SCIENTIA
Series A: *Mathematical Sciences*, Vol. 23 (2012), 87–98
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The Forcing Edge Monophonic Number of a Graph

J. John[a] and P Arul Paul Sudhahar[b]

ABSTRACT. For a connected graph $G = (V, E)$, let a set M be a minimum edge monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum edge monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing edge monophonic number of M , denoted by $f_{m1}(M)$ is the cardinality of a minimum forcing subset of M. The forcing edge monophonic number of G, denoted by $f_{m1}(G)$ is $f_{m1}(G) = min\,\{f_{m1}(M)\}$, where the minimum is taken over all minimum edge monophonic sets M in G. Some general properties satisfied by this concept are studied.The forcing edge monophonic number of certain classes of graphs are determined. It is known that $m(G) \leqslant m_1(G)$, where $m(G)$ and $m_1(G)$ respectively the monophonic number and the edge monophonic number of a connected graph G. However, there is no relation between $f_m(G)$ and $f_{m1}(G)$, where $f_m(G)$ is the forcing monophonic number of a connected graph G. We give realization results for various possibilities of these four parameters.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1,4]. A chord of a path $u_o, u_1, u_2, ..., u_n$ is an edge $u_i u_j$ with $j \geqslant i + 2$. An $u - v$ path is called a monophonic path if it is a chordless path. A monophonic set of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is the minimum monophonic set of G. The monophonic number of a graph is introduced in [2] and further studied in [5, 8]. An edge monophonic set of G is a set $M \subseteq V(G)$ such that every edge of G is

2000 *Mathematics Subject Classification.* Primary 05C038.
Key words and phrases. Monophonic path, Monophonic number, Edge monophonic number, Forcing monophonic number, Forcing edge monophonic number.

contained in a monophonic path joining some pair of vertices in M. The edge monophonic number $m_1(G)$ of G is the minimum order of its edge monophonic sets and any edge monophonic set of order $m_1(G)$ is a minimum edge monophonic set of G. The edge monophonic number of a graph is introduced in [7]. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic number of M, denoted by $f_m(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic number of G, denoted by $f_m(G)$, is $f_m(G) = min\{f_m(M)\}$, where the minimum is taken over all minimum monophonic sets M in G. The forcing geodetic number, the forcing monophonic number and the forcing Steiner number where studied in [3, 6, 9]. A vertex v of G is said to be a monophonic vertex of G if v belongs to every minimum monophonic set of G. A vertex v is an extreme vertex of a graph if the subgraph induced by its neighbors is complete. The following theorems are used in the sequel.

Theorem 1.1. [**5, 7**] Each extreme vertex of G belongs to every monophonic set of G as well as every edge monophonic set of G.

Theorem 1.2. [**7**] For any non trivial tree T, the edge monophonic number equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique minimum edge monophonic set of T.

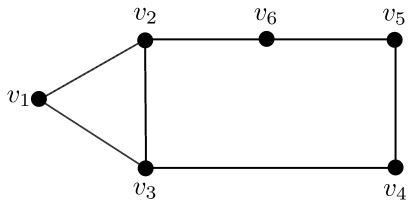
Theorem 1.3. [**6**] Let G be a connected graph and W be the set of all monophonic vertices of G. Then $f_m(G) \leqslant m(G) - |W|$

2. Forcing Edge Monophonic number of a graph

Even though every connected graph contains a minimum edge monophonic set, some connected graph may contain several minimum edge monophonic sets. For each minimum edge monophonic set M in a connected graph G, there is always some subset T of M that uniquely determine M as the minimum edge monophonic set containing T. Such "forcing subsets" will be considered in this paper.

Definition 2.1. Let G be a connected graph and M a minimum edge monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum edge monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing edge monophonic number of M, denoted by $f_{m1}(M)$, is the cardinality of a minimum forcing subset of M. The forcing edge monophonic number of G, denoted by $f_{m1}(G)$, is $f_{m1}(G) = min\{f_{m1}(M)\}$, where the minimum is taken over all minimum edge monophonic sets M in G.

Example 2.1. For the graph G given in Figure 2.1, $M_1 = \{v_1, v_2, v_4\}, M_2 = \{v_1, v_2, v_5\}, M_3 = \{v_1, v_3, v_6\}, M_4 = \{v_1, v_3, v_5\}$ are the only four minimum edge monophonic sets of G.It is clear that $f_{m1}(M_1) = 1, f_{m1}(M_2) = 2, f_{m1}(M_3) = 1, f_{m1}(M_4) = 2$ so that $f_{m1}(G) = 1$.



G

Figure 2.1

The next theorem follows immediately from the definition of the edge monophonic number and the forcing edge monophonic number of a connected graph G.

THEOREM 2.1. For every connected graph $G, 0 \leqslant f_{m1}(G) \leqslant m_1(G)$.

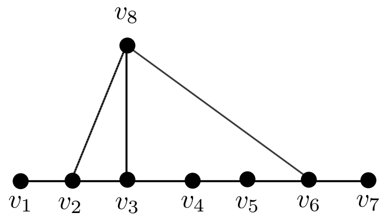
REMARK 2.1. The bounds in Theorem 2.1 are sharp. For the graph $G = K_p$, the vertex set V is the unique minimum edge monophonic set of G so that $f_{m1}(G) = 0$. For the graph G given in Figure 2.1, $m_1(G) = 3$ and $f_{m1}(G) = 1$. Thus $0 < f_{m1}(G) < m_1(G)$. Also for the graph $G = C_4, m_1(G) = 2$ and $f_{m1}(G) = 2$ so that $f_{m1}(G) = m_1(G)$.

In the following we characterize graphs G for which bounds in the Theorem 2.1 attained and also graph for which $f_{m1}(G) = 1$.

THEOREM 2.2. Let G be a connected graph. Then
(a) $f_{m1}(G) = 0$ if and only if G has a unique minimum edge monophonic set.
(b) $f_{m1}(G) = 1$ if and only if G has at least two minimum edge monophonic sets, one of which is a unique minimum edge monophonic set containing one of its elements, and
(c) $f_{m1}(G) = m_1(G)$ if and only if no minimum edge monophonic set of G is the unique minimum edge monophonic set containing any of its proper subsets.

DEFINITION 2.2. A vertex v of G is said to be an edge monophonic vertex of G if v belongs to every minimum edge monophonic set of G.

EXAMPLE 2.2. For the graph G given in Figure 2.2, $M_1 = \{v_l, v_3, v_7\}$ and $M_2 = \{v_l, v_4, v_7\}$ are the only two m_1-sets of G. It is clear that v_1 and v_7 are edge monophonic vertices of G.



G

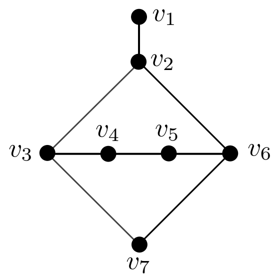
Figure 2.2

THEOREM 2.3. Let G be a connected graph and W be the set of all edge monophonic vertices of G. Then $f_{m1}(G) \leqslant m_1(G) - |W|$

COROLLARY 2.1. If G is a connected graph with k extreme vertices, then $f_{m1}(G) \leqslant m_1(G) - k$.

Proof. This follows from Theorem 1.1 and Theorem 2.3. ∎

REMARK 2.2. The bound in Theorem 2.3 is sharp. For the graph G given in Figure 2.2, $M_1 = \{v_1, v_3, v_7\}, M_2 = \{v_1, v_4, v_7\}$ are the only two m_1-sets so that $m_1(G) = 3$ and $f_{m1}(G) = 1$. Also $W = \{v_1, v_2\}$ is the set of all edge monophonic vertices of G and so $f_{m1}(G) = m_1(G) - |W|$. Also the inequality in Theorem 2.3 can be strict. For the graph G given in Figure 2.3, $M_1 = \{v_l, v_3, v_6\}, M_2 = \{v_1, v_4, v_7\}, M_3 = \{v_1, v_5, v_7\}, M_4 = \{v_1, v_4, v_6\}$ are the only four m_1-sets of G so that $m_1(G) = 3$ and $f_{m1}(G) = 1$. Now v_1 is the only edge monophonic vertex of G and so $f_{m1}(G) < m_1(G) - |W|$



G

Figure 2.3

THEOREM 2.4. For a cycle $G = C_p (p \geqslant 4), M = \{u, v\}$ is a minimum edge monophonic set if and only if u and v are independent.

Proof. Let u and v be two independent vertices of G. It follows that $M = \{u, v\}$ is a minimum edge monophonic set of G. Now, let $M = \{u, v\}$ be a minimum edge monophonic set of G. Suppose that u and v are not independent. Then uv is a chord. Therefore $M = \{u, v\}$ is not an edge monophonic set of G, which is a contradiction. ∎

THEOREM 2.5. For a cycle $G = C_p (p \geqslant 5), f_{m1}(G) = 2$.

THEOREM 2.6. For a complete graph $G = K_p (p \geqslant 2)$ or a non-trivial tree $G = T, f_{m1}(G) = 0$.

Proof. For $G = K_p$, it follows from Theorem 1.1 that the set of all vertices of G is the unique minimum edge monophonic set. Now, it follows from Theorem 2.2(a) that $f_{m1}(G) = 0$. If G is a non-trivial tree, then by Theorem 1.2 the set of all end vertices of G is the unique minimum edge monophonic set of G and so $f_{m1}(G) = 0$ by Theorem 2.2(a). ■

THEOREM 2.7. For the complete bipartite graph $G = K_{m,n} (m, n \geqslant 2)$,

$$f_{m_1}(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

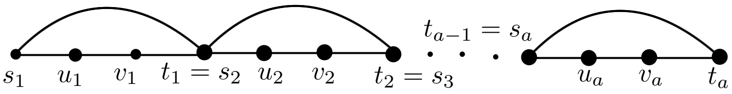
Proof. Let $m, n \geqslant 2$. Without loss of generality, let $m < n$.
Let $U = \{u_1, u_2, ..., u_m\}$ and $W = \{w_1, w_2, ..., w_n\}$ be a bipartition of G. Let $M = U$. We first prove that M is a minimum edge monophonic set of G. Any edge $u_i w_j (1 \leqslant i \leqslant m, i \leqslant j \leqslant n)$ lies on the monophonic path $u_i w_j u_k$ for $k \neq i$ so that M is an edge monophonic set of G. Let T be any set of vertices such that $|T| < |M|$. If $T \subset U$, then there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then for any edges $u_i w_j (1 \leqslant j \leqslant n)$, the only monophonic path containing $u_i w_j$ are $u_i w_j u_k (k \neq i)$ and $w_j u_i w_l (l \neq j)$ and so $u_i w_j$ cannot lie in a monophonic path joining two vertices of T. Thus T is not an edge monophonic set of G. If $T \subset W$, again T is not an edge monophonic set of G by a similar argument. If $T \subseteq U \cup W$ such that T contains at least one vertex from each of U and W, then since $|T| < |M|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge $u_i w_j$ does not lie on a monophonic path connecting two vertices of T so that T is not an edge monophonic set. Thus in any case T is not an edge monophonic set of G. Hence M is a minimum edge monophonic set so that $m_1(G) = |M| = m$. Now, let M_1 be a set of vertices such that $|M_1| = m$. If M_1 is a subset of W, then since $m < n$, there exists a vertex $w_j \in W$ such that $w_j \in M_1$. Then the edge $u_i w_j (1 \leqslant i \leqslant m)$ does not lie on a monophonic path joining a pair of vertices of M_1. If $M_1 \subseteq U \cup W$ such that M_1 contains at least one vertex from each of U and W, then since $M_1 \neq U$, there exists vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin M_1$ and $w_j \notin M_1$ .Then clearly the edge $u_i w_j$ does not lie on a monophonic path joining two vertices of M_1 so that M_1 is not an edge monophonic set of G. It follows that U is the unique minimum edge monophonic set of G. Hence it follows from Theorem 2.2(a) that $f_{m1}(G) = 0$. Now, let $m = n$. Then as in the first part of this theorem, both U and W are minimum edge monophonic sets of G. Now, let M' be any set of vertices such that $|M'| = m$ and $M' \neq U, M' \neq W$. Then there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin M'$ and $w_j \notin M'$. Then as earlier, $M'$is not an edge monophonic set of G. Hence it follows that U and W are

the only two minimum edge monophonic sets of G. Since U is the unique minimum edge monophonic set containing $\{u_i\}$, it follows that $f_{m1}(G) = 1$. ∎

3. Special Graphs

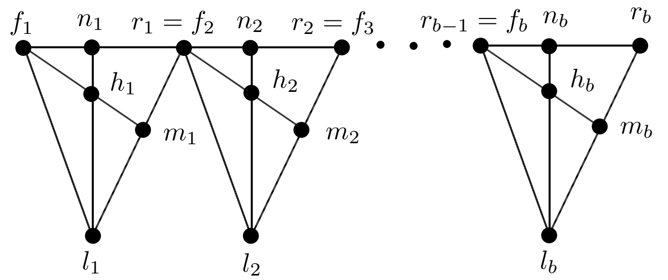
In this section, we present some graphs from which various graphs arising in theorem are generated using identification.

The graph H_a is obtained from the F_i's by identifying the vertices t_{i-1} of F_{i-1} and s_i of $F_i (2 \leqslant i \leqslant a)$, where $F_i : s_i, u_i, v_i, t_i, s_i (1 \leqslant i \leqslant a)$ is a copy of the cycle C_4.



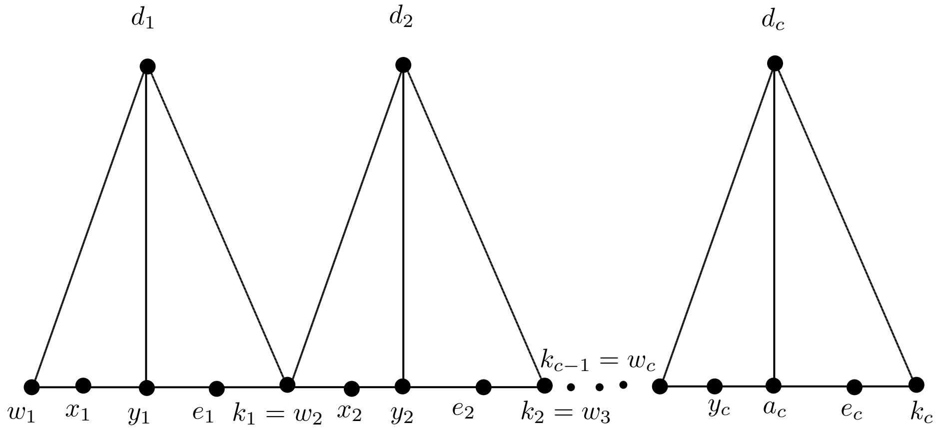
H_a
Figure 3.1

Let $J_i : f_i, l_i, m_i, r_i, n_i, f_i (1 \leqslant i \leqslant b)$ be a copy of the cycle C_5. Let E_i be the graph obtained from J_i by adding a new vertex h_i and the edges $f_ih_i, h_in_i, h_im_i, h_il_i (1 \leqslant i \leqslant b)$. The graph T_b is obtained from E_i's by identifying the vertices r_{i-1} of E_{i-1} and f_i of $E_i (2 \leqslant i \leqslant b)$.



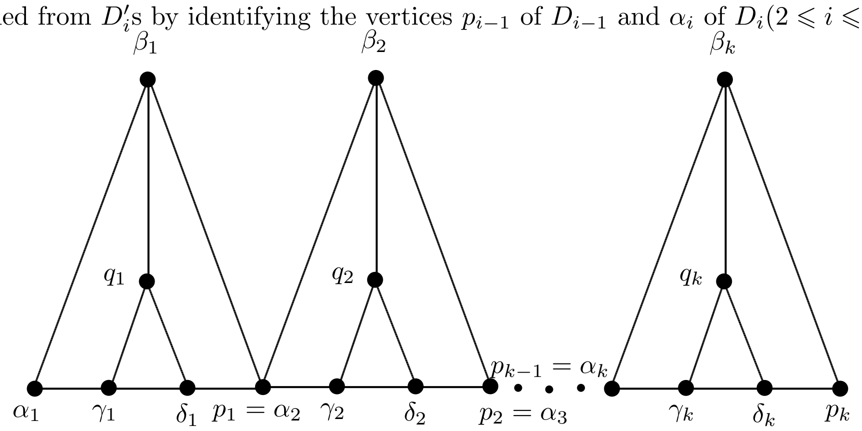
T_b
Figure 3.2

Let $L_i : w_i, x_i, y_i, e_i, k_i, d_i, w_i (1 \leqslant i \leqslant c)$ be a copy of the cycle C_6. Let S_i be the graph obtained from L_i by adding the new edge $d_iy_i (1 \leqslant i \leqslant c)$. The graph L_c is obtained from S_i's by identifying the vertices k_{i-1} of S_{i-1} and w_i of $S_i (2 \leqslant i \leqslant c)$.



L_c
Figure 3.3

Let $Q_i : \alpha_i, \gamma_i, \delta_i, p_i, \beta_i, \alpha_i$ be a copy of C_5. Let D_i be a graph obtained from Q_i by adding a new vertex q_i and the edges $\beta_i q_i, q_i\delta_i, q_i\gamma_i (1 \leqslant i \leqslant k)$. The graph Q_k is obtained from D_i's by identifying the vertices p_{i-1} of D_{i-1} and α_i of $D_i (2 \leqslant i \leqslant k)$.



Q_k
Figure 3.4

4. Some realization results

THEOREM 4.1. For every pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 2$, there exists a connected graph G such that $f_{m1}(G) = f_m(G) = 0$, $m_1(G) = a$ and $m(G) = b$.

Proof. If $a = b$, let $G = K_a$. Then by Theorem 2.6,$f_{m1}(G) = f_m(G) = 0$ and $m_1(G) = m(G) = a$. For $a < b$, let G be the graph obtained from T_{b-a} by adding new vertices $x, z_1, z_2, ..., z_{a-1}$ and joining the edges $xf_1, r_{b-a}z_1, r_{b-a}z_2, ..., r_{b-a}z_{a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{a-1}\}$ be the set of end-vertices of G. Then it is clear that Z is the unique monophonic set of G and so that $m(G) = a$ and $f_m(G) = 0$.We see that Z is not a edge monophonic set of G. Now it is easily seen that $W = Z \cup \{l_1, l_2, ..., l_{b-a}\}$ is the unique m_1-set of G so that $m_1(G) = b$ and $f_{m1}(G) = 0$. ∎

THEOREM 4.2. . For every integers a, b and c with $0 \leqslant a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_m(G) = 0, f_{m1}(G) = a, m(G) = b$ and $m_1(G) = c$.

Proof. Case 1. $a = 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $a \geqslant 1$. Let G be the graph obtained by identifying the vertex k_a of L_a and f_1 of T_{c-b-a} and then adding new vertices $x, z_1, z_2, ..., z_{b-1}$ and joining the edges $w_1, r_{c-b-a}z_1, r_{c-b-a}z_2, ..., r_{c-b-a}z_{b-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-1}\}$be the set of end vertices of G.Then it is clear that Z is a unique monophonic set G so that $m(G) = b$ and $f_m(G) = 0$. Next we show that $m_1(G) = c$. Let M be any edge monophonic set of G. Then by Theorem 1.1, $Z \subseteq M$. It is clear that Z is not an edge monophonic set of G. For $1 \leqslant i \leqslant a$, let $M_i = \{x_i, y_i, e_i\}$. We observe that every m_1-set of G must contain at least one vertex from each M_i and each $l_j (1 \leqslant j \leqslant c - b - a)$ so that $m_1(G) \geqslant b+a+c-b-a = c$. Now, $W = Z \cup \{l_1, l_2, ..., l_{c-b-a}\} \cup \{e_1, e_2, ..., e_a\}$ is an edge monophonic set of G so that $m_1(G) \leqslant b + c - a - b + a = c$. Thus $m_1(G) = c$.Next we show that $f_{m1}(G) = a$. Since every m_1-set contains $Z \cup \{l_1, l_2, ..., l_{c-b-a}\}$, it follows from Theorem 2.3 that $f_{m1}(G) \leqslant m_1(G) - (b + c - b - a) = c - c + a = a$. Now, since $m_1(G) = c$ and every m_1-set contains $Z \cup \{l_1, l_2, ..., l_{c-b-a}\}$, it is easily seen that every m_1-set M is of the form $Z \cup \{l_1, l_2, ..., l_{c-b-a}\} \cup \{p_1, p_2, ..., p_a\}$,where $p_i \in M_i (1 \leqslant i \leqslant a)$. Let T be any proper subset of M with $|T| < a$. Then there exists $p_j (1 \leqslant j \leqslant a)$ such that $p_j \in T$. Let e_j be the vertex of M_j distinct from p_j. Then $W = (M - \{p_j\}) \cup \{e_j\}$ is a m_1-set properly containing T. Thus M is not the unique m_1-set containing T so that T is not a forcing subset of M. This is true for all m_1-sets containing G so that $f_{m1}(G) = a$ ∎

THEOREM 4.3. For every integers a, b and c with $0 \leqslant a < b \leqslant c$ and $b > a + 1$, there exists a connected graph G such that $f_{m1}(G) = 0, f_m(G) = a, m(G) = b$ and $m_1(G) = c$.

Proof. We consider two cases.
Case 1. $a = 0$. Then the graph G constructed in Theorem 4.2 satisfies the requirement of this theorem.

Case 2. $a \geqslant 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from Q_a by adding new vertices $x, z_1, z_2, ..., z_{b-a-1}$ and joining the edges $xw_1, k_az_1, k_az_2, ..., k_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-a-1}\}$. It is clear that Z is not a monophonic set of G. For

$1 \leqslant i \leqslant a$, let $N_i = \{q_i, \gamma_i, \delta_i\}$. We observe that every m-set of G must contain at least one vertex from each N_i so that $m(G) \geqslant b-a+a = b$. Now $W = Z \cup \{q_1, q_2, q_3, ..., q_a\}$ is a monophonic set of G so that $m(G) \leqslant b - a + a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every m-set contains M, it follows from Theorem 1.3 that $f_m(G) \leqslant m(G) - |Z| = b - (b - a) = a$. Now, since $m(G) = b$ and every m-set contains Z, it is easily seen that every m-set M is of the form $Z \cup \{d_1, d_2, d_3, ..., d_a\}$, where $d_i \in N_i (1 \leqslant i \leqslant a)$.Let T be any proper subset of $M$with $|T| < a$ .Then there exists $d_j (1 \leqslant j \leqslant a)$ such that $d_j \in T$ .Let e_j be the vertex of N_j distinct from d_j. Then $W = (M - \{dj\}) \cup \{e_j\}$ is a m-set properly containing T. Thus M is not the unique m-set containing T so that T is not a forcing subset of M. This is true for all m-set of G so that $f_m(G) = a$. Next, we show that $m_1(G) = b$. Let M be any monophonic set of G. Then by Theorem 1.1, $Z \subseteq M$. It is clear that Z is not an edge monophonic set of G. Now $W_1 = Z \cup \{q_1, q_2, ..., q_a\}$ is the unique edge monophonic set of G so that $m_1(G) = b$. It is clear that $f_{m1}(G) = 0$.

Subcase 2b. $b < c$. Let G be the graph obtained by identifying the vertex k_a of Q_a and f_1 of T_{c-b} and adding new vertices $x, z_1, z_2, ..., z_{b-a-1}$ and joining the edges $xw_1, r_{c-b}z_1, r_{c-b}z_2, ..., r_{c-b}z_{b-a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-a-1}\}$ .It is clear that Z is not a monophonic set of G. For $1 \leqslant i \leqslant a$, let $N_i = \{q_i, \gamma_i, \delta_i\}$. We observe that every m-set of G must contain at least one vertex from each N_i so that $m(G) \geqslant a + b - a = b$.Now,$W = Z \cup \{q_1, q_2, ..., q_a\}$ is a monophonic set of G so that $m(G) \leqslant b-a+a = b$. Thus $m(G) = b$. Next we show that $f_m(G) = a$. Since every m-set contains Z, it follows from Theorem 1.3 that $f_m(G) \leqslant m(G)-|Z| = b-(b-a) = a$. Now, since $m(G) = b$ and every m-set contains Z, it is easily seen that every m-set M is of the form $Z \cup \{d_1, d_2, d_3, ..., d_a\}$, where $d_i (1 \leqslant i \leqslant a)$ such that $d_j \in T$. Let e_j be the vertex of N_j distinct from d_j. Then $W = (M - \{dj\}) \cup \{e_j\}$ is a m-set properly containing T. Thus M is not the unique m-set containing T so that T is not a forcing subset of M. This is true for all m-set of G so that $f_m(G) = a$. Next, we show that $m_1(G) = c$. $W = Z \cup \{q_1, q_2, ..., q_a\} \cup \{l_1, l_2, ..., l_{c-b}\}$is the unique m_1 set of G so that $m_1(G) = c$ and $f_{m1}(G) = 0$.

■

Theorem 4.4. For every integers a, b and c with $0 \leqslant a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_{m1}(G) = f_m(G) = a$, $m(G) = b$ and $m_1(G) = c$.

Proof. We consider two cases.
Case 1. $a = 0$, Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $a \geqslant 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from H_a by adding new vertices $x, z_1, z_2, ..., z_{b-a-1}$ and joining the edges $xs_1, t_az_1, t_az_2, ..., t_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-a-1}\}$ .It is clear that Z is not a monophonic set of G. For $1 \leqslant i \leqslant a$,let $N_i = \{u_i, v_i\}$. We observe that every m-set of G must contain at least one vertex from each N_i so that $m(G) \geqslant a + b - a = b$. Now, $W = Z \cup \{u_1, u_2, u_3, ..., u_a\}$ is a monophonic set of G so that $m(G) \leqslant b - a + a = b$. Thus $m(G) = b$. Next, we show that $f_m(G) = a$. Since every m-set contains Z it follows from Theorem 1.3 that $f_m(G) \leqslant m(G) - |Z| = b - (b - a) = a$.Now since $m(G) = b$ and every m-set contains Z, it is easily seen that every m-set M is of the form $Z \cup \{d_1, d_2, d_3, ..., d_a\}$, where $d_i \in N_i (1 \leqslant i \leqslant a)$. Let T be any proper subset of M with $|T| < a$. Then there exists $d_j (1 \leqslant j \leqslant a)$ such that $d_j \in T$. Let e_j be the vertex of N_j distinct from d_j. Then $W = (M - dj) \cup e_j$ is a m-set properly containing T. Thus M is not the unique m-set containing T sot that T is not a forcing subset of M. This is true for all m-sets of G so that $f_m(G) = a$. Similarly we can prove that $m_1(G) = c$ and $f_{m1}(a) = a$.

Subcase 2b. $b < c$. Let G be the graph obtained by identifying the vertices t_a of H_a and f_1 of T_{b-a} and adding the new vertices $x, z_1, z_2, ..., z_{b-a-1}$ and joining the edges $xs_1, r_{c-b}z_1, r_{c-b}z_2, ..., r_{c-b}z_{b-a-1}$. Let $Z = \{x, z_1, z_2, ..., z_{b-a-1}\}$. Then it is clear that Z is not an edge monophonic set. For $1 \leqslant i \leqslant a$, let $N_i = \{u_i, v_i\}$. We observe that every m_1-set of G must contain at least one vertex from each N_i and each $l_j (1 \leqslant j \leqslant c-b)$ so that $m_1(G) \geqslant b-a+a+c-b = c$. Now, $W = Z \cup \{l_1, l_2, ..., l_{c-b}\} \cup \{u_1, u_2, ..., u_a\}$ is an edge monophonic set of G so that $m_1(G) \leqslant b-a+a+c-b = c$. Thus $m_1(G) = c$. Next, we show that $f_{m1}(G) = a$. Since every m_1-set containing $Z \cup \{l_1, l_2, ..., l_{c-b}\}$, it follows from Theorem 2.3 that $f_{m1}(G) \leqslant m_1(G) - (b - a + c - b) = c + a - c = a$. Now, since $m_1(G) = c$ and every m_1-set contains Z, it is easily seen that every m_1-set M is of the form Z$\cup \{l_1, l_2, ..., l_{c-b}\} \cup \{d_1, d_2, ..., d_a\}$ where $d_i \in N_i (1 \leqslant i \leqslant a)$. Let T be any proper subset of $M$with $|T| < a$. Then there exists $d_j (1 \leqslant j \leqslant a)$ such that $dj \in T$. Let e_j be the vertex of N distinct from d_j. Then $W = (M - \{d_j\}) \cup \{e_j\}$ is a m_1-set properly containing T. Thus M is not the unique m_1-set containing T so that T is not a forcing subset of M. This is true for all m_1-sets of G so that $f_{m1}(G) = a$. Next, we show that $m(G) = b$ and $f_m(G) = a$. This follows from Subcase 2a ■

THEOREM 4.5. For every integers a, b, c and d with $0 \leqslant c \leqslant d$,$a \leqslant b \leqslant d$ and $c > a+1$ there exists a connected graph G such that $f_{m1}(G) = a, f_m(G) = b, m(G) = c$ and $m_1(G) = d$.

Proof. We consider four cases.
Case 1. $a = 0, b \geqslant 0$. Then the graph G constructed in Theorem 4.4 satisfies the requirement of this theorem.

Case 2. $a \geqslant 0, b = 0$. Then the graph G constructed in Theorem 4.2 satisfies the requirement of this theorem.

Case 3. $0 \leqslant a = b$. Then the graph G constructed in Theorem 4.3 satisfies the requirement of this theorem.

Case 4. $1 \leqslant a < b$.

Subcase 4a. $c = d$. Let G be the graph obtained by identifying the vertices t_a of H_a and α_1 of Q_{b-a} and adding the new vertices $x, z_1, z_2, ..., z_{c-b-1}$ and joining the edges $xs_1, p_{b-a}z_1, p_{b-a}z_2, ..., p_{b-a}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, ..., z_{c-b-1}\}$.Then it is clear that Z is not an edge monophonic set of G. For $1 \leqslant i \leqslant a$, let $N_i = \{u_i, v_i\}$. We observe that every m_1-set of G must contain at least one vertex from each N_i and $q_j (1 \leqslant j \leqslant b-a)$ so that $m_1(G) \geqslant b-a+a+c-b = c$. Next, let $W = Z \cup \{u_1, u_2, ..., u_a\} \cup \{q_1, q_2, ..., q_{b-a}\}$ is an edge monophonic set of G so that $m_1(G) \leqslant b-a+a+c-b = c$. Thus $m_1(G) = c$. Next, we show that $f_{m1}(G) = a$. Since every m_1-set contains $Z \cup \{q_1, q_2, ..., q_{b-a}\}$, it follows from Theorem 2.3 that $f_{m1}(G) \leqslant m_1(G) - (b-a+c-b) = c+a-c = a$. Now, since $m_1(G) = c$ and every m_1-set contains Z, it is easily seen that every m_1-set M is of the form $Z \cup \{d_1, d_2, ..., d_a\} \cup \{q_1, q_2, ..., q_{b-a}\}$ where $d_i \in N_i (1 \leqslant i \leqslant a)$. Let T be any proper subset of $M$with $|T| < a$. Then there exists $d_j \in N_j (1 \leqslant j \leqslant a)$ such that $d_j \notin T$. Let e_j be the vertex of N distinct from d_j. Then $W = (M - \{d_j\}) \cup \{e_j\}$ is a m_1-set G so that $f_{m1}(G) = a$.Similarly we can prove that $m(G) = c$ and $f_m(G) = b$.

Subcase 4b. $c < d$. Let R be the graph obtained by identifying the vertex t_a of H_a and α_1 of Q_{b-a}. Let G be the graph obtained by identifying the vertices k_{b-a-1} of R and f_1 of T_{d-c} and adding new vertices $x, z_1, z_2, ..., z_{c-b-1}$ and joining the edges $xs_1, r_{d-c}z_1, r_{d-c}z_2, ..., r_{d-c}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, ..., z_{c-b-1}\}$ .Then m_1-set M is of the form $M = Z \cup \{c_1, c_2, c_3, ..., c_a\} \cup \{q_1, q_2, q_3, ..., q_{b-a}\} \cup \{l_1, l_2, l_3, ..., l_{d-c}\}$, where each $c_i \in N_i (1 \leqslant i \leqslant a)$ so that $m_1(G) = d$ and $f_{m1}(G) = a$.The m-set is of the form $M = Z \cup \{c_1, c_2, c_3, ..., c_a\} \cup \{d_1, d_2, d_3, ..., d_{b-a}\}$, where $c_i \in N_i (1 \leqslant i \leqslant a)$ and $d_j \in F_j = \{q_i, \gamma_i, \delta_i\} (1 \leqslant i \leqslant b-a)$ so that $m(G) = c$ and $f_m(G) = d$. ∎

Theorem 4.6. For every integers a, b, c and d with $a \leqslant b \leqslant c \leqslant d$ and $c > b+1$, there exists a connected graph G such that $f_m(G) = a, f_{m1}(G) = b, m(G) = c$ and $m_1(G) = d$.

Proof. Case 1. $a = 0, b \geqslant 0$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $b = 0, a \geqslant 0$. Then the graph G constructed in Theorem 4.2 satisfies the requirement of this theorem.

Case 3. $0 \leqslant a = b$. Then the graph G construed in Theorem 4.3 satisfies the requirement of this theorem.

Case 4. $1 \leqslant a \leqslant b$

Subcase 4a. $c = d$. Then the graph G constructed in Theorem 4.5 satisfies the requirement of this theorem.

Subcase 4b. $c < d$. Let X be the graph obtained by identifying the vertices t_a of H_a and α_1 of Q_{b-a}. Let G be the graph obtained by identifying the vertices p_{b-a} of X and f_1 of T_{d-c} and adding the new vertices $x, z_1, z_2, ..., z_{c-b-1}$ and joining the edges $xw_1, r_{d-c}z_1, r_{d-c}z_2, ..., r_{d-c}z_{c-b-1}$. Let $Z = \{x, z_1, ..., z_{c-b-1}\}$. Then the m_1-set is of the form $M = Z \cup \{c_1, c_2, ..., c_a\} \cup \{q_1, q_2, ..., q_{b-a}\} \cup \{l_1, l_2, ..., l_{d-c}\}$ where$c_i \in N_i = \{u_i, v_i\}$ $(1 \leqslant i \leqslant d - c)$ so that $m_1(G) = d$ and $f_{m1}(G) = b$. The m-set is of the form $Z \cup \{c_1, c_2, ..., c - a\} \cup \{q_1, q_2, ..., q_{b-a}\}$ where $c_i \in N_i = \{u_i, v_i\}$ $(1 \leqslant i \leqslant a)$ so that $m(G) = c$ and $f_m(G) = a$. ∎

References

[1] F. Buckley and F. Harary, Distance in Graphs, *Addition- Wesley, Redwood City, CA*, 1990.

[2] Carmen Hernando, Merce Mora I.M. Pelayo and Carlos Seara, *On monophonic sets in graphs*, (2003) preprint.

[3] G. Chartrand, F. Harary and P. Zhang, *The forcing geodetic number of a graph, Discuss Math. Graph theory*, 19 (1999), 45-58.

[4] F. Harary, Graph theory, *Addison - Wesley*, 1969.

[5] J. John and S. Panchali, *The upper monophonic number of a graph*, International J.Math.Combin.Vol.4(2010) 46 - 52.

[6] J. John and S. Panchali, *The forcing monophonic number of a graph*, International Journal of Mathematical Archive, 3(3)(2012).

[7] J. John and P. Arul Paul Sudhahar, *On the edge monophonic number of a graph*, Filomat (in press).

[8] Mitre C. Dourado, Fabio protti and Jayme L.Szwarefiter, *Algorithmic Aspects of monophonic convexity*, Electronic notes in Discrete Mathematics, 30 (2008) 177-182.

[9] A. P. Santhakumaran and J. John, *The forcing Steiner number of a graph, Discuss. Math. Graph Theory*, 31(2011) 171-181.

Received 27 07 2011 , revised 11 09 2012

[a]Department of Mathematics,
Government College of Engineering, Tirunelveli - 627 007,
India
E-mail address: johnramesh1971@yahoo.co.in

[b]Department of Mathematics, Alagappa Government Arts College,
Karaikudi - 630 003,
India
E-mail address: arulpaulsudhar@gmail.com