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Dynamics of a stochastic predator-prey model with Beddington DeAngelis functional response

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Abstract. This paper is concerned with a stochastic predator-prey model with Beddington DeAngelis functional response. The existence of the global solution and the ultimate boundedness of moments of the solution are proved. Moreover, we also estimate the average in time of the solution.

1. Introduction

In recent years, there have been many attempts to investigate stochastic eco-models. For a systematic review, we refer to L. J. S. Allen [1]. In [3] and [9], authors gave lower and upper growth rate of the solution to the stochastic Lotka-Volterra model

$$x(t) = \operatorname{diag}\big(x_1(t), \cdots, x_n(t)\big)\Big(\big(b + Ax(t)\big)dt + \sigma x(t)dB(t)\Big), \tag{1.1}$$

where $x(t) = \big(x_1(t), \cdots, x_n(t)\big)^T, b = (b_i)_{1\times n}, A = (a_{ij})_{n\times n}, \sigma = (\sigma_{ij})_{n\times n}$. They also estimated the average in time of the moment of the solution. Especially, the asymptotic behavior of the classical predator-prey perturbed by white noise was studied more detail in [4], [5] and [10].

On another direction, the deterministic predator-prey model with Beddington-DeAngelis functional response

$$\begin{cases} dx(t) = x(t)\big(A - Bx(t) - \dfrac{Cy(t)}{\alpha + \beta x(t) + \gamma y(t)}\big)dt \\ dy(t) = y(t)\big(- D + \dfrac{Ex(t)}{\alpha + \beta x(t) + \gamma y(t)}\big)dt \end{cases} \tag{1.2}$$

have received many attention. Here, $x(t)$ and $y(t)$ respectively represent the densities of the prey and the predator at time t, while A is the intrinsic growth rate of the prey, D is the mortality rate of the predator,C is the feeding parameter, E is the conversion efficiency parameter, B and F are the intraspecies interference parameters, β is a

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food weighting factor, that correlates inversely with the prey density at which feeding saturation occurs and α is a normalization coefficient that relates the densities of the predator and prey to the environment in which they interact. All these parameters are positive.

In [2], [6] and [7], a complete classification of the global dynamics of this model was done . However, so far there seems no study on the stochastic predator-prey model with Beddington-DeAngelis functional response. Suppose the functional response is perturbed by white noise, then Equations (1.2) becomes a stochastic differential equations

$$\begin{cases} dx(t) = x(t)\big(A - Bx(t) - \dfrac{Cy(t)}{\alpha + \beta x(t) + \gamma y(t)}\big)dt + \dfrac{\sigma x(t)y(t)}{\alpha + \beta x(t) + \gamma y(t)} dB(t) \\ dy(t) = y(t)\big(- D + \dfrac{Ex(t)}{\alpha + \beta x(t) + \gamma y(t)}\big)dt + \dfrac{\rho x(t)y(t)}{\alpha + \beta x(t) + \gamma y(t)} dB(t), \end{cases} \tag{1.3}$$

where σ and ρ are real constants. The goal of this paper is to prove that Equations (1.3) have the following properties

(1) With probability 1, the solution with the initial value $(x_0, y_0) \in \mathbb{R}^2_+$ will remain in $\mathbb{R}^2_+$.
(2) Every moment of the solution is ultimately bounded.
(3) The average in time of the total population density is lower and upper bounded by positive constants, that is, there exist two positive constants l and L such that,

$$l \leqslant \liminf_{t\to\infty} \frac{1}{t}\int_0^t \big(x(s) + y(s)\big)ds \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t \big(x(s) + y(s)\big)ds \leqslant L, \ a.s.$$

These results reveals the important role that environmental noise plays in population dynamics. In [11], the authors show that the predator or both species will be extinctive in some cases. But, in this paper, the authors proved that a tiny amount of stochastic noise can make both species survive.

The paper is organized as follows. The first section, authors introduce the stochastic predator-prey model with Beddington-DeAngelis functional response. The section 2, we study the non-explosion of the solution to (1.3) and moment estimation. Section 3 give a estimation for a growth rate of the solution and illustrate this result by numerical solution. Moreover, we also estimate the average in time of $x(t) + y(t)$. In section 4, we give some conclusions.

2. The non-explosion of the solution and moment estimation

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets. Let $B(t), t \geqslant 0$ is a scalar Brownian motion defined on this probability space. Denote by $\mathbb{R}^2_+$ the set $\{(x, y) \in \mathbb{R}^2, x > 0, y > 0\}$. Obviously, the coefficients of Equations (1.3) are locally Lipschitz continuous but do not satisfy the linear growth condition. However, we have

THEOREM 2.1. *For any given initial value $(x(0), y(0)) \in \mathbb{R}^2_+$, there is a unique solution $x(t)$ to Equations (1.3) on $t \geqslant 0$ and this solution will remain in R^2_+ almost surely.*

PROOF. Since the coefficients of Equations (1.3) are locally Lipschitz continuous, for any given initial value $(x_0, y_0) \in \mathbb{R}^2_+$, there is a unique local solution x(t) on $t \in [0, \tau_e)$, where τ_e is the explosion time. The solution is global if $\tau_e = \infty$ a.s.
For each $k \in \mathbb{N}$, we define the stopping time $\tau_k = \inf\{t \geqslant 0, x(t) \wedge y(t) < k^{-1}$ or $x(t) \vee y(t) > k\}$ with convention $\inf \emptyset = \infty$. τ_k is increasing, so we put $\tau_\infty = \lim\limits_{k\to\infty} \tau_k$. Denote by $V(x, y)$ the function $x + y - \ln x - \ln y$. Obviously $V(x, y) \geqslant 0$ for all $x > 0, y > 0$. By Ito formula,

$$\begin{aligned} dV(x(t), y(t) =& \Big[\big(x(t)-1\big)\big(A - Bx(t) - \frac{Cy(t)}{\alpha+\beta x(t)+\gamma y(t)}\big) \\ &+ \big(y(t)-1\big)\big(-D + \frac{Ex(t)}{\alpha+\beta x(t)+\gamma y(t)}\big) \\ &+ \frac{\big(\sigma y(t)\big)^2 + \big(\rho x(t)\big)^2}{2\big(\alpha+\beta x(t)+\gamma y(t)\big)^2}\Big]dt \\ &+ \frac{\sigma(x(t)-1)y(t) + \rho(y(t)-1)x(t)}{\alpha+\beta x(t)+\gamma y(t)} dB(t). \end{aligned}$$

It is easy to show that, there exist two constants K_1, K_2 such that $\big(x-1\big)\big(A - \frac{Cy}{\alpha+\beta x+\gamma y}\big) + \big(y-1\big)\big(-D + \frac{Ex}{\alpha+\beta x+\gamma y}\big) \leqslant K_1(1+x)$, and $\frac{\big(\sigma y(t)\big)^2 + \big(\rho x(t)\big)^2}{2\big(\alpha+\beta x(t)+\gamma y(t)\big)^2} \leqslant K_2$ for all $x > 0, y > 0$. Thus, there exists $K > 0$ such that

$$\begin{aligned} &\big(x-1\big)\big(A - Bx - \frac{Cy}{\alpha+\beta x+\gamma y}\big) + \big(y-1\big)\big(-D + \frac{Ex}{\alpha+\beta x+\gamma y}\big) + \\ &\frac{\big(\sigma y(t)\big)^2 + \big(\rho x(t)\big)^2}{2\big(\alpha+\beta x(t)+\gamma y(t)\big)^2} \leqslant K_1(x+1) + K_2 - bx^2 < K \,\forall (x,y) \in \mathbb{R}^2_+. \end{aligned}$$

Consequently, for any $t > 0$, we have
(2.1)

$$\begin{aligned} \mathbb{E}V(x(t\wedge\tau_k), y(t\wedge\tau_k)) =& V(x(0), y(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} \Big[\big(x(s)-1\big)\big(A - Bx(s) - \frac{Cy(s)}{\alpha+\beta x(s)+\gamma y(s)}\big) \\ &+ \big(y(s)-1\big)\big(-D + \frac{Ex(s)}{\alpha+\beta x(s)+\gamma y(s)}\big) + \frac{\big(\sigma y(s)\big)^2 + \big(\rho x(s)\big)^2}{2\big(\alpha+\beta x(s)+\gamma y(s)\big)^2}\Big]ds \\ \leqslant& V(x(0), y(0)) + \mathbb{E}\int_0^{t\wedge\tau_k} K ds = V(x(0), y(0)) + K\mathbb{E}(t\wedge\tau_k). \end{aligned}$$

Suppose $\tau_\infty < \infty$ with a positive probability. It implies the existence of two positive constants ϵ and $T > 0$ such that $\mathbb{P}\{\tau_\infty < T\} > 2\epsilon$. Hence, there is $k_0 \in \mathbb{N}$ such that $\mathbb{P}\{\tau_k < T\} > \epsilon$ for any $k > k_0$.
Put $m_k = (k - \ln k) \wedge (k^{-1} + \ln k)$, then $m_k \to \infty$ as $k \to \infty$ and $V(\tau_k) \geqslant m_k$. By

(2.1), $m_k\epsilon \leqslant m_k\mathbb{P}\{\tau_k < T\} \leqslant \mathbb{E}V(T\wedge\tau_k) \leqslant K\mathbb{E}(T\wedge\tau_k) + V(x(0),y(0)) \leqslant KT + V(x(0),y(0))\,\forall k > k_0$. Let $k\to\infty$ we get a contradiction $\infty \leqslant KT + V(x(0),y(0))$. Hence, $\tau_\infty = \infty$ with probability 1. The proof is complete. □

THEOREM 2.2. *For any $\theta > 0$, there exists $K_\theta > 0$ such that for any given initial value $x(0) > 0, y(0) > 0$, $\limsup_{t\to\infty} \mathbb{E}\big(x^\theta(s) + y^\theta(s)\big) \leqslant K_\theta$.*

PROOF. By Lyapunov's inequality $\mathbb{E}|X^r|^{\frac{1}{r}} \leqslant \mathbb{E}|X^p|^{\frac{1}{p}}\,\forall\, 0 < r < p$, it is sufficient to prove the theorem for $\theta > 2$. Put $V(x,y) = (Ex+Cy+1)^\theta$. In view of Ito formula,

$$\begin{aligned} de^{\theta Dt}V(x(t),y(t)) =& e^{\theta Ds}\Big(\mathcal{L}V(x(t),y(t)) + \theta DV(x(t),y(t))\Big)dt \\ &+ \theta e^{\theta Dt}\big(Ex(t)+Cy(t)+1\big)^{\theta-1}\frac{(E\sigma + C\rho)x(t)y(t)}{\alpha+\beta x(s)+\gamma y(s)}dB(t), \end{aligned} \tag{2.2}$$

where

$$\mathcal{L}V(x,y) = \theta(Ex+Cy+1)^{\theta-1}\Big((EAx - EBx^2 - CDy) + \frac{\theta-1}{2(Ex+Dy+1)}\frac{(E\sigma+C\rho)^2x^2y^2}{(\alpha+\beta x+\gamma y)^2}\Big).$$

Note that $\dfrac{\theta-1}{2(Ex+Cy+1)}\dfrac{(E\sigma+C\rho)^2x^2y^2}{(\alpha+\beta x+\gamma y)^2} \leqslant \dfrac{\theta-1}{2E}\dfrac{(E\sigma+C\rho)^2}{\gamma^2}x\,\forall x > 0, y > 0$. Therefore, $M_1 > 0$ can be found such that for all $x > 0, y > 0$,

$$\begin{aligned} \theta DV(x,y) + \mathcal{L}V(x,y) =& \theta(Ex+Cy+1)^{\theta-1}\Big((E(A+D)x - EBx^2 + D) \\ &+ \frac{\theta-1}{2(Ex+Cy+1)}\frac{(E\sigma+C\rho)^2x^2y^2}{(\alpha+\beta x+\gamma y)^2}\Big) \leqslant M_1. \end{aligned} \tag{2.3}$$

Define the stopping times $t_k = \inf\{t : x(t)\vee y(t) > k\}, k\in\mathbb{N}$. By virtue of (2.3), taking expectations on both sides of (2.2) yields

$$\mathbb{E}e^{\theta D(t\wedge t_k)}V(x(t\wedge t_k), y(t\wedge t_k)) \leqslant V(x_0,y_0) + \frac{M_1}{\theta D}\mathbb{E}\big(e^{\theta D(t\wedge t_k)} - 1\big)$$

Let $k\to\infty$, we get

$$e^{\theta Dt}\mathbb{E}V(x(t),y(t)) \leqslant V(x_0,y_0) + \frac{M_1}{\theta D}(e^{\theta Dt}-1),$$

or

$$\mathbb{E}V(x(t),y(t)) \leqslant V(x_0,y_0)e^{-\theta Dt} + \frac{M_1}{\theta D}(1-e^{-\theta Dt}),$$

which implies

$$\limsup_{t\to\infty}\mathbb{E}V(x(t),y(t)) \leqslant \frac{M_1}{\theta D.}$$

By combining this with the inequality $x^\theta + y^\theta \leqslant (E^{-\theta} + C^{-\theta})V(x,y)$, the proof is complete. □

The following result is a direct corollary of this theorem.

COROLLARY 2.1. *Equations (1.3) is stochastically ultimately bounded in the sense that for any* $\epsilon > 0$, *there is a positive constant* $H = H(e)$ *such that for any initial value* $(x(0), y(0)) \in \mathbb{R}^2_+$, *the solution has the property that* $\limsup_{t\to\infty} \mathbb{P}\{x(t) + y(t) > H\} < \epsilon$.

3. Pathwise estimation

In this section, we always denote by $(x(t), y(t))$ the solution to Equations (1.3) with the initial value $(x(0), y(0)) \in \mathbb{R}^2_+$. The following theorem give a estimation for the growth rate of the solution.

THEOREM 3.1.

$$\limsup_{t\to\infty} \frac{Ex(t) + Cy(t)}{\ln t} \leqslant \frac{(E\sigma + C\rho)^2}{EB\gamma^2}.$$

PROOF. Let $0 < \theta < D$ and put $V(x, y) = Ex + Cy$. In view of Ito formula,

$$\begin{aligned} e^{\theta t}V(x(t), y(t)) =& V(x(0), y(0)) + \int_0^t e^{\theta s}\Big(Ex(s)\big(A + \theta - Bx(s)\big) - C(D - \theta)y(s)\Big)ds \\ &+ \int_0^t e^{\theta s}\frac{(E\sigma + C\rho)x(s)y(s)}{\alpha + \beta x(s) + \gamma y(s)}dB(s). \end{aligned} \tag{3.1}$$

Put $M(t) = \int_0^t e^{\theta s}\frac{(E\sigma + C\rho)x(s)y(s)}{\alpha + \beta x(s) + \gamma y(s)}dB(s)$. It is also known that $M(t)$ is a real-valued continuous local martigale vanishing at $t = 0$ with quadratic form

$$\langle M(t), M(t)\rangle = \int_0^t \frac{e^{2\theta s}\big((E\sigma + C\rho)x(s)y(s)\big)^2}{\big(\alpha + \beta x(s) + \gamma y(s)\big)^2}ds.$$

For each $\lambda > 0$, it follows from the exponential martigale inequality that

$$\mathbb{P}\big\{\sup_{0\leqslant t\leqslant k} M(t) - \lambda e^{-\theta k}\langle M(t), M(t)\rangle > \frac{e^{\theta k}}{\lambda}\ln k\big\} \leqslant k^{-2}.$$

By the Borel-Cantelli lemma, there exists an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega \in \Omega_0$, there exists a $k_0 = k_0(\omega) \in \mathbb{N}$ satisfying

$$M(t) - \lambda e^{-\theta k}\langle M(t), M(t)\rangle < \frac{e^{\theta k}}{\lambda}\ln k\,\forall\, 0 \leqslant t \leqslant k, k \geqslant k_0.$$

On the other hand, for any $0 \leqslant t \leqslant k$,

$$e^{-\theta k}\langle M(t), M(t)\rangle \leqslant \int_0^t \frac{e^{\theta s}\big((E\sigma + C\rho)x(s)y(s)\big)^2}{\big(\alpha + \beta x(s) + \gamma y(s)\big)^2}ds \leqslant \int_0^t \frac{e^{\theta s}(E\sigma + C\rho)^2}{\gamma^2}x^2(s)ds,$$

which results in

$$M(t) \leqslant \int_0^t \frac{e^{\theta s}\lambda(E\sigma + C\rho)^2}{\gamma^2}x^2(s)ds + \frac{e^{\theta k}}{\lambda}\ln k\,\forall\, 0 \leqslant t \leqslant k, k \geqslant k_0. \tag{3.2}$$

It is easy to show that for $\lambda < \dfrac{EB\gamma^2}{(E\sigma + C\rho)^2}$,

$$K_\lambda = \sup_{(x,y)\in\mathbb{R}^2_+} Ex(A+\theta-Bx) - (D-\theta)y + \lambda\frac{(E\sigma+C\rho)^2}{\gamma^2}x^2 < \infty. \tag{3.3}$$

Thus, from (3.1), (3.2) and (3.3) we have

$$e^{\theta t}V(x(t),y(t)) \leqslant V(x_0,y_0) + K_\lambda\int_0^t e^{\theta s}ds + \frac{e^{\theta k}}{\lambda}\ln k\,\forall\, 0\leqslant t\leqslant k, k\geqslant k_0.$$

Consequently, for all $k \geqslant k_0$ and $0 \leqslant t \leqslant k$,

$$V(x(t),y(t)) \leqslant V(x_0,y_0)e^{-\theta t} + \frac{K_\lambda}{\theta}(1-e^{-\theta t}) + \frac{e^{\theta(k-t)}\ln k}{\lambda}.$$

Obviously, if $k \geqslant k_0$ and $k-1 \leqslant t \leqslant k$, the following inequality holds

$$\frac{V(x(t),y(t))}{\ln t} \leqslant \frac{e^{-\theta t}}{\ln(k-1)}\big(V(x_0,y_0) - \frac{K_\lambda}{\theta}\big) + \frac{K_\lambda}{\theta\ln(k-1)} + \frac{e^\theta}{\lambda}\frac{\ln k}{\ln(k-1)}.$$

Let $k \to \infty$, we get $\limsup\limits_{t\to\infty} \dfrac{V(x(t),y(t))}{\ln t} \leqslant \dfrac{e^\theta}{\lambda}$. The required assertion follows from letting $\theta \to 0, \lambda \to \dfrac{EB\gamma^2}{(E\sigma+C\rho)^2}$. □

Example 3.2. We illustrate the above results by the following example. Consider (1.3) with A=1, B=2,C=1, D=1,E=1,F=2, $\alpha = \beta = \gamma = 1$, $\sigma = \rho = 2$.
This numerical solution is displayed in figure 1.

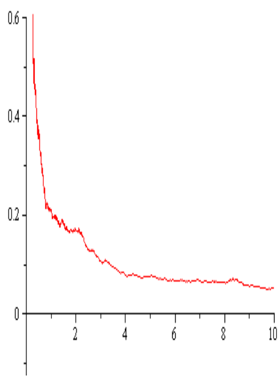

Figure 1. The graph of function $\dfrac{Ex(t)+Cy(t)}{\ln t}$

In the sequel, we estimate the average in time of $x(t) + y(t)$.

Theorem 3.3. *There exists* $L > 0$ *such that* $\limsup \frac{1}{t}\int_0^t \big(x(s)+y(s)\big)ds \leqslant L\, a.s.$

PROOF. Put $V(x,y) = Ex + Cy$. By Ito formula,

(3.4)

$$V(x(t),y(t)) - V(x_0,y_0) = +\int_0^t \big[Ex(s)\big(A - Bx(s)\big) - CDy(s)\big]ds + M(t)$$

$$= -D\int_0^t V(x(s),y(s))ds + \int_0^t \Big(Ex(s)\big(A + D - Bx(s)\big) + \epsilon\frac{\big((E\sigma + C\rho)x(s)y(s)\big)^2}{\big(\alpha + \beta x(s) + \gamma y(s)\big)^2}\Big)ds$$

$$+ M(t) - \epsilon\langle M(t), M(t)\rangle,$$

where $M(t) = \int_0^t \frac{(E\sigma + C\rho)x(s)y(s)}{\alpha + \beta x(s) + \gamma y(s)}dB(t)$ is a real-valued continuous local martigale vanishing at $t = 0$ with quadratic form

$$\langle M(t), M(t)\rangle = \int_0^t \frac{\big((E\sigma + C\rho)x(s)y(s)\big)^2}{\big(\alpha + \beta x(s) + \gamma y(s)\big)^2}ds.$$

It is easy to chose ϵ sufficiently small such that

$$-EBx^2 + \epsilon\frac{((E\sigma + C\rho)xy)^2}{\big(\alpha + \beta x + \gamma y\big)^2} \leqslant -\epsilon x^2 \,\forall (x,y) \in \mathbb{R}_+^2,$$

which implies

$$M_2 = \sup_{(x,y)\in\mathbb{R}_+^2}\Big(Ex(A + D - Bx) + \epsilon\frac{\big((E\sigma + C\rho)xy\big)^2}{(\alpha + \beta x + \gamma y)^2}\Big) < \infty. \tag{3.5}$$

On the other hand, $V(x,y) > 0\,\forall (x,y) \in \mathbb{R}_+^2$. Thus, it follows from (3.4) and (3.5) that

$$D\int_0^t V(x(s),y(s))ds \leqslant V(x_0,y_0) + M_2 t + M(t) - \epsilon\langle M(t), M(t)\rangle. \tag{3.6}$$

Applying the exponential martingale inequality [8, Theorem 7.4, p.44] yields,

$$\mathbb{P}\Big\{\sup_{0\leqslant t\leqslant k}\{M(t) - \epsilon\langle M(t), M(t)\rangle\} > \frac{\ln k}{\epsilon}\Big\} \leqslant \frac{1}{k^2}.$$

By Borel- Cantelli lemma, for almost all ω, there exists a number $k_0 = k_0(\omega)$ such that for all $k > k_0$ and $0 \leqslant t \leqslant k$,

$$M(t) - \epsilon\langle M(t), M(t)\rangle < \frac{\ln k}{\epsilon},$$

which implies that for $k - 1 \leqslant t \leqslant k$

$$\frac{1}{t}\big(M(t) - \epsilon\langle M(t), M(t)\rangle\big) \leqslant \frac{\ln k}{(k-1)\epsilon}.$$

As a result,

$$\limsup_{t\to\infty}\frac{1}{t}\big(M(t) - \epsilon\langle M(t), M(t)\rangle\big) \leqslant \lim_{k\to\infty}\frac{\ln k}{(k-1)\epsilon} = 0. \tag{3.7}$$

Combining (3.6) and (3.7) gets $\limsup\limits_{t\to\infty}\dfrac{1}{t}\int_0^t V(x(s),y(s))ds \leqslant \dfrac{M_2}{D}, a.s.$ Thus

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \big(x(s)+y(s)\big)ds \leqslant L\, a.s,$$

where $L = \dfrac{M_2}{D\min\{E^{-1},C^{-1}\}}$. The proof is complete. ☐

THEOREM 3.4. *There exists $l>0$ such that $\liminf\limits_{t\to\infty}\dfrac{1}{t}\int_0^t \big(x(s)+y(s)\big)ds \geqslant l\, a.s.$*

PROOF. By Ito formula

$$\begin{aligned}\ln x(t)-\ln x(0) &= \int_0^t \big(A - Bx(s) - \frac{Cy(s)}{\alpha+\beta x(s)+\gamma y(s)}\big)ds \\ -\frac{1}{2}\int_0^t \frac{\sigma^2y^2(s)}{\big(\alpha+\beta x(s)+\gamma y(s)\big)^2}ds &+ \int_0^t \frac{\sigma y(s)}{\big(\alpha+\beta x(s)+\gamma y(s)\big)}dB(s)\end{aligned} \tag{3.8}$$

$M(t) = \int_0^t \dfrac{\sigma y(s)}{\big(\alpha+\beta x(s)+\gamma y(s)\big)}dB(s)$ is a continuous martigale vanishing at $t=0$ with quadratic form

$$\langle M(t),M(t)\rangle = \int_0^t \frac{\sigma^2y^2(s)}{\big(\alpha+\beta x(s)+\gamma y(s)\big)^2}ds \leqslant \sigma^2\gamma^{-2}t.$$

By the strong law of large numbers [8, Theorem 1.3.4, p.12],

$$\lim_{t\to\infty}\frac{M(t)}{t} \to 0\, a.s. \tag{3.9}$$

On the other hand, it follows from Theorem (3.1) that

$$\limsup_{t\to\infty}\frac{\ln x(t)-\ln x(0)}{t} \leqslant 0\, a.s. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) yields

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \big(Bx(s)+\frac{Cy(s)}{\alpha+\beta x(s)+\gamma y(s)}+\frac{1}{2}\frac{\sigma^2y^2(s)}{\big(\alpha+\beta x(s)+\gamma y(s)\big)^2}\big)ds \geqslant A.$$

Note that an $H>0$ can be found such that

$$\frac{Cy}{\alpha+\beta x+\gamma y}+\frac{1}{2}\frac{\sigma^2y^2}{\big(\alpha+\beta x+\gamma y\big)^2} \leqslant Hy\,\forall\,(x,y)\in\mathbb{R}^2_+.$$

Thus, $\liminf \dfrac{1}{t}\int_0^t \big(Bx(s)+Hy(s)\big)ds \geqslant A$ with probability 1. This inequality results in the required assertion. ☐

4. Conclusion

Compare with Equations (1.1) which is studied in [3] and [9], Equations (1.3) has more desired properties. It is due to the linear growth rate of the function $\dfrac{x(t)y(t)}{\alpha+\beta x(t)+\gamma y(t)}$. Namely, while Mao et al [9] show that for $\theta < 3$, there exists M_θ such that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}\big[\sum_{i=1}^n x_i^\theta(t)\big] < M_\theta \,\forall x_0\in R_+^n,$$

the solution to Equations (1.3) satisfies $\limsup_{t\to\infty}\mathbb{E}[x^\theta(t)+y^\theta(t)] < K_\theta \,\forall\theta>0$, where K_θ is only dependent on θ. In addition, we show that $\limsup_{t\to\infty}\dfrac{Ex(t)+Cy(t)}{\ln t} \leqslant \dfrac{(E\sigma+C\rho)^2}{EB\gamma^2}$, instead of the estimation $\limsup_{t\to\infty}\dfrac{\ln\sum_{i=1}^n x_i(t)}{\ln t} \leqslant 1$ given by Du and Sam [3]. The estimation $0 < l \leqslant \liminf_{t\to\infty}\dfrac{1}{t}\int_0^t \big(x(s)+y(s)\big)ds \leqslant \limsup_{t\to\infty}\dfrac{1}{t}\int_0^t \big(x(s)+y(s)\big)ds \leqslant L < \infty\, a.s.$ is also an interesting property.

References

[1] L. J. S. Allen, An Introduction To Stochastic Processes With Applications to Biology, Pearson Education Inc., Upper Saddle River, New Jersey 2003.
[2] R.S. Cantrell, C. Cosner, On the dynamics of predatorprey models with the BeddingtonDeAngelis functional response, J. Math. Anal. Appl. 257 (2001) 206-222.
[3] Du, N. H., Sam, V. H., Dynamics of a stochastic Lotka-Volterra model perturbed by white noise, J. Math. Anal. Appl., 324(1)(2006), 82-97.
[4] R. Z. Khasminskii and F. C. Klebaner, Long term behavior of solutions of the Lotka-Volterra system under small random perturbations, Ann. Appl. Probab. 11(3), 952- 963
[5] R. Z. Khasminskii and F. C. Klebaner and R. Liptser, Some results on the LotkaVolterra Model and its small random perturbations, Acta Appl. Math. 78 (2003), 201–206
[6] T.W. Hwang, Global analysis of the predatorprey system with BeddingtonDeAngelis functional response, J. Math. Anal. Appl. 281 (2003), 395- 401.
[7] T.W. Hwang, Uniqueness of limit cycles of the predatorprey system with BeddingtonDeAngelis functional response, J. Math. Anal. Appl. 290 (2004), 113–122.
[8] Mao, X., Stochastic Differential Equations and Applications, Horwood Publishing Chichester (1997).
[9] Mao, X., Sabanis, S., Renshaw, E., Asymptotic behaviour of the stochastic Lotka-Volterra model, J. Math. Anal. Appl, 287(2003), 141-156.
[10] Rudnicki, R., Long-time behaviour of a stochastic prey-predator model, J. Math. Anal. Appl, 108(2003), 93- 107.

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