

The integrals in Gradshteyn and Ryzhik. Part 24: Polylogarithm functions

Kim McInturff ^a and Victor H. Moll ^b

ABSTRACT. The table of Gradshteyn and Ryzhik contains some integrals that can be evaluated using the polylogarithm function. A small selection of examples is discussed.

1. Introduction

The table of integrals [2] contains many entries that are expressible in terms of the *polylogarithm function*

$$(1.1) \quad \operatorname{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

In this paper we describe the evaluation of some of them. The series (1.1) converges for $|z| < 1$ and $\operatorname{Re} s > 1$. The integral representation

$$(1.2) \quad \operatorname{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} dx}{e^x - z}$$

provides an analytic extension to \mathbb{C} . Here $\Gamma(s)$ is the classical *gamma function* defined by

$$(1.3) \quad \Gamma(s) := \int_0^{\infty} x^{s-1} e^{-x} dx.$$

The polylogarithm function is a generalization of the *Riemann zeta function*

$$(1.4) \quad \zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s} = \operatorname{Li}_s(1).$$

A second special value is given by

$$(1.5) \quad \operatorname{Li}_s(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = -(1 - 2^{1-s})\zeta(s),$$

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the last equality being obtained by splitting the sum according to the parity of the summation index.

The first result is an identity between an integral and a series coming from the evaluation of the polylogarithm at two values on the unit circle. Many of the entries presented here are special cases. This is a classical result, the proof is presented here in order to keep the paper as self-contained as possible.

THEOREM 1.1. *Let $\nu \in \mathbb{C}$ with $\operatorname{Re} \nu > 0$ and $0 < t < \pi$. Then*

$$(1.6) \quad \int_0^\infty \frac{x^{\nu-1} dx}{\cosh x - \cos t} = \frac{2\Gamma(\nu)}{\sin t} \sum_{k=1}^\infty \frac{\sin kt}{k^\nu}.$$

PROOF. The integral representation (1.2) gives

$$\begin{aligned} i [\operatorname{Li}_s(e^{-it}) - \operatorname{Li}_s(e^{it})] &= \frac{i}{\Gamma(s)} \int_0^\infty x^{s-1} \left[\frac{1}{e^{x+it} - 1} - \frac{1}{e^{x-it} - 1} \right] dx \\ &= \frac{\sin t}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} dx}{\cosh x - \cos t}. \end{aligned}$$

The series representation (1.1) gives

$$\begin{aligned} i [\operatorname{Li}_s(e^{-it}) - \operatorname{Li}_s(e^{it})] &= 2 \sum_{k=1}^\infty \frac{e^{ikt} - e^{-ikt}}{2ik^s} \\ &= 2 \sum_{k=1}^\infty \frac{\sin kt}{k^s}. \end{aligned}$$

This proves the result. □

COROLLARY 1.1. *Let $\nu \in \mathbb{C}$ with $\operatorname{Re} \nu > 0$ and $0 < t < \pi$. Then*

$$(1.7) \quad \int_0^\infty \frac{x^{\nu-1} dx}{\cosh x + \cos t} = \frac{2\Gamma(\nu)}{\sin t} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin kt}{k^\nu}.$$

PROOF. Replace t by $\pi - t$ in the statement of Theorem 1.1. □

This corollary appears as entry **3.531.7** in [2].

REMARK 1.1. In the special case $\nu = 2$, the series in the corollary appears in the expansion of the Lobachevsky function

$$(1.8) \quad L(t) := - \int_0^t \ln \cos s ds = t \ln 2 - \frac{1}{2} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin 2kt}{k^2}, \quad 0 < t < \frac{\pi}{2}.$$

This special case of the corollary can be stated as

$$(1.9) \quad \int_0^\infty \frac{x dx}{\cosh x + \cos 2t} = \frac{4(t \ln 2 - L(t))}{\sin 2t}, \quad 0 < t < \frac{\pi}{2}.$$

This is entry **3.531.2** of [2]. Observe that this is written as

$$(1.10) \quad \int_0^\infty \frac{x dx}{\cosh 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t}, \quad 0 < t < \frac{\pi}{2}.$$

The fact that this is the only entry in Section 3.531 with $\cosh 2x$ instead of $\cosh x$ can lead to confusion.

2. Some examples from the table by Gradshteyn and Ryzhik

This section presents the evaluation of some entries from the table [2] by making specific choices for the parameters ν and t in Theorem 1.1 and Corollary 1.1. Naturally a closed-form for the integral is obtained in those cases for which the series can be evaluated.

EXAMPLE 2.1. Take $\nu = 2$ and $t = \pi/3$. Theorem 1.1 gives

$$\begin{aligned} \int_0^\infty \frac{x dx}{\cosh x - \frac{1}{2}} &= \frac{2\Gamma(2)}{\sin \pi/3} \sum_{k=1}^\infty \frac{\sin(k\pi/3)}{k^2} \\ &= \frac{4}{\sqrt{3}} \sum_{k=1}^\infty \frac{\sin(k\pi/3)}{k^2}. \end{aligned}$$

The function $\sin(\pi k/3)$ is periodic, with period 6, and repeating values $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0$. Therefore

$$\sum_{k=1}^\infty \frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} = \frac{\sqrt{3}}{2} \left(\sum_{k=0}^\infty \frac{1}{(6k+1)^2} + \sum_{k=0}^\infty \frac{1}{(6k+2)^2} - \sum_{k=0}^\infty \frac{1}{(6k+4)^2} - \sum_{k=0}^\infty \frac{1}{(6k+5)^2} \right).$$

To evaluate this series, recall the series representation of the *digamma* function $\psi(x) = \Gamma'(x)/\Gamma(x)$, given by

$$(2.1) \quad \psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{x}{k(x+k)}, \quad \text{for } x > 0.$$

Differentiation yields

$$(2.2) \quad \psi'(x) = \sum_{k=0}^\infty \frac{1}{(x+k)^2}, \quad \text{for } x > 0,$$

and we obtain

$$\sum_{k=0}^\infty \frac{1}{(6k+j)^2} = \frac{1}{36} \sum_{k=0}^\infty \frac{1}{\left(k + \frac{j}{6}\right)^2} = \frac{1}{36} \psi' \left(\frac{j}{6} \right).$$

This provides the expression

$$(2.3) \quad \sum_{k=1}^\infty \frac{\sin\left(\frac{\pi k}{3}\right)}{k^2} = \frac{\sqrt{3}}{72} \left(\psi' \left(\frac{1}{6} \right) + \psi' \left(\frac{2}{6} \right) - \psi' \left(\frac{4}{6} \right) - \psi' \left(\frac{5}{6} \right) \right).$$

The identities

$$(2.4) \quad \psi(1-x) = \psi(x) + \pi \cot \pi x, \quad \text{for } 0 < x < 1,$$

and

$$(2.5) \quad \psi(2x) = \frac{1}{2} \left(\psi(x) + \psi\left(x + \frac{1}{2}\right) \right) + \ln 2,$$

produce

$$\psi' \left(\frac{1}{6} \right) = 5\psi' \left(\frac{1}{3} \right) - \frac{4\pi^2}{3}, \quad \psi' \left(\frac{2}{3} \right) = -\psi' \left(\frac{1}{3} \right) + \frac{4\pi^2}{3}, \quad \psi' \left(\frac{5}{6} \right) = -5\psi' \left(\frac{1}{3} \right) + \frac{16\pi^2}{3}.$$

It follows that

$$(2.6) \quad \int_0^\infty \frac{x dx}{\cosh x - \frac{1}{2}} = \frac{2}{3}\psi' \left(\frac{1}{3} \right) - \frac{4\pi^2}{9}.$$

This example appears as entry **3.531.1**. The value stated there is given in terms of the Lobachevsky function using (1.9):

$$(2.7) \quad \int_0^\infty \frac{x dx}{\cosh x - \frac{1}{2}} = \frac{8}{\sqrt{3}} \left(\frac{\pi}{3} \ln 2 - L \left(\frac{\pi}{3} \right) \right).$$

Comparing these two evaluations gives

$$(2.8) \quad L \left(\frac{\pi}{3} \right) = -\frac{1}{4\sqrt{3}}\psi' \left(\frac{1}{3} \right) + \frac{\pi^2}{6\sqrt{3}} + \frac{\pi}{3} \ln 2.$$

This example also appears as entry **3.418.1** in the form

$$(2.9) \quad \int_0^\infty \frac{x dx}{e^x + e^{-x} - 1} = \frac{1}{3} \left[\psi' \left(\frac{1}{3} \right) - \frac{2\pi^2}{3} \right].$$

EXAMPLE 2.2. Entry **3.514.1** in [2] is

$$(2.10) \quad \int_0^\infty \frac{dx}{\cosh ax + \cos t} = \frac{t}{a \sin t}, \quad \text{for } 0 < t < \pi, a > 0.$$

The case of arbitrary $a > 0$ is equivalent to the special case $a = 1$. This follows from the change of variables $ax \mapsto x$. The integral

$$(2.11) \quad \int_0^\infty \frac{dx}{\cosh x + \cos t} = \frac{t}{\sin t}, \quad \text{for } 0 < t < \pi,$$

is now evaluated by elementary methods.

The next sequence of identities gives the result:

$$\begin{aligned} \int_0^\infty \frac{dx}{\cosh x + \cos t} &= 2 \int_0^\infty \frac{e^x dx}{e^{2x} + 2e^x \cos t + 1} \\ &= 2 \int_1^\infty \frac{dr}{r^2 + 2r \cos t + 1} \\ &= 2 \int_{1+\cos t}^\infty \frac{du}{u^2 + \sin^2 t} \\ &= \frac{2}{\sin t} \int_{\cot(t/2)}^\infty \frac{dv}{v^2 + 1} \\ &= \frac{t}{\sin t}. \end{aligned}$$

EXAMPLE 2.3. The exponential generating function for the Bernoulli polynomials $B_n(x)$ is $te^{xt}/(e^t - 1)$, so for real x and t with $0 < |t| < 2\pi$,

$$(2.12) \quad \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

For $n = 2m + 1$ an odd integer, these polynomials have a Fourier sine series given by

$$(2.13) \quad \frac{2^{2m}\pi^{2m+1}(-1)^m}{(2m+1)!} B_{2m+1}\left(\frac{t}{2\pi} + \frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2m+1}} \sin kt, \text{ for } |t| < \pi.$$

For example, $n = 3$ gives

$$(2.14) \quad \frac{t(\pi^2 - t^2)}{12} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^3}, \text{ for } |t| < \pi,$$

and $n = 5$ gives

$$(2.15) \quad \frac{t(\pi^2 - t^2)(7\pi^2 - 3t^2)}{720} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^5}, \text{ for } |t| < \pi.$$

These representations and Corollary 1.1 give the evaluations

$$(2.16) \quad \int_0^{\infty} \frac{x^2 dx}{\cosh x + \cos t} = \frac{t(\pi^2 - t^2)}{3 \sin t}, \text{ for } 0 < t < \pi,$$

and

$$(2.17) \quad \int_0^{\infty} \frac{x^4 dx}{\cosh x + \cos t} = \frac{t(\pi^2 - t^2)(7\pi^2 - 3t^2)}{15 \sin t}, \text{ for } 0 < t < \pi.$$

These integrals appear as entries **3.531.3** and **3.531.4**, respectively. The Fourier sine series

$$(2.18) \quad \frac{t}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k}, \text{ for } |t| < \pi,$$

shows that the evaluation given in Example 2.2 is also part of this family.

EXAMPLE 2.4. The limiting case $t \rightarrow 0$ in Corollary 1.1 gives, for $\nu \neq 2$, the evaluation

$$(2.19) \quad \int_0^{\infty} \frac{x^{\nu-1} dx}{\cosh x + 1} = 2(1 - 2^{2-\nu})\Gamma(\nu)\zeta(\nu - 1).$$

The proof uses the elementary limit $\sin kt/\sin t \rightarrow k$ as $t \rightarrow 0$ and (1.5). The identity (2.19) is part of entry **3.531.6**. An alternative direct proof is presented next.

The integral representation

$$(2.20) \quad \int_0^{\infty} \frac{x^{s-1} dx}{e^{px} + 1} = \frac{(1 - 2^{1-s})}{p^s} \Gamma(s)\zeta(s)$$

appears as entry **9.513.1** in [2] and it is established in [3] and in [1].

Differentiating with respect to p gives

$$(2.21) \quad \int_0^{\infty} \frac{x^s e^{px} dx}{(e^{px} + 1)^2} = \frac{s(1 - 2^{1-s})}{p^{1+s}} \Gamma(s)\zeta(s)$$

and $p = 1/2$ produces

$$(2.22) \quad \int_0^\infty \frac{x^s dx}{e^{x/2} + e^{-x/2} + 2} = 2^{s+1}(1 - 2^{1-s})\Gamma(s+1)\zeta(s).$$

The change of variables $u = x/2$ and $\nu = s + 1$ give the result.

The limiting case $\nu \rightarrow 2$

$$(2.23) \quad \int_0^\infty \frac{x dx}{\cosh x + 1} = 2 \ln 2$$

that is also part of entry **3.531.6**, appears from the limiting behavior

$$(2.24) \quad \zeta(s) = \frac{1}{s-1} + F(s)$$

where $F(s)$ is an entire function.

EXAMPLE 2.5. Let $t = 2\pi a$ in Theorem 1.1 and take $\nu = 2m + 1$ with $m \in \mathbb{N}$ to obtain

$$(2.25) \quad \int_0^\infty \frac{x^{2m} dx}{\cosh x - \cos 2\pi a} = \frac{2(2m)!}{\sin 2\pi a} \sum_{k=1}^\infty \frac{\sin 2\pi ka}{k^{2m+1}}.$$

This is entry **3.531.5** in [2]. The hypotheses of the theorem restrict a to $0 < a < 1/2$, but the symmetry about $a = 1/2$ implies that (2.25) also holds for $1/2 < a < 1$.

In the special case $a = \frac{1}{2}$, replacing $\sin 2\pi ka / \sin 2\pi a$ by its limiting value, produces

$$(2.26) \quad \int_0^\infty \frac{x^{2m} dx}{\cosh x + 1} = 2(1 - 2^{1-2m})(2m)!\zeta(2m),$$

in agreement with (2.19). For positive integer m , the relation

$$(2.27) \quad \zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!}$$

expresses the integral in (2.26) in terms of the Bernoulli numbers B_{2m} as

$$(2.28) \quad \int_0^\infty \frac{x^{2m} dx}{\cosh x + 1} = 2(2^{2m-1} - 1)\pi^{2m}|B_{2m}|.$$

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^a5433 THAMES CT, SANTA BARBARA,
CA 93111
U.S.A
E-mail address: mcint@cox.net

^bDEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY,
NEW ORLEANS, LA 70118
U.S.A
E-mail address: vhm@math.tulane.edu