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On the boundary control of a parabolic system coupling KS-KdV and Heat equations

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Abstract. This paper presents a control problem for a one-dimensional nonlinear parabolic system. The system consists of a Kuramoto-Sivashinsky-Korteweg de Vries equation coupled to a heat equation. The problem of boundary controllability is discussed. The local null-controllability of the system is proven. The proof is based on a Carleman estimates approach to deal with the linearized system around the origin. A local inversion theorem is applied to get the result for the nonlinear system.

1. Introduction

A one-dimensional model for turbulence and wave propagation in reaction-diffusion systems is given by the Kuramoto-Sivashinsky (KS) equation, which reads as

$$u_t + \gamma u_{xxxx} + a u_{xx} + u u_x = 0. \tag{1.1}$$

This equation, where γ and a are coefficients accounting for the long-wave instability and the short-wave dissipation respectively, was derived in various physical contexts [**14, 15, 21**]. By adding a linear third-order term, Benney in [**5**] takes into account dissipative effects in the Kuramoto-Sivashinsky-Korteweg-de Vries (KS-KdV) equation

$$u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = 0. \tag{1.2}$$

Equation (1.2) allows to study various nonlinear dissipative waves. When looking for solitary-pulse solutions of (1.2) on the whole line, one finds out that they are unstable. In order to combine dissipative and dispersive features, and to simultaneously support stable solitary-pulse, a one-dimensional model consisting of a KS-KdV equation, linearly coupled to an extra dissipative equation, was proposed in [**17**]. The model has the form

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x, \\ v_t - \Gamma v_{xx} + c v_x = u_x, \end{cases} \tag{1.3}$$

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where the dissipative parameter (effective diffusion coefficient) $\Gamma > 0$ accounts for stabilization and c is the group-velocity mismatch between the wave modes. Generalizations of this system to the two-dimensional case has been considered. In [**18**], the authors use Zakharov-Kuznetsov equations and in [**19**] Kadomtsev-Petviashvili equations.

In this work we are interested in the study of controllability properties of system (1.3) posed on a bounded interval with boundary control inputs h_1, h_2, h_3, and initial data u_0, v_0. Thus, the system considered is written as

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x, & (x,t) \in (0,1) \times (0,T), \\ v_t - \Gamma v_{xx} + c v_x + v^2 = u_x, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = h_1(t), \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = h_2(t), \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = h_3(t), \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1), \end{cases} \tag{1.4}$$

where we have added a quadratic term into the heat equation. Here, we assume that

$$a, \gamma, \text{ and } \Gamma \text{ are positive constants while } c \text{ may have any sign.} \tag{1.5}$$

We address the problem of steering the solutions of system (1.4) to the rest. More precisely, given $T > 0$ and an appropriate space X, we say that system (1.4) is *null controllable* if for any initial condition $(u_0, v_0) \in X$, there exist boundary controls h_1, h_2, h_3 such that the solution of (1.4) satisfies $u(T,\cdot) = v(T,\cdot) = 0$. We say that the *local null controllability* holds if we can find controls as above whenever $\|(u_0, v_0)\|_X$ is small enough. In this paper we will prove this last property for system (1.4), which couples a heat equation and a KS-like equation.

On one hand, the null controllability for the heat equation is well known from [**16**] by Lebeau and Robbiano. The same result can be proven by using a global Carleman estimate as in [**10**] by Fursikov and Imanuvilov. On the other hand, recently the null controllability of the Kuramoto-Sivashinsky has been proven in [**7**] (see also [**6**] about the linearized KS equation). The approach used is the same as that by Fursikov and Imanuvilov. In this paper we combine the results for these two equations in order to get the following theorem.

THEOREM 1.1. Let $T > 0$. There exists $\delta > 0$ such that for any $(u_0, v_0) \in H^{-2}(0,1) \times H^{-1}(0,1)$ with

$$\|u_0\|_{H^{-2}(0,1)} + \|v_0\|_{H^{-1}(0,1)} < \delta,$$

there exist $h_1, h_2, h_3 \in L^2(0,T)$ such that the corresponding solution

$$(u,v) \in C([0,T], H^{-2}(0,1) \times H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1) \times L^2(0,1))$$

of (1.4) satisfies

$$u(T,\cdot) = v(T,\cdot) = 0.$$

The paper is organized as follows. In Section 2, the well-posedness framework is stated for system (1.4). The proof of Theorem 1.1 is given in Section 3. The linearized system is studied in Section 3.1 by using a Carleman estimates approach. The final result for the nonlinear system is obtained by means of a local inversion theorem in Section 3.2.

REMARK 1.2. As stated in the survey [**4**], the study of the controllability for systems of parabolic equations is rather recent. Let us mention, in the case of internal control of coupled reaction-diffusion equations, the articles [**2, 3, 12, 13**] and the references therein. In those papers, the main tool is a Carleman estimate. In the one-dimensional case, the boundary control of two coupled heat equations has been considered in [**9**]. The authors use the moment method and prove the existence of an appropriate biorthogonal family of L^2-functions.

2. Well-posedness

This section is devoted to the proof of well-posedness results for the equations we are concerned here. We state results for both the linear and nonlinear systems.

2.1. Linear homogeneous system. Next theorem states the existence and uniqueness of solutions for the linear system with homogeneous boundary data.

THEOREM 2.1. Let a, γ, Γ and c as in (1.5), $(u_0, v_0) \in H_0^2(0,1) \times H_0^1(0,1)$ and $f_1, f_2 \in L^2(0,T; L^2(0,1))$. Then, for any $T > 0$ the linear system

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x + f_1, & (x,t) \in (0,1) \times (0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x + f_2, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = 0, \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

has a unique solution (u,v) such that

$$(u,v) \in L^2(0,T; H^4(0,1) \times H^2(0,1)) \cap C([0,T]; H_0^2(0,1) \times H_0^1(0,1)). \tag{2.2}$$

Moreover, there exists $C > 0$ such that

$$\|(u,v)\|_{C(H_0^2 \times H_0^1) \cap L^2(H^4 \times H^2)} \leqslant C \Big\{ \|(f_1, f_2)\|_{L^2(L^2)^2} + \|(u_0, v_0)\|_{H_0^2 \times H_0^1} \Big\} \tag{2.3}$$

for all $(u_0, v_0) \in H_0^2(0,1) \times H_0^1(0,1)$ and $f_1, f_2 \in L^2(0,T; L^2(0,1))$.

REMARK 2.2. We have introduced for $m, s \in \mathbb{Z}$, the notation

$$H^m(H^s) := H^m(0,T; H^s(0,1)), \quad C(H^s) := C([0,T], H^s(0,1)).$$

The proof of Theorem 2.1 will be done by applying the Faedo-Galerkin method with a special basis formed by the solutions of the spectral problem associated with the operator Δ^2. Therefore, the following result will be needed.

LEMMA 2.3. There exists a set of positive real numbers $(\mu_j)_{j \in \mathbb{N}}$ such that the corresponding solutions $(w_j)_{j \in \mathbb{N}}$ of the problem

$$\begin{cases} \Delta^2 w_j(x) = \mu_j w_j(x), & x \in (0,1), \\ w_j(0) = w_j'(0) = w_j(1) = w_j'(1) = 0, \end{cases} \tag{2.4}$$

form a basis in $H^4(0,1) \cap H_0^2(0,1)$, which is orthonormal in $L^2(0,1)$.

We note that, since the operator Δ^2 is simultaneously positive and self-adjoint, the assertions of Lemma 2.3 follow from classical results (see, for instance, [**8**]).

Proof of Theorem 2.1. We split this proof into five steps.

Step 1: Approximate solution. Let $(w_j)_{j\in\mathbb{N}}$ be the basis of $H^4(0,1) \cap H_0^2(0,1)$ given by Lemma 2.3 and $V_N = \langle w_1, w_2, \ldots, w_N\rangle$ the subspace spanned by the N first eigenfunctions w_j. We formulate the approximate problem as follows. Find $u^N, v^N \in C^1((0,T), V_N)$, i.e.,

$$u^N(t) = \sum_{i=1}^N g_i^N(t) w_i, \quad \text{and} \quad v^N(t) = \sum_{i=1}^N h_i^N(t) w_i,$$

where $g_i^N(t)$ and $h_i^N(t)$ are solutions of the system of ordinary differential equations given by

(2.5)
$$\begin{cases} (u_t^N(t), w_j) + \gamma(u_{xxxx}^N(t), w_j) + (u_{xxx}^N(t), w_j) + a(u_{xx}^N(t), w_j) = (v_x^N(t), w_j) + (f_1, w_j) \\ (v_t^N(t), w_j) - \Gamma(v_{xx}^N(t), w_j) + c(v_x^N(t), w_j) = (u_x^N(t), w_j) + (f_2, w_j) \\ u^N(0) = u_0^N, \quad v_0^N(0) = v_0^N, \end{cases}$$

for $j = 1, \ldots, N$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(0,1)$ and $(u_0^N)_{N\in\mathbb{N}}$, $(v_0^N)_{N\in\mathbb{N}}$ are sequences such that $u_0^N \to u_0$ and $v_0^N \to v_0$ as $N \to \infty$, strongly in $H_0^2(0,1)$ and $H_0^1(0,1)$ respectively.

According to Caratheodory's theorem, system (2.5) has a local solution on $[0, t_N)$ and its extension to $[0,T]$ is a consequence of the estimates given below.

Step 2: Energy estimates. We first replace w_j by u^N in the first equation of (2.5) and w_j by v^N in the second one. In order to make the reading easier we omit the indices and simply denote u^N by u and v^N by v.

After integration by parts we can add both equations to obtain

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_0^1 (u^2+v^2)dx + \gamma \int_0^1 |u_{xx}|^2 dx \quad &+ \quad \Gamma \int_0^1 |v_x|^2 dx \\ &= \quad - \quad a\int_0^1 u_{xx} u dx + \int_0^1 f_1 u dx + \int_0^1 f_2 v dx. \end{aligned}$$

Applying Hölder's inequality at the right hand side of the identity above it follows that

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_0^1 (u^2+v^2)dx + \frac{\gamma}{2}\int_0^1 |u_{xx}|^2 dx + \Gamma \int_0^1 |v_x|^2 dx \\ \leqslant \left(\frac{a^2}{2\gamma} + \frac{1}{2}\right)\int_0^1 (u^2+v^2)dx + \frac{1}{2}\int_0^1 (f_1^2+f_2^2)dx. \end{aligned}$$

Then, from Gronwall's inequality and the convergences of u_0^N and v_0^N, we get

$$\begin{aligned}(2.6)\quad \int_0^1 (u^2+v^2)dx + \int_0^T\int_0^1 |u_{xx}|^2 dxdt + \int_0^T\int_0^1 |v_x|^2 dxdt \\ \leqslant C_1\big(\|(u_0,v_0)\|^2_{L^2\times L^2} + \|(f_1,f_2)\|^2_{L^2(L^2)^2}\big),\end{aligned}$$

where $C_1>0$ does not depend on N and $t\in[0,T]$.

Using $w_j = \partial_x^4 \mu_j^{-1} w_j$ in the first equation of (2.5), multiplying by $g_j^N(t)$ and summing from $j=1$ to N, we obtain the identity

$$(2.7)\quad \int_0^1 (u_t + u_{xxx} + au_{xx})u_{xxxx}dx + \gamma\int_0^1 |u_{xxxx}|^2 dx -$$

$$\int_0^1 v_x u_{xxxx}dx = \int_0^1 f_1 u_{xxxx}dx.$$

The next steps are devoted to estimate the terms appearing in the left hand side of the identity above. It will be done in several steps.

(i) Performing integration by parts, we have

$$\int_0^1 u_t u_{xxxx}dx = \frac{d}{dt}\frac{1}{2}\int_0^1 |u_{xx}|^2 dx.$$

(ii) For any $\varepsilon>0$, there exists $c(\varepsilon)>0$ satisfying

$$||u_{xxx}||^2 \leqslant \varepsilon||u_{xxxx}||^2 + c(\varepsilon)||u||^2.$$

Then,

$$\int_0^1 u_{xxx}u_{xxxx}dx \geqslant -\left(\frac{\gamma}{2\delta}+\frac{\delta\varepsilon}{2\gamma}\right)\int_0^1 |u_{xxxx}|^2 dx - \frac{\delta c(\varepsilon)}{2\gamma}\int_0^1 u^2 dx,$$

for any $\delta>0$.

(iii) The remaining terms can be estimated as follows

$$\int_0^1 a u_{xx}u_{xxxx}dx \geqslant -\frac{\gamma}{2\delta}\int_0^1 |u_{xxxx}|^2 dx - \frac{\delta a^2}{2\gamma}\int_0^1 |u_{xx}|^2 dx$$

$$\int_0^1 v_x u_{xxxx}dx \geqslant -\frac{\gamma}{2\delta}\int_0^1 |u_{xxxx}|^2 dx - \frac{\delta}{2\gamma}\int_0^1 |v_x|^2 dx,$$

$$\int_0^1 f_1 u_{xxxx}dx \geqslant -\frac{\gamma}{2\delta}\int_0^1 |u_{xxxx}|^2 dx - \frac{\delta}{2\gamma}\int_0^1 f_1^2 dx,$$

for any $\delta>0$.

Then, using previous computations (from (i) up to (iii)) in (2.7) we deduce that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_0^1 |u_{xx}|^2dx \quad &+ \quad \gamma\left(1-\frac{2}{\delta}-\frac{\varepsilon\delta}{2\gamma^2}\right)\int_0^1 |u_{xxxx}|^2dx \\ &\leqslant \quad \frac{\delta}{2\gamma}\int_0^1 |v_x|^2dx + \frac{\delta c(\varepsilon)}{2\gamma}\int_0^1 u^2dx + \frac{a^2\delta}{2\gamma}\int_0^1 |u_{xx}|^2dx + \frac{\delta}{2\gamma}\int_0^1 f_1^2dx.\end{aligned}$$

We choose $\varepsilon = \frac{4\gamma^2}{\delta^2}$, with δ sufficiently large. We can combine (2.6) and Gronwall's inequality to obtain

$$\begin{aligned}&\int_0^1 |u_{xx}|^2dx + \int_0^T\int_0^1 |u_{xxxx}|^2dxdt \qquad (2.8)\\ &\leqslant C_2\left(\|(u_0,v_0)\|^2_{H_0^2\times L^2} + \|(f_1,f_2)\|^2_{L^2(L^2)^2}\right),\end{aligned}$$

where C_2 is a positive constant that does not depend on N.

Now, we obtain a uniform bound for the term v^N, which is simply denoted by v. In order to do that, we replace w_j by v_{xx} in the second equation of (2.5) and perform integration by parts to obtain

$$\frac{d}{dt}\frac{1}{2}\int_0^1 |v_x|^2dx + \Gamma\int_0^1 |v_{xx}|^2dx = c\int_0^1 v_xv_{xx}dx - \int_0^1 u_xv_{xx}dx - \int_0^1 f_2v_{xx}dx.$$

Thus, for any $\delta > 0$, we get

$$\frac{d}{dt}\frac{1}{2}\int_0^1 |v_x|^2dx + \left(\Gamma-\frac{3\delta}{2}\right)\int_0^1 |v_{xx}|^2dx \leqslant \frac{c^2}{2\delta}\int_0^1 |v_x|^2dx + \frac{1}{2\delta}\int_0^1 |u_x|^2dx + \frac{1}{2\delta}\int_0^1 f_2^2dx.$$

Consequently, choosing $\delta > 0$ sufficiently small and using Gronwall's inequality the previous estimates allow us to conclude that

$$\begin{aligned}&\int_0^1 |v_x|^2dx + \int_0^T\int_0^1 |v_{xx}|^2dxdt \qquad (2.9)\\ &\leqslant C\left(\|(u_0,v_0)\|^2_{L^2\times H_0^1} + \|(f_1,f_2)\|^2_{L^2(L^2)^2}\right),\end{aligned}$$

where C is a positive constant independent on N.

Step 3: Convergence of the approximate solutions. According to (2.6), (2.8) and (2.9), the sequences $(u^N)_{N\geqslant 1}$ and $(v^N)_{N\geqslant 1}$ satisfy

$$(2.10)\qquad \begin{cases}(u^N)_{N\geqslant 1} & \text{is bounded in } L^\infty(0,T;H_0^2(0,1))\cap L^2(0,T;H^4(0,1)),\\ (u_t^N)_{N\geqslant 1} & \text{is bounded in } L^2(0,T;L^2(0,1)),\\ (v^N)_{N\geqslant 1} & \text{is bounded in } L^\infty(0,T;H_0^1(0,1))\cap L^2(0,T;H^2(0,1)),\\ (v_t^N)_{N\geqslant 1} & \text{is bounded in } L^2(0,T;L^2(0,1)),\end{cases}$$

for all $T > 0$. The boundeness given by (2.10) allows to extract subsequences of $(u^N)_{N\geqslant 1}$ and $(v^N)_{N\geqslant 1}$, still denoted by the the same index N, such that

$$\begin{cases} u^N \rightharpoonup u & \text{weakly } * \text{ in } L^\infty(0,T;H_0^2(0,1)) \cap L^2(0,T;H^4(0,1)),\\ u_t^N \rightharpoonup u_t & \text{weakly } * \text{ in } L^2(0,T;L^2(0,1)),\\ v^N \rightharpoonup v & \text{weakly } * \text{ in } L^\infty(0,T;H_0^1(0,1)) \cap L^2(0,T;H^2(0,1)),\\ v_t^N \rightharpoonup v_t & \text{weakly } * \text{ in } L^2(0,T;L^2(0,1)),\end{cases} \tag{2.11}$$

as $N \to \infty$.

Observe that (2.11) allows to pass to the limit in the linear terms of the approximate problem (2.5) to deduce that (u,v) satisfies (2.2) and

$$\int_0^T (u_t(t) + u_{xxx}(t) + au_{xx}(t) + \gamma u_{xxxx}(t) - v_x(t) - f_1, w(t))_{L^2(0,1)} dt = 0$$

$$\int_0^T (v_t(t) + cv_x(t) - \Gamma v_{xx}(t) - v_x(t) - f_2, w(t))_{L^2(0,1)} dt = 0,$$

for all $w \in L^2(0,T;L^2(0,1))$. Thus, the above identities hold for $w \in D'((0,T)\times(0,1))$ and since $(u_t + u_{xxx} + au_{xx} + \gamma u_{xxxx} - v_x - f_1)$, and $(v_t + cv_x - \Gamma v_{xx} - v_x - f_2)$, belong to $L^2((0,T)\times(0,1))$ the above identities hold a. e. in $(0,1)\times(0,T)$, for any $T > 0$.

Step 4: Verifications of the initial data. From the previous steps, we obtain some subsequences of $(u^N)_{N\geqslant 1}$ and $(v^N)_{N\geqslant 1}$, still denoted by the the same index N, such that

$$\int_0^T \left(u^N(t), w(t)\right)_{L^2(0,1)} \, dt \longrightarrow \int_0^T (u(t), w(t))_{L^2(0,1)} \, dt \tag{2.12}$$

and

$$\int_0^T \left(u_t^N(t), w(t)\right)_{L^2(0,1)} \, dt \longrightarrow \int_0^T (u_t(t), w(t))_{L^2(0,1)} \, dt, \tag{2.13}$$

for any $w \in L^2(0,T;L^2(0,1))$, as $N \to \infty$.

Let $\theta \in C^1([0,T])$, such that $\theta(0) = 1$ and $\theta(T) = 0$. For any $z \in L^2(0,1)$, we take $w(t) = \theta'(t)z$ in (2.12) and $w(t) = \theta(t)z$ in (2.13). Thus, we obtain

$$\int_0^T \frac{d}{dt}[(u^N(t), z)_{L^2(0,1)}\theta(t)] \, dt \longrightarrow \int_0^T \frac{d}{dt}[(u(t), z)_{L^2(0,1)}\theta(t)] \, dt, \quad \text{as } N \to \infty.$$

Consequently, for any $z \in L^2(0,1)$,

$$\left(u^N(0), z\right)_{L^2(0,1)} \longrightarrow (u(0), z)_{L^2(0,1)}, \quad \text{as } N \to \infty. \tag{2.14}$$

From (2.5), we know that $u^N(0) = u_0^N$ and that $u_0^N \to u^0$ as $N \to \infty$. By using (2.14) we conclude that $u(0) = u_0$. Analougously, we obtain $v(0) = v_0$.

Step 5: Uniqueness. Let (u^1, v^1) and (u^2, v^2) be two solutions of the problem, corresponding to the same initial data (u_0, v_0) and sources terms (f_1, f_2). Then,

$u = u^1 - u^2$ and $v = v^1 - v^2$ satisfy

$$\begin{cases} u_t + u_{xxx} + au_{xx} + \gamma u_{xxxx} = v_x, & (x,t) \in (0,1) \times (0,T), \\ v_t + cv_x - \Gamma v_{xx} = u_x, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u_x(0,t) = u(1,t) = u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = v(1,t) = 0, & t \in (0,T), \\ u(x,0) = 0, \quad v(x,0) = 0, & x \in (0,1). \end{cases} \tag{2.15}$$

Then, combining Gronwall's inequality we can proceed as in the previous estimates to deduce that $u = v = 0$, i. e., $u^1 = u^2$ and $v^1 = v^2$.

Finally, inequality (2.3) follows directly from estimates (2.8) and (2.9). This completes the proof of Theorem 2.1. □

2.2. Adjoint system. A time-backward equation will appear in the next section, when applying the transposition method. Moreover, this equation will be used when studying control properties of system (1.4). It is called adjoint equation and reads as

$$\begin{cases} -\varphi_t + \gamma\varphi_{xxxx} + a\varphi_{xx} - \varphi_{xxx} = -\psi_x + g_1, \\ -\psi_t - \Gamma\psi_{xx} - c\psi_x = -\varphi_x + g_2, \\ \varphi(t,0) = 0, \quad \varphi(t,1) = 0, \\ \varphi_x(t,0) = 0, \quad \varphi_x(t,1) = 0, \\ \psi(t,0) = 0, \quad \psi(t,1) = 0, \\ \varphi(T,x) = \varphi_T, \quad \psi(T,x) = \psi_T. \end{cases} \tag{2.16}$$

By performing the change $T - t$ by t, we obtain system (2.1) and we can apply Theorem 2.1. Thus, we get the well posedness of the adjoint system.

Proposition 2.4. Let $(\varphi_T, \psi_T) \in H_0^2(0,1) \times H_0^1(0,1)$ and $(g_1, g_2) \in L^2(0,T; L^2(0,1))^2$. System (2.16) has a unique solution $(\varphi, \psi) \in C(H_0^2 \times H_0^1) \cap L^2(H^4 \times H^2)$. Moreover, there exists $C > 0$ such that

$$\|(\varphi,\psi)\|_{C(H_0^2 \times H_0^1) \cap L^2(H^4 \times H^2)} \leqslant C\Big\{\|(g_1,g_2)\|_{L^2(L^2)^2} + \|(\varphi_T,\psi_T)\|_{H_0^2 \times H_0^1}\Big\}. \tag{2.17}$$

In order to deal with L^2-regular boundary data, we present the next regularity result. Taking into account the continuous embeddings $H^2(0,1) \subset C^1([0,1])$ and $H^4(0,1) \subset C^3([0,1])$, from Proposition 2.4, we directly obtain:

Proposition 2.5. There exists $C > 0$ such that any (φ, ψ) solution of (2.16) satisfies

$$\begin{aligned} &\|\psi_x(\cdot,0)\|_{L^2(0,T)} + \|\varphi_{xxx}(\cdot,0)\|_{L^2(0,T)} + \|\varphi_{xx}(\cdot,0)\|_{L^2(0,T)} \\ &\qquad\qquad \leqslant C\Big\{\|(g_1,g_2)\|_{L^2(L^2)^2} + \|(\varphi_T,\psi_T)\|_{H_0^2(0,1) \times H_0^1(0,1)}\Big\}. \end{aligned} \tag{2.18}$$

2.3. Linear non-homogeneous system. We have to deal with boundary data in the space $L^2(0,T)$. Let us define what we mean by a solution of the linear control system

$$
\begin{cases}
u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x + f_1, & (x,t) \in (0,1)\times(0,T),\\
v_t - \Gamma v_{xx} + c v_x = u_x + f_2, & (x,t) \in (0,1)\times(0,T),\\
u(0,t) = h_1(t), \quad u(1,t) = 0, & t \in (0,T),\\
u_x(0,t) = h_2(t), \quad u_x(1,t) = 0, & t \in (0,T),\\
v(0,t) = h_3(t), \quad v(1,t) = 0, & t \in (0,T),\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1).
\end{cases} \tag{2.19}
$$

DEFINITION 2.6. Let $u_0 \in H^{-2}(0,1)$, $v_0 \in H^{-1}(0,1)$, $h_1, h_2, h_3 \in L^2(0,T)$, $f_1 \in L^1(W^{-1,1})$, $f_2 \in L^1(L^1)$. A solution of the system (2.19) is a couple $(u,v) \in L^2(0,T;L^2(0,1))^2$ such that for any $g_1, g_2 \in L^2(0,T;L^2(0,1))$,

$$
\begin{aligned}
\int_0^T\int_0^1 u(t,x)g_1(t,x) + \int_0^T\int_0^1 v(t,x)g_2(t,x) &= \langle u_0, \varphi(0,x)\rangle_{H^{-2},H_0^2} + \langle v_0, \psi(0,x)\rangle_{H^{-1},H_0^1}\\
&- \gamma\int_0^T h_1(t)\varphi_{xxx}(t,0)\,dt + \gamma\int_0^T h_2(t)\varphi_{xx}(t,0)\,dt + \Gamma\int_0^T h_3(t)\psi_x(t,0)\,dt\\
&+ \langle f_1(t,x), \varphi(t,x)\rangle_{L^1(W^{-1,1}),L^\infty(W^{1,\infty})} + \langle f_2(t,x), \psi(t,x)\rangle_{L^1(L^1),L^\infty(L^\infty)},
\end{aligned} \tag{2.20}
$$

where (φ,ψ) is the solution of

$$
\begin{cases}
-\varphi_t + \gamma\varphi_{xxxx} + a\varphi_{xx} - \varphi_{xxx} = -\psi_x + g_1,\\
-\psi_t - \Gamma\psi_{xx} - c\psi_x = -\varphi_x + g_2,\\
\varphi(t,0) = 0, \quad \varphi(t,1) = 0,\\
\psi(t,0) = 0, \quad \psi(t,1) = 0,\\
\varphi(T,x) = 0, \quad \psi(T,x) = 0.
\end{cases} \tag{2.21}
$$

REMARK 2.7. As usual, $\langle\cdot,\cdot\rangle_{X,Y}$ stands for the duality product between two spaces X and Y.

Next theorem establishes existence and uniqueness of solutions of system (2.19).

THEOREM 2.8. Let $u_0 \in H^{-2}(0,1)$, $v_0 \in H^{-1}(0,1)$, $h_1, h_2, h_3 \in L^2(0,T)$, and $f_1 \in L^1(W^{-1,1})$, $f_2 \in L^1(L^1)$. Then, there exists a unique solution $(u,v) \in L^2(0,T;L^2(0,1))^2$ of system (2.19). Moreover $(u,v) \in C([0,T], H^{-2}(0,1)\times H^{-1}(0,1))$.

PROOF. The right-hand side of (2.20) defines a linear bounded functional

$$
L_h^0 : (g_1,g_2) \in L^2(0,T;L^2(0,1))^2 \longmapsto L_h^0(g_1,g_2) \in \mathbb{R},
$$

and therefore, from the Riesz representation theorem, we obtain the existence and uniqueness of a solution $(u,v) \in L^2(0,T;L^2(0,1))^2$. We can prove that Proposition 2.4 and all the above results on system (2.16) still hold if $(g_1,g_2) \in L^1(0,T;H_0^2(0,1)\times H_0^1(0,1))$. Thus, the right hand side of (2.20) also defines a linear bounded functional

$$
L_h^0 : (g_1,g_2) \in L^1(0,T;H_0^2(0,1)\times H_0^1(0,1)) \longmapsto L_h^0(g_1,g_2) \in \mathbb{R},
$$

which gives the extra regularity stated in the theorem.

□

2.4. Nonlinear system.

THEOREM 2.9. There exists a positive real number r such that for any $u_0 \in H^{-2}(0,1)$, $v_0 \in H^{-1}(0,1)$, h_1, h_2 and $h_3 \in L^2(0,T)$ satisfying

$$\|(u_0,v_0)\|_{H^{-2}\times H^{-1}(0,1)} + \|(h_1,h_2,h_3)\|_{L^2(0,T)^3} \leqslant r, \tag{2.22}$$

the nonlinear equation (1.4) has a unique solution $(u,v) \in L^2(0,T;L^2(0,1))^2$. Moreover $(u,v) \in C([0,T],H^{-2}(0,1)\times H^{-1}(0,1))$.

PROOF. Let us consider u_0, v_0, h_1, h_2 and h_3 satisfying (2.22) for $r > 0$ to be chosen later.

We define the following map

$$\Pi : (\ell,m) \in L^2(0,T;L^2(0,1))^2 \longmapsto (u,v) \in L^2(0,T;L^2(0,1))^2 \tag{2.23}$$

where (u,v) is the solution of

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} = v_x - \ell\ell_x, & (x,t) \in (0,1)\times(0,T), \\ v_t - \Gamma v_{xx} + c v_x = u_x - m^2, & (x,t) \in (0,1)\times(0,T), \\ u(0,t) = h_1(t), \quad u(1,t) = 0, & t \in (0,T), \\ u_x(0,t) = h_2(t), \quad u_x(1,t) = 0, & t \in (0,T), \\ v(0,t) = h_3(t), \quad v(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in (0,1). \end{cases} \tag{2.24}$$

Let us notice that $(\tilde{u},\tilde{v})$ is a fixed point of this map Π if and only if $(\tilde{u},\tilde{v})$ is a solution of our nonlinear control system (1.4).

From Theorem 2.8 and by using

$$\|\ell\ell_x\|_{L^1(W^{-1,1})} \leqslant \frac{1}{2}\|\ell\|^2_{L^2(L^2)}$$

and

$$\|m^2\|_{L^1(L^1)} = \|m\|^2_{L^2(L^2)}$$

we get

$$\|\Pi(\ell,m)\|_{L^2(L^2))^2} \leqslant C\Big(\|u_0\|_{H^{-2}(0,1)} + \|v_0\|_{H^{-1}(0,1)} + \|(h_1,h_2,h_3)\|_{L^2(0,T)^3} + \|\ell\|^2_{L^2(L^2)} + \|m\|^2_{L^2(L^2)}\Big).$$

For each $R > 0$, let us denote the ball of radius R and centered at the origin by

$$B(0,R) := \{(\ell,m) \in L^2(0,T;L^2(0,1))^2;\ \|(\ell,m)\|_{L^2(L^2)^2} \leqslant R\}.$$

We see that if $r > 0$ and $R > 0$ are chosen such that $C(r+2R^2) \leqslant R$, we obtain that $\Pi|_{B(0,R)} \subset B(0,R)$. Let us verify that we can choose R such that Π is a contraction. Let (ℓ,m) and $(\tilde{\ell},\tilde{m}) \in L^2(0,T;L^2(0,1))^2$. The couple $(\hat{u},\hat{v}) = (\Pi(\tilde{\ell},\tilde{m}) - \Pi(\ell,m))$ is the solution of

$$\begin{cases} \hat{u}_t + \gamma\hat{u}_{xxxx} + \hat{u}_{xxx} + a\hat{u}_{xx} = \hat{v}_x + \ell\ell_x - \tilde{\ell}\tilde{\ell}_x, & (x,t) \in (0,1)\times(0,T), \\ \hat{v}_t - \Gamma\hat{v}_{xx} + c\hat{v}_x = \hat{u}_x + m^2 - \tilde{m}^2, & (x,t) \in (0,1)\times(0,T), \\ \hat{u}(0,t) = 0, \quad \hat{u}(1,t) = 0, & t \in (0,T), \\ \hat{u}_x(0,t) = 0, \quad \hat{u}_x(1,t) = 0, & t \in (0,T), \\ \hat{v}(0,t) = 0, \quad \hat{v}(1,t) = 0, & t \in (0,T), \\ \hat{u}(x,0) = 0, \quad \hat{v}(x,0) = 0, & x \in (0,1). \end{cases}$$

From Theorem 2.8, we get

$$\|\Pi(\tilde{\ell},\tilde{m}) - \Pi(\ell,m)\|_{L^2(L^2)} \leqslant C\Big\{\|\tilde{\ell}\,\tilde{\ell}_x - \ell\ell_x\|_{L^1(W^{-1,1})} + \|\tilde{m}^2 - m^2\|_{L^1(L^1)}\Big\}.$$

By using that

$$\|\bar{\ell}\,\bar{\ell}_x - \ell\ell_x\|_{L^1(W^{-1,1})} = \frac{1}{2}\|\bar{\ell}^2 - \ell^2\|_{L^1(L^1)} \leqslant \frac{1}{2}\|\bar{\ell} + \ell\|_{L^2(L^2)}\|\bar{\ell} - \ell\|_{L^2(L^2)}$$

and

$$\|\tilde{m}^2 - m^2\|_{L^1(L^1)} \leqslant \|\tilde{m} + m\|_{L^2(L^2)}\|\tilde{m} - m\|_{L^2(L^2)}$$

we obtain

$$\|\Pi(\tilde{\ell},\tilde{m}) - \Pi(\ell,m)\|_{L^2(L^2)} \leqslant 2CR\Big\{\|\tilde{\ell} - \ell\|_{L^2(L^2)} + \|\tilde{m} - m\|_{L^2(L^2)}\Big\}$$

and therefore the map Π is a contraction if $2CR < 1$. By applying the Banach fixed point theorem, we can conclude that Π has a unique fixed point which is the solution of equation (1.4) in $L^2(0,T;L^2(0,1))^2$. Now, once we have (u,v) the solution of equation (1.4), we see that (u,v) satisfies equation (2.19) with source terms $f_1 = -uu_x$, and $f_2 = -v^2$, which belong to $L^1(0,T;W^{-1,1}(0,1))$ and $L^1(0,T;L^1(0,1))$ respectively. We apply Thereom 2.8 and we get the extra regularity $(u,v) \in C([0,T],H^{-2}(0,1)\times H^{-1}(0,1))$. □

3. Null controllability

3.1. Linear control system. In this section, we study the boundary control of the linear system (2.19).

Let us take a well-posedness framework $(U_1,U_2,U_3,X_1,X_2,Y,Z)$ for this system. By this we mean that given $h_j \in U_j$ for $j = 1,2,3$, $(f_1,f_2) \in Y = Y_1 \times Y_2$, $u_0 \in X_1$ and $v_0 \in X_2$ then there exists a unique $(u,v) \in Z = Z_1 \times Z_2$ solution of equation (2.19).

This system is said to be null controllable if for any state $u_0 \in X_1$, $v_0 \in X_2$ and for any $(f_1,f_2) \in Y_1 \times Y_2$, one can find controls $h_k \in U_k$, $k = 1,2,3$, such that the solution (u,v) of (2.19) satisfies $u(T) = v(T) = 0$. It is a well-known fact that by duality, this null-controllability property is equivalent to the existence of a constant $C > 0$ such that

$$\begin{aligned} \|(\varphi,\psi)\|_{Y^*} + \|\varphi(0,x)\|_{X_1^*} + \|\psi(0,x)\|_{X_2^*} \leqslant C\big(&\|(g_1,g_2)\|_{Z^*} \\ &+ \|\varphi_{xxx}(t,0)\|_{U_1^*} + \|\varphi_{xx}(t,0)\|_{U_2^*} + \|\psi_x(t,0)\|_{U_3^*}\big) \end{aligned} \tag{3.1}$$

for every $\varphi_T \in X_1^*$, $\psi_T \in X_2^*$ and $g \in Z^*$, where * stands for dual space and (φ, ψ) is the solution of the adjoint linear system (2.16).

Inequality (3.1) is called an *observability inequality* for system (2.16).

In this section we prove a Carleman estimate for system (2.16). Then, we use it in order to prove the observability inequality (3.1) within an appropriate well-posedness framework. Thus, we get the null-controllability of system (2.19).

We shall use an abbreviated notation for the derivatives and integrals. We write, for k integer, w_{kx} instead of $\frac{\partial^k w}{\partial x^k}$ and $\iint$ instead of $\int_0^T \int_0^1$, avoiding the symbols $dxdt$ in the last case.

We take a function $\beta \in C^2([0,1])$ satisfying

$$0 < \delta \leqslant \frac{d^k \beta}{dx^k}(x), \quad \forall x \in [0,1], \text{ for } k = 0,1 \tag{3.2}$$

and

$$\frac{d^2 \beta}{dx^2}(x) \leqslant -\delta < 0, \quad \forall x \in [0,1], \tag{3.3}$$

for some positive constant δ.

Carleman estimates for heat equation are well known (see [**10**] for example). We recall a one-parameter estimate, which holds because we are in the one-dimensional case, in the following theorem.

Theorem 3.1. (See [**10**]). Let $\mu(t,x) = \frac{\beta(x)}{t(T-t)}$ with $\beta \in C^2([0,1])$ satisfying (3.2) and (3.3). There exist $C > 0$ and $s_0 > 0$ such that

$$\begin{aligned} s^3 \iint |\psi|^2 e^{-2s\mu} \mu^3 + s \iint |\psi_x|^2 e^{-2s\mu} \mu \\ \leqslant C\Big(\iint |L\psi|^2 e^{-2s\mu} + s \int_0^T |\psi_x(0,t)|^2 e^{-2s\mu(0,t)} dt \Big) \end{aligned} \tag{3.4}$$

for every $s \geqslant s_0$ and $\psi \in C^\infty([0,T] \times [0,1])$, where $L\psi = -\psi_t - \Gamma \psi_{xx} - c\psi_x$.

On the other hand, a Carleman estimate for the KS equation has been studied in [**7**].

Theorem 3.2. (See [**7**]). Let $\eta(t,x) = \frac{\beta(x)}{t(T-t)}$ with $\beta \in C^4([0,1])$ satisfying (3.2) and (3.3). There exist $C > 0$ and $\lambda_0 > 0$ such that

$$\begin{aligned} \lambda^7 \iint |\varphi|^2 e^{-2\lambda\eta} \eta^7 + \lambda^5 \iint |\varphi_x|^2 e^{-2\lambda\eta} \eta^5 + \lambda^3 \iint |\varphi_{2x}|^2 e^{-2\lambda\eta} \eta^3 + \lambda \iint |\varphi_{3x}|^2 e^{-2\lambda\eta} \eta \\ \leqslant C\Big(\iint |P\varphi|^2 e^{-2\lambda\eta} + \lambda^3 \int_0^T |\varphi_{2x}(0,t)|^2 e^{-2\lambda\eta(0,t)} \eta_x^3(0,t) dt \\ + \lambda \int_0^T |\varphi_{3x}(0,t)|^2 e^{-2\lambda\eta(0,t)} \eta_x(0,t) dt \Big) \end{aligned} \tag{3.5}$$

for every $\lambda \geqslant \lambda_0$ and $\varphi \in C^\infty([0,T] \times [0,1])$, where $P\varphi = -\varphi_t + \gamma \varphi_{xxxx} + a\varphi_{xx} - \varphi_{xxx} - \varphi_x$.

In order to get an estimate for system (2.16) let us recall that, if (φ, ψ) is a solution, then $P\varphi = -\psi_x + g_1$ and $L\psi = -\varphi_x + g_2$. We apply the Carleman estimates (3.4) and (3.5) (with the same weight function and $s = \lambda$) to the corresponding equation in the system, and we add both estimates. Taking the parameter λ large enough, the integrals involving the coupling terms ψ_x and φ_x are absorbed by the left hand side of the inequality. Hence we obtain the following result.

THEOREM 3.3. Let β and η be as in Theorem 3.2. There exist $C > 0$ and $\lambda_0 > 0$ such that

(3.6)
$$\lambda^7 \iint |\varphi|^2 e^{-2\lambda\eta}\eta^7 + \lambda^5 \iint |\varphi_x|^2 e^{-2\lambda\eta}\eta^5 + \lambda^3 \iint |\varphi_{2x}|^2 e^{-2\lambda\eta}\eta^3 + \lambda \iint |\varphi_{3x}|^2 e^{-2\lambda\eta}\eta$$
$$\lambda^3 \iint |\psi|^2 e^{-2\lambda\eta}\eta^3 + \lambda \iint |\psi|^2 e^{-2\lambda\eta}\eta \leqslant C\Big(\iint (|g_1|^2 + |g_2|^2) e^{-2\lambda\eta}$$
$$+\lambda^3 \int_0^T |\varphi_{2x}(0,t)|^2 e^{-2\lambda\eta(0,t)}\eta_x^3(0,t)dt + \lambda \int_0^T |\varphi_{3x}(0,t)|^2 e^{-2\lambda\eta(0,t)}\eta_x(0,t)dt$$
$$+\lambda \int_0^T |\psi_x(0,t)|^2 e^{-2s\eta(0,t)}dt\Big)$$

for every $\lambda \geqslant \lambda_0$ and $g_j \in L^2((0,T); L^2(0,1))$, $j = 1, 2$, where (φ, ψ) is the solution of equation (2.16).

We prove the following energy estimate for equation (2.16).

LEMMA 3.4. If $g_1, g_2 \in L^2((0,T); L^2(0,1))$ and φ, ψ are the solutions of system (2.16) then

$$-\frac{d}{dt}\int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dx \leqslant C \int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dx + \tag{3.7}$$
$$\leqslant \int_0^1 (g_1^2 + g_2^2)dx$$

for every $t \in [0, T]$.

PROOF. Multiplying the first and the second equations of system (2.16) by φ and ψ respectively and integrating in $(0, 1)$ we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\varphi(x,t)|^2 dx + \gamma \int_0^1 |\varphi_{xx}(x,t)|^2 dx + a \int_0^1 \varphi_{xx}(x,t)\varphi(x,t)dx \tag{3.8}$$
$$= -\int_0^1 \psi_x(x,t)\varphi(x,t)dx + \int_0^1 \varphi(x,t)g_1(x,t)dx$$

and

(3.9)
$$-\frac{1}{2}\frac{d}{dt}\int_0^1 |\psi(x,t)|^2 dx + \Gamma \int_0^1 |\psi_x(x,t)|^2 dx = -\int_0^1 \varphi_x(x,t)\psi(x,t)dx \int_0^1 \psi(x,t)g_2(x,t)dx$$

for each $t \in [0,T]$. Adding (3.8) and (3.9) and using that

$$-a \int \varphi_{xx}\varphi \leqslant \gamma \int |\varphi_{xx}|^2 + \frac{a^2}{\gamma} \int |\varphi|^2,$$

we get (3.7). □

LEMMA 3.5. If $g_1, g_2 \in L^2((0,T); L^2(0,1))$ and φ, ψ are the solutions of system (2.16), we define $\Phi(t) = \int_0^1 (\varphi(t)^2 + \psi(t)^2)dx$. Then,

$$\|\Phi\|_{L^\infty(0,\frac{T}{2})} \leqslant C \left(\|\Phi\|^2_{L^2(\frac{T}{2},\frac{3T}{4})} + \int_0^{\frac{3T}{4}} \int_0^1 (g_1^2 + g_2^2) dx dt \right). \tag{3.10}$$

PROOF. Let $z \in C^\infty(0,T)$ be such that $z(t) = 1$ for all $t \in [0, T/2]$ and $z(t) = 0$ for all $t \in [3T/4, T]$. Multiplying (3.7) by z we obtain

$$-\frac{d}{dt}(z\Phi) \leqslant Cz\Phi - z_t\Phi + z\int_0^1 (g_1^2 + g_2^2)dx. \tag{3.11}$$

For each $t \in [0,T]$ we apply Gronwall inequality in $[t,T]$. Since $z(T) = 0$ we obtain

$$z(t)\Phi(t) \leqslant C \int_t^T h(s)ds \tag{3.12}$$

where $h(t) = z(t)\int_0^1 (g_1^2 + g_2^2)dx - z_t(t)\int_0^1 (g_1^2 + g_2^2)dx$.

From (3.12) we deduce (3.10). □

We introduce a weight function $\tilde\eta(x,t) = \beta(x)\phi(t)$ where $\beta \in C^4([0,1])$ satisfies hypothesis (3.2)-(3.3) and ϕ is defined by

$$\phi(t) = \begin{cases} \dfrac{4}{T^2} & \text{if} \quad 0 \leqslant t < T/2, \\ \dfrac{1}{t(T-t)} & \text{if} \quad T/2 \leqslant t \leqslant T. \end{cases}$$

PROPOSITION 3.6. There exist $\lambda_0, C > 0$ such that for every $\lambda \geqslant \lambda_0$ the solution (φ, ψ) of (2.16) satisfies

$$\begin{aligned} &\iint |\psi|^2 e^{-2\lambda\tilde\eta}\tilde\eta^3 + \int_0^1 |\psi(x,0)|^2 dx + \iint |\varphi|^2 e^{-2\lambda\tilde\eta}\tilde\eta^7 \\ &+ \int_0^1 |\varphi(x,0)|^2 dx \leqslant C\Big(\iint (|g_1|^2 + |g_2|^2)e^{-2\lambda\tilde\eta} + \int_0^T |\varphi_{2x}(0,t)|^2 e^{-2\lambda\tilde\eta(0,t)}\tilde\eta_x(0,t)^3 dt \\ &+ \int_0^T |\varphi_{3x}(0,t)|^2 e^{-2\lambda\tilde\eta(0,t)}\tilde\eta_x(0,t)dt + \int_0^T |\psi_x(0,t)|^2 e^{-2s\tilde\eta(0,t)}dt\Big) \end{aligned} \tag{3.13}$$

for every g_1, g_2 such that $\iint e^{-2\lambda\tilde\eta}(g_1^2 + g_2^2) < \infty$.

Proof. From Lemma 3.5 and the fact that $e^{-2\lambda\tilde{\eta}} \geqslant \delta > 0$ for $t \in [0, 3T/4]$ we obtain

$$
\begin{aligned}
&\int_0^1 (\varphi(x,0)^2 + \psi(x,0)^2)dx + \int_0^{T/2}\int_0^1 \tilde{\eta}^7 e^{-2\lambda\tilde{\eta}}\varphi(x,t)^2 dxdt \\
&+ \int_0^{T/2}\int_0^1 \tilde{\eta}^3 e^{-2\lambda\tilde{\eta}}\psi(x,t)^2 dxdt \leqslant \|(\varphi,\psi)\|_{L^\infty(0,T/2;L^2(0,1))^2} \\
&\qquad \leqslant C\|(\varphi,\psi)\|_{L^\infty(T/2,3T/4;L^2(0,1))^2} + \\
&\qquad C\|(g_1,g_2)\|_{L^\infty(0,3T/4;L^2(0,1))^2} \\
&\qquad \leqslant C\int_{T/2}^{3T/4}\int_0^1 (\varphi(x,t)^2 + \psi(x,t)^2)dxdt \\
&\qquad + C\int_0^{\frac{3T}{4}}\int_0^1 e^{-2\lambda\tilde{\eta}}(g_1^2 + g_2^2)dxdt.
\end{aligned} \tag{3.14}
$$

On the other hand, we have $\tilde{\eta}(x,t) = \eta(x,t)$ if $t \in [T/2, T]$ and $\tilde{\eta}(x,t) \leqslant \eta(x,t)$ if $t \in [0, T/2]$. Using that $\eta(0,0)^k e^{-2\lambda\tilde{\eta}(0,0)} = 0$ for $k = 3, 7$ and Carleman estimate (3.6) we deduce that

$$
\begin{aligned}
&\int_{T/2}^T\int_0^1 |\psi|^2 e^{-2\lambda\tilde{\eta}}\tilde{\eta}^3 + \int_{T/2}^T\int_0^1 |\varphi|^2 e^{-2\lambda\tilde{\eta}}\tilde{\eta}^7 \leqslant C\Big(\iint (|g_1|^2 + |g_2|^2)e^{-2\lambda\tilde{\eta}} \\
&+ \int_0^T |\varphi_{2x}(0,t)|^2 e^{-2\lambda\tilde{\eta}(0,t)}\tilde{\eta}_x(0,t)^3 dt + \int_0^T |\varphi_{3x}(0,t)|^2 e^{-2\lambda\tilde{\eta}(0,t)}\tilde{\eta}_x(0,t)dt \\
&\qquad + \int_0^T |\psi_x(0,t)|^2 e^{-2s\tilde{\eta}(0,t)}dt\Big).
\end{aligned} \tag{3.15}
$$

Inequality (3.13) is obtained from (3.14) and (3.15). □

We will assume an additional property of the weight function. We will ask the function $\beta(x)$ to satisfy

$$
\max_{x\in[0,1]} \beta(x) < 2 \min_{x\in[0,1]} \beta(x), \tag{3.16}
$$

as well as, the stated conditions (3.2) and (3.3). Thus, we define

$$
r_1 := \min_{x\in[0,1]} \beta(x), \quad r_2 := \max_{x\in[0,1]} \beta(x) \tag{3.17}
$$

and we take $r \in \mathbb{R}$ such that $r_2 < r < 2r_1$. Through the rest of the paper we will denote by ρ the function defined by

$$
\rho(t) = e^{-\frac{r}{(T-t)}} \tag{3.18}
$$

for $t \in (0, T)$.

PROPOSITION 3.7. There exists $C > 0$ such that the solutions (φ, ψ) of (2.16), satisfy

(3.19)
$$\|(\rho\varphi, \rho\psi)\|_{L^\infty(H_0^2)\times L^\infty(H_0^1)} \leqslant C\Big(\iint (|g_1|^2 + |g_2|^2) e^{-\frac{r_1}{T-t}} + \int_0^T |\psi_x(t,0)|^2 e^{-\frac{2r_1}{T-t}} dt$$
$$+ \int_0^T |\varphi_{2x}(t,0)|^2 e^{-\frac{2r_1}{T-t}} (T-t)^{-3} dt + \int_0^T |\varphi_{3x}(t,0)|^2 e^{-\frac{2r_1}{T-t}} (T-t)^{-1} dt\Big)$$

for every g such that $\displaystyle\iint |g|^2 e^{-\frac{2r}{T-t}} < \infty$.

PROOF. Define $\tilde{\varphi} = \rho\varphi$, $\tilde{\psi} = \rho\psi$. Notice that $\tilde{\varphi}$, $\tilde{\psi}$ satisfy system (2.16) with $\varphi_T = \psi_T = 0$ and with the functions $(\rho g_1 - \rho_t \varphi)$ and $(\rho g_2 - \rho_t \psi)$ at the right-hand side instead of g_1 and g_2, respectively. Thanks to Proposition 2.4 we have

$$\|(\rho\varphi, \rho\psi)\|_{L^\infty(H_0^2)\times L^\infty(H_0^1)} \leqslant C\Big(\|(\rho g_1, \rho g_2)\|_{L^2(L^2)} + \|(\rho_t\varphi, \rho_t\psi)\|_{L^2(L^2)} \Big).$$

We can easily check the existence of some positive constants C_1 and C_2 such that

$$\iint |\rho g_j|^2 \leqslant C \iint |g_j|^2 e^{-\frac{2r}{T-t}},$$

$$\iint |\rho_t \varphi|^2 \leqslant C \iint |\varphi|^2 e^{-\frac{2r}{T-t}} (T-t)^{-4} \leqslant C \iint |\varphi|^2 e^{-\frac{2r_2}{T-t}}$$

and in the same manner

$$\iint |\rho_t \psi|^2 \leqslant C \iint |\psi|^2 e^{-\frac{2r_2}{T-t}}.$$

Therefore, by using (3.13), we get (3.19). □

Inequality (3.19) directly implies an observability inequality like (3.1) in some weighted spaces. In order to precise that, we introduce the following notations.

DEFINITION 3.8. Given $T > 0$ and a function $\eta : (0,T) \longrightarrow \mathbb{R}^+$, we denote

$$L_t^2(\eta) := \Big\{ f\,;\, \int_0^T |f(t)|^2 \eta(t)\, dt < \infty \Big\}$$

and

$$L_{tx}^2(\eta) := \Big\{ f\,;\, \int_0^T \int_0^1 |f(x,t)|^2 \eta(t)\, dxdt < \infty \Big\}$$

endowed with their natural norms.

Taking into account the continuous embeddings $H_0^1(0,1) \hookrightarrow L^\infty(0,1)$ and $H_0^2(0,1) \hookrightarrow W^{1,\infty}(0,1)$, inequality (3.19) gives us the observability (3.1) in the spaces

$$U_1 = L_t^2\left(e^{\frac{2r_1}{T-t}}\right), \quad U_2 = L_t^2\left(e^{\frac{2r_1}{T-t}}(T-t)\right), \quad U_3 = L_t^2\left(e^{\frac{2r_1}{T-t}}(T-t)^3\right),$$

$$X_1 = H^{-2}(0,1), \quad X_2 = H^{-1}(0,1), \quad Z = L_{tx}^2(e^{\frac{2r_1}{T-t}})^2,$$

$$Y_1 = \{y; \rho^{-1} y \in L^1(0,T; W^{-1,1}(0,1))\} \tag{3.20}$$

and

$$Y_2 = \{y; \rho^{-1} y \in L^1(0,T; L^1(0,1))\}.$$

Thus, by duality we obtain the null controllability result in this functional framework. Next proposition gives extra information on the decay of the controlled trajectories, which will be useful later in dealing with the nonlinear equation.

Proposition 3.9. For each $f_1 \in Y_1$, $f_2 \in Y_2$, $u_0 \in H^{-2}(0,1)$, $v_0 \in H^{-1}(0,1)$ there exist controls $h_j \in U_j$ for $j = 1,2,3$, such that the solution (u,v) of system (2.19) belongs to $L^2_{tx}(e^{\frac{2r_1}{T-t}})^2$ and satisfies

$$(T-t)^2 e^{\frac{r_1}{T-t}} (u,v) \in C([0,T]; H^{-2}(0,1) \times H^{-1}(0,1)).$$

In particular, $u(T) = v(T) = 0$.

Proof. The existence of the solution $(u,v) \in L^2_{tx}(e^{\frac{2r_1}{T-t}})^2$ is a direct consequence of the inequality (3.19) and the controllability-observability duality.

On the other hand, let us define $\tilde{u} = (T-t)^2 e^{\frac{r_1}{T-t}} u$, $\tilde{v} = (T-t)^2 e^{\frac{r_1}{T-t}} v$, $\tilde{f}_j = (T-t)^2 e^{\frac{r_1}{T-t}} f_j$ for $j = 1,2$ and $\tilde{h}_j = (T-t)^2 e^{\frac{r_1}{T-t}} h_j$ for $j = 1,2,3$.

Then we have that $\tilde{h}_j \in L^2(0,T)$ and $(\tilde{u}, \tilde{v})$ satisfy

$$\tilde{u}_t + \gamma \tilde{u}_{xxxx} + a\tilde{u}_{xx} + \tilde{u}_{xxx} = \tilde{v}_x + \tilde{f}_1 - 2(T-t) e^{\frac{r_1}{T-t}} u + r_1 e^{\frac{r_1}{T-t}} u, \tag{3.21}$$

$$\tilde{v}_t - \Gamma \tilde{v}_{xx} + c\tilde{v}_x = \tilde{u}_x + \tilde{f}_2 - 2(T-t) e^{\frac{r_1}{T-t}} v + r_1 e^{\frac{r_1}{T-t}} v.$$

By using that $(u,v) \in L^2_{tx}(e^{\frac{2r_1}{T-t}})^2$ and that $f_j \in Y_j$ for $j = 1,2$, we obtain (recall that $r_1 < r_2 < r < 2r_1$)

$$(\tilde{f}_1, \tilde{f}_2) \in L^1(0,T; W^{-1,1}(0,1)) \times L^1(0,T; L^1(0,1))$$

and that

$$-2(T-t) e^{\frac{r_1}{T-t}} u + r_1 e^{\frac{r_1}{T-t}} u - 2(T-t) e^{\frac{r_1}{T-t}} v + r_1 e^{\frac{r_1}{T-t}} \in L^2(0,T; L^2(0,1)).$$

Hence, by applying Theorem 2.8 we obtain the desired result. □

3.2. Nonlinear control system. We shall prove the null controllability of the nonlinear system. We will deduce that result from the null controllability of the linear equation by using a local inversion theorem.

In order to obtain Theorem 1.1, we apply the following result.

Theorem 3.10. (see [**1**]) Let E and G be two Banach spaces and let $\Lambda : E \to G$ satisfy $\Lambda \in C^1(E; G)$. Assume that $\hat{e} \in E$, $\Lambda(\hat{e}) = \hat{g}$, and $\Lambda'(\hat{e}) : E \to G$ is surjective. Then, there exists $\delta > 0$ such that, for every $g \in G$ satisfying $\|g - \hat{g}\|_G < \delta$, there exists some $e \in E$ solution of the equation $\Lambda(e) = g$.

Let us define some appropriate spaces E, G and a map Λ whose surjectivity is equivalent to the null controllability for the KS equation. We denote

$$L_1(u,v) = u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} - v_x,$$

$$L_2(u,v) = v_t - \Gamma v_{xx} + c v_x - u_x.$$

Keeping in mind (3.20) we define the spaces

$$E := \Big\{ (u,v) \in Z\, : L_1(u,v) \in Y_1, L_2(u,v) \in Y_2, \text{ and } (T-t)^2 e^{\frac{r_1}{T-t}}(u,v) \in C(H^{-2}\times H^{-1}). \Big\}$$

and

$$G := H^{-2}(0,1) \times Y_1 \times H^{-1}(0,1) \times Y_2.$$

The map Λ is given by

$$\begin{aligned} \Lambda : E \longrightarrow \quad & G, \\ (u,v) \longmapsto \quad & \big(u(0,\cdot), L_1(u,v) + u u_x, v(0,\cdot), L_2(u,v) + v^2\big). \end{aligned}$$

We now prove that Λ is well-defined. First, we have to verify that $uu_x \in Y_1$ for each $(u,v) \in E$.

We have

$$\begin{aligned} uu_x \in Y_1 \quad & \Longleftrightarrow \quad e^{\frac{r}{(T-t)}} uu_x \in L^1(0,T;W^{-1,1}(0,1)) \\ & \Longleftrightarrow \quad e^{\frac{r}{(T-t)}} |u|^2 \in L^1(0,T;L^1(0,1)) \\ & \Longleftrightarrow \quad \int_0^T \int_0^1 |u|^2 e^{\frac{r}{(T-t)}}\, dxdt < \infty. \end{aligned}$$

Therefore, as $r < 2r_1$ (see (3.16)-(3.17)) and $u \in L^2_{xt}(e^{\frac{2r_1}{(T-t)}})$, we see that $(u,v) \in E$ implies $uu_x \in Y_1$.

In the same way, let us see that $v^2 \in Y_2$ for each $(u,v) \in E$. In fact,

$$\int_0^T \int_0^1 |v|^2 e^{\frac{r}{(T-t)}}\, dxdt \leqslant \int_0^T \int_0^1 |v|^2 e^{\frac{2r_1}{(T-t)}}\, dxdt < \infty$$

and then $v^2 e^{\frac{2r_1}{(T-t)}} \in L^1(0,T;L^1(0,1)) \subset L^1(0,T;W^{-1,1}(0,1))$, which means that $v^2 \in Y_2$.

Notice that each one of the maps $(u_1,u_2) \mapsto \frac{1}{2}(u_1u_2)_x \in Y_1$, and $(v_1,v_2) \mapsto (v_1v_2) \in Y_2$ is a bilinear continuous map and consequently Λ is a C^1 map.

As the functions $(u,v) \in E$ satisfy $u(T) = v(T) = 0$, the local surjectivity of Λ around the origin is equivalent to the local null controllability of system (1.4). Thus, by Theorem 3.10, the proof of Theorem 1.1 will be ended if we prove that the map $\Lambda'(0)$ is surjective.

Proposition 3.11. The map $\Lambda'(0) : E \to G$ is surjective.

Proof. It is easy to see that this map is given by

$$\begin{aligned} \Lambda'(0) : E \longrightarrow \quad & G, \\ (u,v) \longmapsto \quad & (u(0,\cdot), L_1(u,v), v(0,\cdot), L_2(u,v)), \end{aligned}$$

and therefore its surjectivity is equivalent to the null controllability of the linearized equation with source terms lying in $Y_1 \times Y_2$. This control property was proved in Proposition 3.9. □

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