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The quasi-Hadamard product of certain subclasses of p-valent functions with negative coefficients

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Abstract. The authors establishes certain results concerning the quasi-Hadamard product of certain analytic and p-valent functions with negative coefficients in the open unit disc.

1. Introduction

Throughout the paper, let the functions of the form

$$f(z) = a_p z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (a_p > 0;\ a_{p+k} \geqslant 0)\,, \tag{1.1}$$

$$f_i(z) = a_{p,i} z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k} \qquad (a_{p,i} > 0;\ a_{p+k,i} \geqslant 0)\,, \tag{1.2}$$

$$g(z) = b_p z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \qquad (b_p > 0;\ b_{p+k} \geqslant 0)\,, \tag{1.3}$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{k=1}^{\infty} b_{p+k,j} z^{p+k} \qquad (b_{p,j} > 0;\ b_{p+k,j} \geqslant 0)\,, \tag{1.4}$$

where $p, i, j \in \mathbb{N} = \{1, 2, \ldots\}$, be analytic and p-valent in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}\,.$$

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Let $\mathcal{S}_p^*(\alpha,\beta,\lambda)$ denote the class of functions f of the form (1.1) and satisfying

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{\alpha\frac{zf'(z)}{f(z)}+p-\lambda(\alpha+1)}\right|<\beta \tag{1.5}$$

for some α $(0\leqslant\alpha\leqslant1)$, β $(0<\beta\leqslant1)$, λ $(0\leqslant\lambda<p)$ and for all $z\in\mathbb{U}$.

Also let $\mathcal{C}_p^*(\alpha,\beta,\lambda)$ denote the class of functions of the form (1.1) such that $\frac{1}{p}zf'\in\mathcal{S}_p^*(\alpha,\beta,\lambda)$.

We note that when $a_p=\alpha=\beta=1$, the classes $\mathcal{S}_p^*(1,1,\lambda)=\mathcal{T}^*(p,\lambda)$ and $\mathcal{C}_p^*(1,1,\lambda)=\mathcal{C}(p,\lambda)$ are studied by Owa [**12**]. When $p=1$ and $\lambda=0$, the classes $\mathcal{S}_1^*(\alpha,\beta,0)=\mathcal{S}_0(\alpha,\beta)$ and $\mathcal{C}_1^*(\alpha,\beta,0)=\mathcal{C}_0(\alpha,\beta)$ are studied by Owa [**11**] and Aouf [**2**].

Also, the class $\mathcal{S}_p(\alpha,\beta,\lambda)$ consists of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}$$

and satisfying the condition (1.5) was studied by Owa and Aouf [**13**].

Using similar arguments as given by Owa [**12**], Aouf et al. [**1**] gave the following analogous results for functions in the classes $\mathcal{S}_p^*(\alpha,\beta,\lambda)$ and $\mathcal{C}_p^*(\alpha,\beta,\lambda)$.

A function f defined by (1.1) belongs to the class $\mathcal{S}_p^*(\alpha,\beta,\lambda)$ if and only if

$$\sum_{k=1}^{\infty}\left[\{k(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\}a_{p+k}\right]\leqslant\beta(1+\alpha)(p-\lambda)a_p \tag{1.6}$$

and f defined by (1.1) belongs to the class $\mathcal{C}_p^*(\alpha,\beta,\lambda)$ if and only if

$$\sum_{k=1}^{\infty}\left[\left(\frac{p+k}{p}\right)\{k(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\}a_{p+k}\right]\leqslant\beta(1+\alpha)(p-\lambda)a_p. \tag{1.7}$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Kumar ([**7**], [**8**] and [**9**]), Aouf et al. [**1**], Hossen [**6**], Darwish [**3**], Sekine [**14**] and Goyal and Goswami [**5**].

Accordingly, the quasi-Hadamard product of two functions f and g is defined by

$$f*g(z)=a_pb_pz^p-\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}. \tag{1.8}$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

Let $\psi_p(z)$ be a fixed function of the form

$$\psi_p(z)=a_pz^p-\sum_{k=1}^{\infty}c_{p+k}z^{p+k}\qquad(a_p>0;\ c_{p+k}\geqslant c_{p+1}>0;\ k\geqslant1). \tag{1.9}$$

For $p=1$, we have the function $\psi_1(z)=\psi(z)$ defined by Frasin and Aouf [**4**].

Using the function defined by (1.9), we now define the following new classes.

DEFINITION 1.1. A function $f \in \mathcal{M}^0_{\psi_p}(c_{p+k},\delta)$ $(c_{p+k} \geqslant c_{p+1} > 0;\ k \geqslant 1)$ if and only if

$$\sum_{k=1}^{\infty} c_{p+k} a_{p+k} \leqslant \delta a_p, \tag{1.10}$$

where $\delta > 0$.

DEFINITION 1.2. A function $f \in \mathcal{N}^0_{\psi_p}(c_{p+k},\delta)$ $(c_{p+k} \geqslant c_{p+1} > 0;\ k \geqslant 1)$ if and only if

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) c_{p+k} a_{p+k} \leqslant \delta a_p, \tag{1.11}$$

where $\delta > 0$.

Also, we introduce the following class of analytic functions which plays an important role in the discussion that follows.

DEFINITION 1.3. A function $f \in \mathcal{B}^l_{\psi_p}(c_{p+k},\delta)$ $(c_{p+k} \geqslant c_{p+1} > 0;\ k \geqslant 1)$ if and only if

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^l c_{p+k} a_{p+k} \leqslant \delta a_p, \tag{1.12}$$

where $\delta > 0$ and l is any fixed nonnegative real number.

It is easy to check that various subclasses of analytic and multivalent functions can be represented as $\mathcal{B}^l_{\psi}(c_{p+k},\delta)$ for suitable choices of p, c_{p+k}, δ and l studied by various authors. For example:

(1) (i) $\mathcal{B}^0_{\psi_p}(\{k(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\},\ \beta(1+\alpha)(p-\lambda)) \equiv \mathcal{S}^*_p(\alpha,\beta,\lambda)$ (Aouf et al. [**1**]);

(ii) $\mathcal{B}^0_{\psi_p}\left(\left(\frac{p+k}{p}\right)\{k(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\},\ \beta(1+\alpha)(p-\lambda)\right) \equiv \mathcal{C}^*_p(\alpha,\beta,\lambda)$ (Aouf et al. [**1**]);

(iii) $\mathcal{B}^l_{\psi_p}(\{k(1+\alpha\beta)+\beta(1+\alpha)(p-\lambda)\},\ \beta(1+\alpha)(p-\lambda)) \equiv \mathcal{S}^*_{p,l}(\alpha,\beta,\lambda)$ (Aouf et al. [**1**]);

(2) (i) $\mathcal{B}^0_{\psi_1}(\{k+\beta(1+\alpha(k+1))\},\ \beta(1+\alpha)) \equiv \mathcal{S}_0(\alpha,\beta)$ (Owa [**11**]);

(ii) $\mathcal{B}^0_{\psi_1}((k+1)\{k+\beta(1+\alpha(k+1))\},\ \beta(1+\alpha)) \equiv \mathcal{C}_0(\alpha,\beta)$ (Owa [**11**]);

(iii) $\mathcal{B}^l_{\psi_1}(\{k+\beta(1+\alpha(k+1))\},\ \beta(1+\alpha)) \equiv \mathcal{S}_l(\alpha,\beta)$ (Aouf [**2**]);

(3) (i) $\mathcal{B}^0_{\psi_1}(k+1-\alpha,\ 1-\alpha) \equiv \mathcal{ST}^*_0(\alpha)$ (Silverman [**15**]);

(ii) $\mathcal{B}^0_{\psi_1}((k+1)(k+1-\alpha),\ 1-\alpha) \equiv \mathcal{C}_0(\alpha)$ (Silverman [**15**]);

(iii) $\mathcal{B}^l_{\psi_1}(k+1-\alpha,\ 1-\alpha) \equiv \mathcal{S}^*_l(\alpha)$ (Kumar [**9**]).

(4) *(i)* $\mathcal{B}^0_{\psi_1}\left(\left\{(1-\beta)(k+1)-\alpha\beta\right\},\ \beta(1-\alpha)\right) \equiv \mathcal{S}^*_0(\alpha,\beta)$ (Owa [**10**]).
(ii) $\mathcal{B}^0_{\psi_1}\left((k+1)\left\{(1-\beta)(k+1)-\alpha\beta\right\},\ \beta(1-\alpha)\right) \equiv \mathcal{C}^*_0(\alpha,\beta)$ (Darwish [**3**]).
(iii) $\mathcal{B}^l_{\psi_1}\left(\left\{(1-\beta)(k+1)-\alpha\beta\right\},\ \beta(1-\alpha)\right) \equiv \mathcal{S}^*_l(\alpha,\beta)$ (Darwish [**3**]).
(5) $\mathcal{B}^l_{\psi_1}(c_{1+k},\delta) \equiv \mathcal{B}^l_{\psi}(c_{1+k},\delta)$ (Frasin and Aouf [**4**]).

Evidently, $\mathcal{B}^1_{\psi_p}(c_{p+k},\delta) \equiv \mathcal{N}^0_{\psi_p}(c_{p+k},\delta)$ and, for $l=0$, $\mathcal{B}^0_{\psi_p}(c_{p+k},\delta)$ is identical to $\mathcal{M}^0_{\psi_p}(c_{p+k},\delta)$. Further, $\mathcal{B}^l_{\psi_p}(c_{p+k},\delta) \subset \mathcal{B}^h_{\psi_p}(c_{p+k},\delta)$ if $l > h \geqslant 0$, the containment being proper. Whence, for any positive integer l, we have the inclusion relation

$$\mathcal{B}^l_{\psi_p}(c_{p+k},\delta) \subset \mathcal{B}^{l-1}_{\psi_p}(c_{p+k},\delta) \subset \cdots \subset \mathcal{B}^2_{\psi_p}(c_{p+k},\delta) \subset \mathcal{N}^0_{\psi_p}(c_{p+k},\delta) \subset \mathcal{M}^0_{\psi_p}(c_{p+k},\delta).$$

We note that for every nonnegative real number l, the class $\mathcal{B}^l_{\psi_p}(c_{p+k},\delta)$ is nonempty as the functions of the form

$$f(z) = a_p z^p - \sum_{k=1}^{\infty} \frac{\delta a_p}{\left(\frac{p+k}{p}\right)^l c_{p+k}} \lambda_{p+k} z^{p+k} \qquad (z \in \mathbb{U}),$$

where $a_p > 0$, $\lambda_{p+k} \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_{p+k} \leqslant 1$, satisfy the inequality (1.12).

In this work, we establish certain results concerning the quasi-Hadamard product of functions belonging to the classes $\mathcal{B}^l_{\psi_p}(c_{p+k},\delta)$, $\mathcal{M}^0_{\psi_p}(c_{p+k},\delta)$ and $\mathcal{N}^0_{\psi_p}(c_{p+k},\delta)$.

2. The main theorem

THEOREM 2.1. *Let the functions f_i defined by* (1.2) *be in the class $\mathcal{N}^0_{\psi_p}(c_{p+k},\delta)$ for every $i = 1,2,\ldots,r$; and let the functions g_j defined by* (1.4) *be in the class $\mathcal{M}^0_{\psi_p}(c_{p+k},\delta)$ for every $j = 1,2,\ldots,s$. If $c_{p+k} \geqslant \left(\frac{p+k}{p}\right)\delta$, then the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_s(z)$ belongs to the class $\mathcal{B}^{2r+s-1}_{\psi_p}(c_{p+k},\delta)$.*

PROOF. We denote the quasi-Hadamard product $f_1 * f_2 * \cdots * f_r * g_1 * g_2 * \cdots * g_s(z)$ by the function $h(z)$, for the sake of convenience.

Clearly,

$$h(z) = \left\{\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s} b_{p,j}\right\} z^p - \sum_{k=1}^{\infty} \left\{\prod_{i=1}^{r} a_{p+k,i} \prod_{j=1}^{s} b_{p+k,j}\right\} z^{p+k}. \tag{2.1}$$

To prove the theorem, we need to show that

$$\sum_{k=1}^{\infty} \left[\left(\frac{p+k}{p}\right)^{2r+s-1} c_{p+k} \left\{\prod_{i=1}^{r} a_{p+k,i} \prod_{j=1}^{s} b_{p+k,j}\right\}\right] \leqslant \delta \left\{\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s} b_{p,j}\right\}. \tag{2.2}$$

Since $f_i \in \mathcal{N}^0_{\psi_p}(c_{p+k},\delta)$, we have

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) c_{p+k} a_{p+k,i} \leqslant \delta a_{p,i} \tag{2.3}$$

for every $i = 1, 2, \ldots, r$. Therefore

$$a_{p+k,i} \leqslant \frac{\delta}{\left(\frac{p+k}{p}\right) c_{p+k}} a_{p,i}$$

for every $i = 1, 2, \ldots, r$. The right side of the above inequality is not greater than $\left(\frac{p+k}{p}\right)^{-2} a_{p,i}$. Hence

$$a_{p+k,i} \leqslant \left(\frac{p+k}{p}\right)^{-2} a_{p,i}, \tag{2.4}$$

for every $i = 1, 2, \ldots, r$. Similarly, for $g_j \in \mathcal{M}^0_{\psi_p}(c_{p+k}, \delta)$, we have

$$\sum_{k=1}^{\infty} c_{p+k} b_{p+k,j} \leqslant \delta b_{p,j} \tag{2.5}$$

for every $j = 1, 2, \ldots, s$. Hence we obtain

$$b_{p+k,j} \leqslant \left(\frac{p+k}{p}\right)^{-1} b_{p,j}, \tag{2.6}$$

for every $j = 1, 2, \ldots, s$.

Using (2.4) for $i = 1, 2, \ldots, r$, (2.6) for $j = 1, 2, \ldots, s-1$ and (2.5) for $j = s$, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} \left[\left(\frac{p+k}{p}\right)^{2r+s-1} c_{p+k} \left\{ \prod_{i=1}^{r} a_{p+k,i} \prod_{j=1}^{s} b_{p+k,j} \right\} \right] \\
\leqslant\ & \sum_{k=1}^{\infty} \left[\left(\frac{p+k}{p}\right)^{2r+s-1} c_{p+k} b_{p+k,s} \left\{ \left(\frac{p+k}{p}\right)^{-2r} \left(\frac{p+k}{p}\right)^{-(s-1)} \right\} \left(\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s-1} b_{p,j} \right) \right] \\
=\ & \left(\sum_{k=1}^{\infty} c_{p+k} b_{p+k,s} \right) \left(\prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s-1} b_{p,j} \right) \\
\leqslant\ & \delta \left\{ \prod_{i=1}^{r} a_{p,i} \prod_{j=1}^{s} b_{p,j} \right\},
\end{aligned}$$

and therefore $h \in \mathcal{B}^{2r+s-1}_{\psi_p}(c_{p+k}, \delta)$, completing the proof of the theorem. $\square$

We note that the required estimate can be also obtained by using (2.4) for $i = 1, 2, \ldots, r-1$, (2.6) for $j = 1, 2, \ldots, s$, and (2.3) for $i = r$.

Now we discuss some applications of Theorem 2.1. Taking into account the quasi-Hadamard product of functions $f_1, f_2, \ldots, f_r$ only, in the proof of Theorem 2.1, and using (2.4) for $i = 1, 2, \ldots, r-1$ and (2.3) for $i = r$ we are led to

Corollary 2.1. *Let the functions f_i defined by* (1.2) *belong to the class* $\mathcal{N}_{\psi_p}^{0}\left(c_{p+k},\delta\right)$ *for every* $i=1,2,\ldots,r$. *Then the quasi-Hadamard product* $f_1 * f_2 * \cdots * f_r(z)$ *belongs to the class* $\mathcal{B}_{\psi_p}^{2r-1}\left(c_{p+k},\delta\right)$.

Next, taking into account the quasi-Hadamard product of the functions $g_1, g_2, \ldots, g_s$ only, in the proof of Theorem 2.1, and using (2.6) for $j=1,2,\ldots,s-1$, and (2.5) for $j=s$, we have

Corollary 2.2. *Let the functions g_j defined by* (1.4) *belong to the class* $\mathcal{M}_{\psi_p}^{0}\left(c_{p+k},\delta\right)$ *for every* $j=1,2,\ldots,s$. *Then the quasi-Hadamard product* $g_1 * g_2 * \cdots * g_s(z)$ *belongs to the class* $\mathcal{B}_{\psi_p}^{s-1}\left(c_{p+k},\delta\right)$.

Remark. **(i)** Taking $c_{p+k}=k\left(1+\alpha\beta\right)+\beta\left(1+\alpha\right)\left(p-\lambda\right)$ and $\delta=\beta\left(1+\alpha\right)\left(p-\lambda\right)$, $(0\leqslant\alpha\leqslant 1,\ 0<\beta\leqslant 1,\ 0\leqslant\lambda<p)$, in the above theorem, we obtain the main result given by Aouf et al. [**1**].

(ii) Taking $p=1$ in the above theorem, we obtain the main result given by Frasin and Aouf [**4**].

(iii) Taking $p=1$ and $c_{1+k}=k+\beta\left(1+\alpha\left(k+1\right)\right)$ and $\delta=\beta\left(1+\alpha\right)$, $(0\leqslant\alpha\leqslant 1,\ 0<\beta\leqslant 1)$, in the above theorem, we obtain the main result given by Aouf [**2**].

(iv) Taking $p=1$ and $c_{1+k}=k+1-\alpha$ and $\delta=1-\alpha$, $(0\leqslant\alpha<1)$, in the above theorem, we obtain the main result given by Kumar [**9**].

(v) Taking $p=1$ and $c_{1+k}=\left(1-\beta\right)\left(k+1\right)-\alpha\beta$ and $\delta=\beta\left(1-\alpha\right)$, $\left(0\leqslant\alpha<1,\ 0<\beta\leqslant\frac{1}{2}\right)$, in the above theorem, we obtain the main result given by Darwish [**3**].

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