple root.

PROPOSITION 3.2. *For $0 < \lambda < 1/m$, the polynomial $F_{\lambda,m}(z)$ has no multiple unimodular roots.*

PROOF. A multiple root $z = r \in \mathbb{C}$ satisfies both

$$(3.2) \quad F_{\lambda,m}(r) = (1-\lambda)r^{m+1} - (1+\lambda)r^m + (1+\lambda)r - (1-\lambda) = 0, \text{ and}$$

$$(3.3) \quad F'_{\lambda,m}(r) = (m+1)(1-\lambda)r^m - m(1+\lambda)r^{m-1} + (1+\lambda) = 0.$$

Multiply (3.2) by $m+1$ and (3.3) by r and subtract to obtain

$$(3.4) \quad r^m = mr - \frac{(m+1)(1-\lambda)}{1+\lambda}.$$

Replace in (3.2) to obtain

$$(3.5) \quad m(1-\lambda^2)r^2 - 2(m-2\lambda+m\lambda^2)r + m(1-\lambda^2) = 0.$$

This equation (in the variable r) has discriminant is $\Delta = -16\lambda(m-\lambda)(1-m\lambda)$. If $\lambda < 0$, then $\Delta > 0$ and the unimodular root is real. Therefore $r = \pm 1$ and these are easy to rule out using (3.5). In the range $0 < \lambda < 1/m$, square (3.4) to get

$$(3.6) \quad (1+\lambda)^2 r^{2m} = m^2(1+\lambda)^2 r^2 - 2m(m+1)(1-\lambda^2)r + (m+1)^2(1-\lambda)^2.$$

From (3.5), it follows that

$$(3.7) \quad r^2 = \frac{2(m-2\lambda+m\lambda^2)}{m(1-\lambda^2)}r - 1,$$

and replacing in (3.6) gives

$$(3.8) \quad (1+\lambda)^2(1-\lambda)r^{2m} = 2m(1+\lambda)(2m\lambda - \lambda^2 - 1)r + (1-\lambda)(1-\lambda-2m\lambda)(1-\lambda+2m).$$

From (3.4) it follows that $r = \frac{1}{m}r^m + \frac{(m+1)(1-\lambda)}{m(1+\lambda)}$, and replacing in (3.8) gives $(1-\lambda^2)r^{2m} - 2(2m\lambda - \lambda^2 - 1)r^m + (1-\lambda^2) = 0$. This is now solved to arrive at

$$(3.9) \quad r^m - 1 = \frac{2(m\lambda - 1) \pm 2iE}{1 - \lambda^2}.$$

The solution of (3.5) implies

$$(3.10) \quad r - 1 = \frac{2\lambda(m\lambda - 1) \pm 2iE}{m(1 - \lambda^2)},$$

where $E = \sqrt{\lambda(m-\lambda)(1-m\lambda)} > 0$. Then

$$(3.11) \quad \left| \frac{r^m - 1}{r - 1} \right| = m \left| \frac{(m\lambda - 1) \pm iE}{\lambda(m\lambda - 1) \pm iE} \right| = m \sqrt{\frac{(m\lambda - 1)^2 + E^2}{\lambda^2(m\lambda - 1)^2 E^2}} > m,$$

using $\lambda < 1$. On the other hand $|r| = 1$, so that

$$(3.12) \quad \left| \frac{r^m - 1}{r - 1} \right| = |r^{m-1} + \dots + 1| \leq m.$$

This contradiction completes the proof. \square

COROLLARY 3.1. *For $0 \leq \lambda < 1/m$, all roots of $F_{\lambda,m}(z)$ are unimodular.*

Case 3: $\lambda = 1/m$.

The equation $F_{1/m,m}(z) = 0$ is written as

$$(3.13) \quad \frac{1}{m}F_{1/m,m}(z) = (m-1)(z^{m+1} - 1) - (m+1)z(z^{m-1} - 1) = 0.$$

The next result came as a pleasant surprise.

THEOREM 3.1. *The value $z = 1$ is a triple root of $F_{1/m,m}(z) = 0$. The quotient polynomial $F_{1/m,m}(z)/(z-1)^3$ is given by the logconcave polynomial $A(x)$ defined in (1.1); that is,*

$$F_m(z) = (z-1)^3 A(x) = (z-1)^3 \sum_{r=1}^m r(m-r)z^r.$$

PROOF. It has already been established that $z = 1$ is a root of any $F_{\lambda,m}(z)$. A direct computation shows that $F'_{1/m,m}(1) = 0$, $F''_{1/m,m}(1) = 0$ and $F'''_{1/m,m}(1) = m^2 - 1$. Therefore $z = 1$ is a triple root.

Observe that $F_{1/m,m}(z) = m(m-1)(z^{m+1} - 1) - m(m+1)z(z^{m-1} - 1)$ means

$$(3.14) \quad \frac{F_{1/m,m}(z)}{m(z-1)} = (m-1) \sum_{j=0}^m z^j - (m+1) \sum_{j=0}^{m-2} z^{j+1}.$$

It follows that $F_{1/m,m}(z) = m(z-1) \sum_{j=0}^m \alpha_j z^j$ with

$$\alpha_j = \begin{cases} m-1 & \text{if } j = 0 \text{ or } j = m \\ -2 & \text{if } 1 \leq j \leq m-1. \end{cases}$$

The relation $\alpha_0 + \alpha_1 + \cdots + \alpha_m = 0$ yields

$$\begin{aligned} F_{1/m,m}(z) &= m(z-1) \sum_{j=0}^m \alpha_j (z^j - 1) = m(z-1)^2 \sum_{j=1}^m \alpha_j \sum_{k=0}^{j-1} z^k \\ &= m(z-1)^2 \sum_{k=0}^{m-1} \left(\sum_{j=k}^{m-1} \alpha_{j+1} \right) z^k. \end{aligned}$$

Thus $\sum_{j=k+1}^m \alpha_j = -\sum_{j=0}^k \alpha_j = -(m-1-2k)$ gives $-\frac{F_{1/m,m}(z)}{m(z-1)^2} = \sum_{k=0}^{m-1} (m-1-2k)z^k$.

The value $\sum_{k=0}^{m-1} (m-1-2k) = 0$ shows that

$$F_{1/m,m}(z) = -m(z-1)^2 \sum_{k=0}^{m-1} (m-1-2k)(z^k-1) = -m(z-1)^3 \sum_{r=0}^{m-2} \left(\sum_{k=r}^{m-2} (m-2k-3) \right) z^r.$$

Evaluating the internal sum implies the final result. \square

Case 4: $1/m < \lambda$.

A new technique is introduced for this range of λ , which can be used to furnish alternative proofs of the results for $\lambda \leq 1/m$ presented earlier.

THEOREM 3.2. *The polynomial $H_{\lambda,m}(z)$ has at least $m-2$ unimodular roots.*

Recall $H_{\lambda,m}(z) = F_{\lambda,m}(z)/(z-1)$. Then

$$(3.15) \quad H_{\lambda,m}(z) = (1-\lambda)z^m - 2\lambda(z + \cdots + z^{m-1}) + (1-\lambda).$$

This time, the roots are grouped according to the parity of m . Details are given when m is even, the case m odd is left to the reader.

Assume m is even, say $m = 2n$. Denote $u^{(k)} = z^k + 1/z^k$. Then

$$(3.16) \quad z^{-n} H_{\lambda,2n}(z) = -2\lambda + (1-\lambda)u^{(n)} - 2\lambda \sum_{k=1}^{n-1} u^{(k)}.$$

The key observation is the content of the next statement.

LEMMA 3.1. *A complex number $z = e^{i\theta}$ is a root of $H_{\lambda,2n}(z) = 0$ if and only if the equation*

$$(3.17) \quad (1+\lambda) \sin\left(\frac{2n-1}{2}\theta\right) = (1-\lambda) \sin\left(\frac{2n+1}{2}\theta\right)$$

has a real solution θ .

PROOF. Let $u = z + 1/z$ and replace it in (3.16) to obtain

$$(3.18) \quad \Phi(\theta) := \frac{1}{2} - \frac{1-\lambda}{2\lambda} \cos n\theta + \sum_{k=1}^{n-1} \cos k\theta = 0.$$

A use of the identity $\frac{1}{2} + \sum_{k=1}^{n-1} \cos k\theta = \frac{\sin((2n-1)\theta/2)}{2\sin\theta/2}$ completes the proof. \square

The equation (3.17) may be expressed as

$$(3.19) \quad (1 + \lambda)U_{2n-2}(w) = (1 - \lambda)U_{2n}(w)$$

where $w = \cos\theta/2$ and U_n is the Chebyshev polynomial of the second kind. The interlacing of the roots of the polynomials $\{U_n\}_n$ is now employed to conclude that $H_{\lambda,m}(z) = 0$ has at least $m - 2$ roots on the unit circle. Indeed, the zeros of the left-hand side of (3.17) are given by $\left\{0, \frac{2\pi}{2n-1}, \frac{4\pi}{2n-1}, \dots, \frac{2(2n-1)\pi}{2n-1}\right\}$ and those of the right-hand side of (3.17) are $\left\{0, \frac{2\pi}{2n+1}, \frac{4\pi}{2n+1}, \dots, \frac{2(2n+1)\pi}{2n+1}\right\}$. From these explicit values, and the interlacing of zeros of $\{U_n(w)\}_n$, it becomes certain that there are at least $2n - 2$ crossings, which automatically renders as many roots of modulus 1 for $H_{\lambda,m}(z) = 0$. Observe that neither $\theta = 0$ nor $\theta = 2\pi$ contribute to the roots of $H_{\lambda,m}$, unless $\lambda = 1/m$.

The next result involves a head count of real roots. Note that for $\lambda \neq 1$, the polynomial $F_{\lambda,m}$ is of degree $m + 1$, so the number of real roots is at most 3, with $z = 1$ always present.

LEMMA 3.2. *Assume $1/m < \lambda < 1$. Aside from the roots at $z = 1$, the polynomial $F_{\lambda,m}(z)$ has exactly two other positive real zeros. It follows that $F_{\lambda,m}(z)$ has three real roots and $m - 2$ unimodular roots.*

PROOF. The first step is to verify that $F_{\lambda,m}(z)$ has at least three real zeros. This follows directly from the following facts: $F_{\lambda,m}(0) = \lambda - 1 < 0$, the slope at $z = 1$ is $F'_{\lambda,m}(0) = -2m(\lambda - 1/m) < 0$ and $F_{\lambda,m}(z) \sim (1 - \lambda)z^{m+1}$ as $z \rightarrow \infty$.

Descartes' rule of signs [2] states that the number of positive roots of a polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by a multiple of 2. At present, the sequence of coefficients is $1 - \lambda, -(1 + \lambda), 1 + \lambda, -(1 - \lambda)$. In particular, for $\lambda < 1$ there are three sign changes. Thus, the number of positive roots is either 3 or 1. It follows that there are exactly three positive roots, as claimed. \square

LEMMA 3.3. *Assume $\lambda > 1$ is real and m is an odd integer. Then if $1 < \lambda < m$, the polynomial $F_{\lambda,m}(z)$ has $z = -1$ as a root and two other negative real roots. The remaining $m - 2$ roots are unimodular. For $\lambda \geq m$, the value $z = -1$ is a triple root and the remaining $m - 2$ roots are unimodular.*

PROOF. The result follows directly from the identity $\lambda F_{1/\lambda,m}(-z) = -F_{\lambda,m}(z)$. \square

LEMMA 3.4. *Assume $\lambda > 1$ is real and m is an even integer. Then $F_{\lambda,m}(z)$ has two distinct negative roots and $z = 1$ as the only positive root. The remaining $m - 2$ roots are unimodular.*

PROOF. The expression

$$(3.20) \quad F_{\lambda,2n}(-z) = (\lambda - 1)z^{2n+1} - (\lambda + 1)z^{2n} - (\lambda + 1)z + (\lambda - 1)$$

and the data $F_{\lambda,2n}(0) = \lambda - 1 > 0$, $F_{\lambda,2n}(-1) = -4$ and $F_{\lambda,2n}(z) \sim (\lambda - 1)z^{2n+1}$ leads to the assertion. \square

LEMMA 3.5. *Assume $1/m < \lambda < 1$ is real and m is an even integer. Then $F_{\lambda,m}(z)$ has two distinct positive roots aside from $z = 1$. The remaining $m - 2$ roots are unimodular.*

PROOF. Observe that

$$(3.21) \quad G(z) = \frac{F_{\lambda,2n}(z)}{z - 1} = (1 - \lambda)z^{2n} - 2\lambda(z + z^2 + \cdots + z^{2n-1}) + (1 - \lambda).$$

Then $G(0) = 1 - \lambda > 0$, $G(1) = 2(1 - m\lambda) < 0$ and $G(z) \sim (1 - \lambda)z^{2n}$ yield the statement. \square

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