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Odd behavior of a four Bessel integral

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Abstract. The integral $\int_0^\infty x^\mu J_\nu(ax)J_\nu(bx)K_\nu(ax)K_\nu(bx)\,\mathrm{d}x$ is evaluated and shown to be independent of ν for $\mu = 3$.

1. Introduction

Definite integrals containing the product of four Bessel and related functions have been of interest for some time [**1, 2, 3, 4, 5**] and a number are tabulated in standard references [**6, 7**]. They are readily reduced to generalized hypergeometric functions of various sorts and symbolic algorithms, such as those incorporated in Mathematica and Maple can evaluate them. However, none of those the author has encountered in the literature has the feature discussed below so it seems reasonable to add this one more.

2. Calculation

Consider, for $a > 0$,

$$f(s,\nu,a) = \int_0^\infty x^{3-4s} J_{2\nu}(x)K_{2\nu}(x)J_{2\nu}(ax)K_{2\nu}(ax)\,\mathrm{d}x. \tag{2.1}$$

By using any of the methods employed in the first five references (the one used here was that of [**5**]: expressing the integrand of (1) as the product of two Meijer G-functions of the type

$$G_{04}^{30}\left(\frac{x^4}{64}|\frac{1}{2}\nu, \frac{1}{2}, 0, -\frac{1}{2}\nu\right) = \sqrt{2\pi}J_\nu(x)K_\nu(x) \tag{2.2}$$

(Meijer's function is a generalized hypergeometric function defined by a contour integral, rather than a series[**6**]) and invoking the last formula in [**6**]) one has

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(2.3)

$$2^{4s}f(s,\nu,a) = \frac{\Gamma(2-2s)\Gamma(\nu-s+1)}{2\nu\Gamma(\nu+s)}\,{}_4F_3\left(\left.\begin{matrix}1-s,\frac{3}{2}-s,1-\nu-s,1+\nu-s\\ \frac{1}{2},1-\nu,1+\nu\end{matrix}\right| a^4\right)$$

$$+a^{4\nu}\frac{\Gamma(-2\nu)\Gamma(2\nu-2s+2)\Gamma(2\nu-s+1)}{\Gamma(2\nu+1)\Gamma(s)}\,{}_4F_3\left(\left.\begin{matrix}1-s,1+\nu-s,\frac{3}{2}+\nu-s,1+2\nu-s\\ \frac{1}{2}+\nu,1+\nu,1+2\nu\end{matrix}\right| a^4\right)$$

$$-2a^2\frac{\Gamma(3-2s)\Gamma(\frac{3}{2}+\nu-s)}{(4\nu^2-1)\Gamma(\nu+s-\frac{1}{2})}\,{}_4F_3\left(\left.\begin{matrix}\frac{3}{2}-s,2-s,\frac{3}{2}-\nu-s,\frac{3}{2}+\nu-s\\ \frac{3}{2},\frac{3}{2}-\nu,\frac{3}{2}+\nu\end{matrix}\right| a^4\right).$$

In the limit $s \to 0$, (2.3) reduces to

$$f(0,\nu,a) = \frac{1}{2}\left\{{}_2F_1\left(\left.\begin{matrix}1,\frac{3}{2}\\ \frac{1}{2}\end{matrix}\right| a^4\right) - 2a^2\,{}_1F_0\left(\left.\begin{matrix}2\\ \end{matrix}\right| a^4\right)\right\} = \frac{1}{2(a^2+1)^2} \tag{2.4}$$

and is independent of the index ν. For the author this was unexpected and, if one compares the graphs of the integrand for $\nu = 0$ and $\nu = 1$, say, is quite mysterious.

This phenomenonis not unknown, for

$$\int_0^\infty xJ_\nu(ax)K_\nu(ax)\,\mathrm{d}x = \frac{1}{2a^2} \tag{2.5}$$

independently of $\nu > -1$, $a > 0$. (A key feature in (2.3) is the factor $\Gamma(s)$ in the denominator, which eliminates the second term, and the fortuitous coincidence of upper and lower parameters in the other two terms ; the case (4) is more elementary.) A natural conjecture to make is that

$$\int_0^\infty x^s J_\nu(x)K_\nu(x)J_\nu(ax)K_\nu(ax)J_\nu(bx)K_\nu(bx)\,\mathrm{d}x \tag{2.6}$$

is independent of ν for $s = 5$ and this fails to check out, nor for a number of reasonable integer guesses for s. Thus, (2.4) and (2.5) appear to be isolated cases. In passing, we note that in this investigation it was found that

$${}_4F_3\left(\left.\begin{matrix}2,\nu+1/2,\nu+2,2\nu+2\\ \nu+1,\nu+5/2,2\nu+1\end{matrix}\right| -1\right) = \frac{2\nu+3}{8(\nu+1)} \tag{2.7}$$

an apparently previously unknown hypergeometric summation.

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