SCIENTIA
Series A: *Mathematical Sciences*, Vol. 22 (2012), 33–46
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Mock theta functions to mock theta conjectures

M.P. Chaudhary

Abstract. Mathematical genius Srinivasa Ramanujan's second last gift to the mathematicians was his ingeneous discovery of the mock theta functions of order third, fifth and seventh. Earlier it was considered that mock theta functions were last ideas discovered by Ramanujan, but it is recently investigated and established that his last discovery was cranks - not mock theta functions [7;28]. Here, we discuss in brief about developments as well as advancements on mock theta functions during last more than ninety years, and its further extensions to "mock theta conjectures". The summary of two known proof for the "mock theta conjectures" given by Hickerson [9] and Folsom [1] respectively, are also discussed.

The concept of mock theta functions were ever first studied and introduced by Ramanujan in his las letter he wrote to G .H. Hardy, a number theorist at Trinity College, University of Cambridge, England, United Kingdom, on dated January 20, 1920, [25, pp.354-355]. In this letter, Ramanujan given a list of seventeen mock theta functions, together with identities they satisfy. Ramanujan divided his list of functions of "orders" 3, 5 or 7, but did not given its proper definition, however he explained what he meant by a mock theta function. There is still no formal definition of "order", but known identities for these mock theta functions make it clear that they are related to the "order" 3, 5 or 7. They were first studied by G. N. Watson [18;19], who was able to find representations of them which allowed him to study their behaviour under the fundamental transformations of the modular group [18]. G. E. Andrews proved that Ramanujan's identities for the third and fifth order mock theta functions by means of certain general theorems.

Now we are writing all mock theta funtions of orders 3, 5 and 7, in their original forms as initially introduced by Ramanujan in his letter to Hardy [24, pp 130-131], as

Third order mock theta functions:

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots \tag{1}$$

2000 *Mathematics Subject Classification.* Primary 11F27, 11F37, 11F50, 17B67, 33D15.
Key words and phrases. Mock theta functions, Heck identities, Maass forms.

$$\psi(q)=\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^3)}+\frac{q^9}{(1-q)(1-q^3)(1-q^5)}+\cdots \tag{2}$$

$$\chi(q)=1+\frac{q}{1-q+q^2}+\frac{q^4}{(1-q+q^2)(1-q^2+q^4)}+\cdots \tag{3}$$

These are related to $f(q)$, as shown below

$$2\phi(-q)-f(q)=f(q)+4\psi(-q)=\frac{1-2q+2q^4-2q^9+\cdots}{(1+q)(1+q^2)(1+q^3)} \qquad 3(a)$$

Ramanujan also define $f(q)$, as[24, p.129]

$$f(q)=1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\cdots \qquad 3(b)$$

Fifth order mock theta functions:

$$f(q)=1+\frac{q}{1+q}+\frac{q^4}{(1+q)(1+q^2)}+\frac{q^9}{(1+q)(1+q^2)(1+q^3)}+\cdots \tag{4}$$

$$\phi(q)=1+q(1+q)+q^4(1+q)(1+q^3)+q^9(1+q)(1+q^3)(1+q^5)+\cdots \tag{5}$$

$$\psi(q)=q+q^3(1+q)+q^6(1+q)(1+q^2)+q^{10}(1+q)(1+q^2)(1+q^3)+\cdots \tag{6}$$

$$\begin{aligned}\chi(q)&=1+\frac{q}{1-q^2}+\frac{q^2}{(1-q^3)(1-q^4)}+\frac{q^3}{(1-q^4)(1-q^5)(1-q^7)}+\cdots\\&=1+\left[\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\cdots\right]\end{aligned} \tag{7}$$

$$F(q)=1+\frac{q^2}{1-q}+\frac{q^8}{(1-q)(1-q^3)}+\cdots \tag{8}$$

$$\phi(-q)+\chi(q)=2F(q)$$

$$\begin{aligned}f(-q)+2F(q^2)-2=\phi(-q^2)+\psi(-q)&=2\phi(-q^2)-f(q)\\&=\frac{1-2q+2q^4-2q^9}{(1-q)(1-q^4)(1-q^6)(1-q^9)}\end{aligned} \qquad 8(a)$$

$$\psi(q)-F(q)+1=q.\frac{1+q^2+q^6+Q^{12}+\cdots}{(1-q^8)((1-q^{12})((1-q^{28})} \qquad 8(b)$$

Ramanujan, again recorded following 5th order mock theta functions

$$f(q)=1+\frac{q^2}{1+q}+\frac{q^6}{(1+q)(1+q^2)}+\frac{q^{12}}{(1+q)(1+q^2)(1+q^3)}+\cdots \tag{9}$$

$$\phi(q)=q+q^4(1+q)+q^9(1+q)(1+q^2)+\cdots \tag{10}$$

$$\psi(q)=1+q(1+q)+q^3(1+q)(1+q^2)+q^6(1+q)(1+q^2)(1+q^3)+\cdots \tag{11}$$

$$\begin{aligned}\chi(q)=\frac{1}{1-q}+\frac{q}{(1-q^2)(1-q^3)}+\frac{q^2}{(1-q^3)(1-q^4)(1-q^5)}+\\+\frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}+\cdots\end{aligned} \tag{12}$$

$$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)} + \cdots \tag{13}$$

Seventh order mock theta functions:

{Seventh order mock theta functions were studied by Selberg [2], who found asymptotic expansion for their coefficients. Andrews [12], also studied these functions. Hickerson [9], found interpretations of many of these functions as the quotients of indefinite theta series through modular forms of weight $\frac{1}{2}$. These three functions are perhaps the most mysterious in all of Ramanujan's work.}

$$1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^8)} + \cdots \tag{14}$$

$$\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)} + \cdots \tag{15}$$

$$\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)} + \cdots \tag{17}$$

Above three identities given by equations (19)-(21), are not related to each other.

Since 1920, considering base of above theta functions extensive research works have been made and a large number of research papers have been produced by researchers until last decade of twenty century. But, a general definition for mock theta functions were needed by researchers. Finally, Andrews and Hickerson [14], ever first introduced a formal definition for mock theta functions, and it states as;

A mock theta function is a function of the complex variable q, defined a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions;

(i). infinitely many roots of unity are exponential singularities,

(ii). for every root of unity ξ there is a theta function $\theta_\xi(q)$ such that the difference $f(q) - \theta_\xi(q)$ is bounded as $\theta \mapsto \xi$ radially (presumably with only finitely many of the θ_ξ being different),

(iii). there is no theta function that works for all ξ, i.e. f is not the sum of two functions, one of which is a theta function and the other a function which is bounded in all roots of unity.

Note: when Ramanujan refers to theta functions, he means sums, products, and quotients of series of the form $\sum_{n\in Z} \varepsilon^n q^{an^2+bn}$ with $a, b \in Q$ and $\varepsilon = -1, 1$.

The seventeen functions given by Ramanujan indeed satisfy conditions (i) and (ii) [2,18,19], still no proof has ever been given that they also satisfy condition (iii).

Watson proved a very week form of condition (iii) for the "third order mock theta functions" [18].

Andrews work on fifth order mock theta functions [12]:

First of all in 1976, Ramanujan's lost notebook discovered by Andrews. In it we find many results for the mock theta functions beyond those already contains in Ramanujan's last letter to Hardy. Then the 5th (and also 7th) order functions have been more of a problem. In one of his paper related to 5th order mock theta functions, Watson [18, p.274] states, as

I have failed to construct a complete and exact transformation theory of the functions, on the lines of the transformation theory of the functions of the third order, and in view of the complexity of all the series which are involved, I am becoming some what skeptical concerning the existence of an exact transformation theory for function of the fifth order

As given below, eight of the ten 5th order mock theta functions of Ramanujan [25, pp.354-355]will be related to double sums of the Heck modular form type. We are not able to give such formulas for either $\chi_0(q)$ or $\chi_1(q)$[18, pp. 277-278]. But, since $\chi_0(q) = 2F_0(q) - \phi_0(-q)$ and $\chi_1(q) = 2F_1(q) + \phi^{-1}\phi_1(-q)$ [18, pp. 277 and 279], we see that each of $\chi_0(q)$ and $\chi_1(q)$ is a linear combination of functions that do appear in our main results. Every equation between (18)to(25) will provide the definition and also the appropriate Heck representation of the fifth order mock theta function under certain condition.

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^j q^{\frac{n(5n+1)}{2} - j^2} (1 - q^{4n+2}),\ |j| \leqslant n \tag{18}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^n q^{5n^2+2n-(\frac{j+1}{2})} (1 + q^{6n+3}) \tag{19}$$

$$1 + 2\psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{(\frac{n+1}{2})}$$

$$= \frac{(-q)_\infty}{(q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{2n^2+n} - 2\sum_{n=1}^{\infty} (-1)^j q^{\frac{n(5n-1)}{2} - \frac{j(3j+1)}{2} j} (1 - q^n) \right),\ |j| < n \tag{20}$$

$$\phi_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2} = \frac{(-q;q^2)}{(q^2;q^2)} \sum_{n=0}^{\infty} (-1)^j q^{5n^2+2n-3j^2-j} (1 - q^{6n+3})\ ,|j| \leqslant n \tag{21}$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n} = \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} (-1)^j q^{\frac{n(5n+3)}{2} - j^2} (1 - q^{2n+1})\ ,|j| \leqslant n \tag{22}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^n q^{5n^2+4n-(\frac{j+1}{2})} (1 + q^{2n+1}) \tag{23}$$

$$\psi_1(q)=\sum_{n=0}^{\infty}(-q)_n q^{\binom{n+1}{2}}=\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^j q^{\frac{n(5n+3)}{2}-\frac{j(3j+1)}{2}}(1-q^{2n+1})\ ,|j|\leqslant n \quad (24)$$

$$\phi_1(q)=\sum_{n=0}^{\infty}(-q;q^2)_n q^{(n+1)^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^j q^{5n^2+4n-3j^2-j}(1+q^{2n+1})\ ,|j|\leqslant n \quad (25)$$

Zwegers [23,p.72] has also recorded all above identities. But his identities, parallel to identities given by equations (20)and (25)are slightly different, as

$$1+2\psi_0(q)=\frac{(-q)_\infty}{(q)_\infty}\left(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+n}-2\sum_{n=1}^{\infty}(-1)^j q^{\frac{5}{2}n^2-\frac{1}{2}n-\frac{3}{2}j^2-\frac{1}{2}j}(1-q^n)\right),|j|<n \quad (25a)$$

$$\phi_1(q)=q\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^j q^{5n^2+4n-3j^2-j}(1-q^{2n+1})\ ,|j|\leqslant n \quad (25b)$$

In [12,p.125,Theorem 9], Andrews has recorded exactly all equations (18)to(25). In [23, p.72], Zwegers has recorded equations (18)to(19) and (21)to(24), and on the place of equations (20) and (25) in [12], he has recorded equations (20a) and (25a) in [23].

Zwegers has considered identities given by equations (18)to(25) as the definitions of the mock theta functions, and he write four of above identities given by equations (18)to(21) in more suitable form by following six equations [23, p.73-74], as

$$2\eta(\tau)q^{-\frac{1}{60}}f_0(q)$$
$$=\sum_{\nu\in\begin{pmatrix}1/10\\0\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}0\\1/4\end{pmatrix})} \quad (26)$$

$$2\eta(\tau)q^{\frac{11}{60}}f_1(q)$$
$$=\sum_{\nu\in\begin{pmatrix}3/10\\0\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}0\\1/4\end{pmatrix})} \quad (27)$$

$$2\eta(\tau)q^{-\frac{1}{240}}(-1+F_0(q^{\frac{1}{2}}))$$
$$=\sum_{\nu\in\begin{pmatrix}1/5\\1/4\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}1/2\\1\end{pmatrix})} \quad (28)$$

$$2\eta(\tau)q^{\frac{71}{240}}F_1(q^{\frac{1}{2}})$$
$$= \sum_{\nu\in\begin{pmatrix}2/5\\1/4\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}1/2\\2\end{pmatrix})} \tag{29}$$

$$2\eta(\tau)\zeta_8^{-1}q^{-\frac{1}{240}}(-1+F_0(-q^{\frac{1}{2}}))$$
$$= \sum_{\nu\in\begin{pmatrix}1/5\\1/4\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}0\\1/4\end{pmatrix})} \tag{30}$$

$$2\eta(\tau)\zeta_8^{-1}q^{\frac{71}{240}}F_1(-q^{\frac{1}{2}})$$
$$= \sum_{\nu\in\begin{pmatrix}2/5\\1/4\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}0\\1/4\end{pmatrix})} \tag{31}$$

with $A=\begin{bmatrix}5 & 0\\0 & -2\end{bmatrix}, c_1=\begin{bmatrix}2\\5\end{bmatrix}, c_2=\begin{bmatrix}-2\\5\end{bmatrix}$

Zwegers has written above six mock theta functions given by equations (26)to(31), in to a single vector-valued mock theta function, as

$$F_{5,1}(\tau)=\begin{bmatrix} q^{-\frac{1}{60}}f_0(q)\\ q^{\frac{11}{60}}f_1(q)\\ q^{-\frac{1}{240}}(-1+F_0(q^{\frac{1}{2}}))\\ q^{\frac{71}{240}}F_1(q^{\frac{1}{2}})\\ q^{-\frac{1}{240}}(-1+F_0(-q^{\frac{1}{2}}))\\ q^{\frac{71}{240}}F_1(-q^{\frac{1}{2}}) \end{bmatrix} \tag{32}$$

Zwegers [23, p.77-79], also write corresponding to other four identities given by equations (22)to(25), in more suitable form by following six equations, as

$$2\zeta_{12}^{-1}\frac{\eta(\tau)^2}{\eta(2\tau)}q^{-\frac{1}{60}}\psi_0(q)$$
$$= \sum_{\nu\in\begin{pmatrix}1/10\\1/6\end{pmatrix}+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\begin{pmatrix}0\\1/6\end{pmatrix})} \tag{33}$$

$$2\zeta_{12}^{-1}\frac{\eta(\tau)^2}{\eta(2\tau)}q^{\frac{11}{60}}\psi_1(q)$$

$$= \sum_{\nu\in\left(\begin{array}{c}3/10\\1/6\end{array}\right)+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\left(\begin{array}{c}0\\1/6\end{array}\right))} \quad (34)$$

$$2\zeta_{60}\frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})}q^{-\frac{1}{240}}\varphi_0(q^{-\frac{1}{2}})$$

$$= \sum_{\nu\in\left(\begin{array}{c}1/5\\1/6\end{array}\right)+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\left(\begin{array}{c}1/10\\1/6\end{array}\right))} \quad (35)$$

$$-2\zeta_{60}^{7}\frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})}q^{-\frac{49}{240}}\varphi_1(q^{-\frac{1}{2}})$$

$$= \sum_{\nu\in\left(\begin{array}{c}2/5\\1/6\end{array}\right)+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\left(\begin{array}{c}1/10\\1/6\end{array}\right))} \quad (36)$$

$$2\zeta_{16}^{-1}\frac{\eta(\tau)^2}{\eta(\frac{\tau+1}{2})}q^{-\frac{1}{240}}\varphi_0(q^{\frac{1}{2}})$$

$$= \sum_{\nu\in\left(\begin{array}{c}1/5\\1/6\end{array}\right)+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\left(\begin{array}{c}0\\1/6\end{array}\right))} \quad (37)$$

$$2\zeta_{16}^{-1}\frac{\eta(\tau)^2}{\eta(\frac{\tau+1}{2})}q^{-\frac{49}{240}}\varphi_1(q^{\frac{1}{2}})$$

$$= \sum_{\nu\in\left(\begin{array}{c}2/5\\1/6\end{array}\right)+Z^2}\{sgn(B(\nu,c_1))-sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\left(\begin{array}{c}0\\1/6\end{array}\right))} \quad (38)$$

with $A=\left[\begin{array}{cc}5&0\\0&-3\end{array}\right], c_1=\left[\begin{array}{c}3\\5\end{array}\right], c_2=\left[\begin{array}{c}-3\\5\end{array}\right]$

Zwegers has written above six mock theta functions given by equations (33)to(38), in to a single vector-valued mock theta function, as

$$F_{5,2}(\tau) = \begin{bmatrix} 2q^{-\frac{1}{60}}\psi_0(q) \\ 2q^{\frac{11}{60}}\psi_1(q) \\ q^{-\frac{1}{240}}\varphi_0(-q^{\frac{1}{2}}) \\ -q^{-\frac{49}{240}}\varphi_1(-q^{\frac{1}{2}}) \\ q^{-\frac{1}{240}}\varphi_0(q^{\frac{1}{2}}) \\ q^{-\frac{49}{240}}\varphi_1(q^{\frac{1}{2}}) \end{bmatrix} \tag{39}$$

In [23,p.67 and 10,p.666], we find the following identities (slightly re-written), for the seventh order mock theta functions, as

$$(q)_\infty f_0(q) = (\sum_{r,s\geqslant 0} - \sum_{r,s<0})(-1)^{r+s}q^{\frac{3}{2}r^2+4rs+\frac{3}{2}s^2+\frac{1}{2}r+\frac{1}{2}s} \tag{40}$$

$$(q)_\infty f_1(q) = (\sum_{r,s\geqslant 0} - \sum_{r,s<0})(-1)^{r+s}q^{\frac{3}{2}r^2+4rs+\frac{3}{2}s^2+\frac{5}{2}r+\frac{5}{2}s+1} \tag{41}$$

$$(q)_\infty f_2(q) = (\sum_{r,s\geqslant 0} - \sum_{r,s<0})(-1)^{r+s}q^{\frac{3}{2}r^2+4rs+\frac{3}{2}s^2+\frac{3}{2}r+\frac{3}{2}s} \tag{42}$$

In [23,p.67], above three identities given by equations (40)to(42), re-written, as

$$2\eta(\tau)\zeta_{14}q^{-\frac{1}{168}}f_0(q) = \sum_{\nu\in\frac{1}{14}e+Z^r}\{sgn(B(\nu,c_1)) - sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{1}{14}e)} \tag{43}$$

$$2\eta(\tau)\zeta_{14}q^{\frac{47}{168}}f_2(q) = \sum_{\nu\in\frac{3}{14}e+Z^r}\{sgn(B(\nu,c_1)) - sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{5}{14}e)} \tag{44}$$

$$2\eta(\tau)\zeta_{14}q^{-\frac{25}{168}}f_1(q) = \sum_{\nu\in\frac{5}{14}e+Z^r}\{sgn(B(\nu,c_1)) - sgn(B(\nu,c_2))\}e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{3}{14}e)} \tag{45}$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$, with $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$, $c_1 = c_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have $B(c_1, c_2) = -28$ and $Q(c_1) = Q(c_2) = -\frac{21}{2}$, such that $c_1 \in C_Q, c_2 \in C_Q$.

Zwegers represent above three mock theta functions, given by equations (43)to(45),into a single vector- valued mock theta function[23, p.67], as

$$F_7(\tau) = \begin{bmatrix} q^{-\frac{1}{168}}f_0(q) \\ q^{\frac{47}{168}}f_2(q) \\ q^{-\frac{25}{168}}f_1(q) \end{bmatrix} \tag{46}$$

Zwegers also recorded vector- valued third order mock theta function [23, p.84], as

$$F(\tau) = \begin{bmatrix} q^{-\frac{1}{24}} f(q) \\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}) \\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}) \end{bmatrix} \tag{47}$$

Some new mock theta functions:

During last three decades number of papers related to mock theta functions have been appeared in various scientific research journals, and some new mock theta functions have been studied by researchers as following

Second order mock theta functions:

In 2007, McIntosh [32], studied the following second order mock theta functions

$$A(q) = \sum_{n\geqslant 0} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)^2_{n+1}} = \sum_{n\geqslant 0} \frac{q^{(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}} \tag{48}$$

$$B(q) = \sum_{n\geqslant 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)^2_{n+1}} = \sum_{n\geqslant 0} \frac{q^{(n)}(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{49}$$

$$\mu(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)^2_n} \tag{50}$$

The function μ was recorded by Ramanujan in his lost notebook [24]

Sixth order mock theta functions:

In his lost notebook [24], Ramanujan recorded sixth order mock theta functions (total seven in numbers), and he also given eleven identities for them, which were proved by Andrews and Hickerson [14]. Ramanujan's two identities relate ϕ and ψ at different arguments, four express ϕ and ψ in terms of Appell-Lerch series, five express the fifth order mock theta functions relate ϕ and ψ, recently Berndt and Chan [5] have discovered two new sixth order mock theta functions. Hence, we have following sixth order mock theta functions;

$$\phi(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}} \tag{51}$$

$$\psi(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}} \tag{52}$$

$$\rho(q) = \sum_{n\geqslant 0} \frac{q^{\frac{n(n+1)}{2}}(-q;q)_n}{(q;q^2)_{n+1}} \tag{53}$$

$$\sigma(q) = \sum_{n\geqslant 0} \frac{q^{\frac{(n+1)(n+2)}{2}}(-q;q)_n}{(q;q^2)_{n+1}} \tag{54}$$

$$\lambda(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n} \tag{55}$$

$$2\mu(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \tag{56}$$

$$\gamma(q) = \sum_{n\geqslant 0} \frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \tag{57}$$

$$\phi_-(q) = \sum_{n\geqslant 1} \frac{q^n(-q;q)_{2n-1}}{(q;q^2)_n} \tag{58}$$

$$\psi_-(q) = \sum_{n\geqslant 1} \frac{q^n(-q;q)_{2n-2}}{(q;q^2)_n} \tag{59}$$

Eight order mock theta functions:

Gordon and Mclntosh [8],discovered eight order mock theta functions (total eight in number), and found five linear relations involving them, expressed four of the functions as Appell-Lerch sums, and also express their transformations under the modular group.

$$S_0(q) = \sum_{n\geqslant 0} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{60}$$

$$S_1(q) = \sum_{n\geqslant 0} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{61}$$

$$T_0(q) = \sum_{n\geqslant 0} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{62}$$

$$T_1(q) = \sum_{n\geqslant 0} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{63}$$

$$U_0(q) = \sum_{n\geqslant 0} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \tag{64}$$

$$U_1(q) = \sum_{n\geqslant 0} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \tag{65}$$

$$V_0(q) = -1 + 2\sum_{n\geqslant 0} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = -1 + 2\sum_{n\geqslant 0} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}} \tag{66}$$

$$V_1(q) = \sum_{n\geqslant 0} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n\geqslant 0} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}} \tag{67}$$

Ramanujan already recorded two of above functions represented by U_0 and V_1, [24,p.8, equation(1)and p.29, equation(6)]. Second order mock theta functions given by equations (48) to (50) and eight order mock theta functions given by equations (64)to(69) are having following relations between them

$$U_0(q) - 2U_1(q) = \mu(q) \tag{68}$$

$$V_0(q) - V_0(-q) = 4qB(q^2) \tag{69}$$

$$V_1(q) + V_1(-q) = 2A(q^2) \tag{70}$$

Tenth order mock theta functions:

Ramanujan recorded following tenth order mock theta functions (total four in number)in his lost notebook [24,p.9], and also introduced some relations between them

$$\phi(q) = \sum_{n\geqslant 0} \frac{q^{\frac{n(n+1)}{2}}}{(q;q^2)_{n+1}} \tag{71}$$

$$\psi(q) = \sum_{n\geqslant 0} \frac{q^{\frac{(n+1)(n+2)}{2}}}{(q;q^2)_{n+1}} \tag{72}$$

$$X(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}} \tag{73}$$

$$\chi(q) = \sum_{n\geqslant 0} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}} \tag{74}$$

All above tenth order mock theta functions given by equations (71)to(74), were proved by Y.-S. Choi [33-36]

History of mock theta conjectures:

Ramanujan's "lost" notebook discovered by Andrews in 1976, contains a number of q-series identities involving Ramanujan's original mock theta functions (total seventeen in number). The results in his letter found the basis on which the intermittent study of these functions has proceeded [12,18,19]. Ramanujan included in his letter [25, pp.354-355] four separate class of mock theta functions - (i). one class of third order mock theta functions, (ii). two class of fifth order mock theta functions, and (iii). one class of seven order mock theta functions.In his "lost" notebook [24, pp. 18-20], Ramanujan stated representative of the two classes of five identities, also recorded by Berndt [4, p.10], as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} + 2 - 2\sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \tag{75}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n} = \frac{(q^5;q^5)_\infty (q^5;q^{10})_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} + \frac{2}{q} - \frac{2}{q}\sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n} \tag{76}$$

Watson [19] define, as

$$\chi_0(q) = \sum_{n\geqslant 0} \frac{q^{n^2+n}}{(q^{n+1};q)_n} \tag{77}$$

$$\chi_1(q) = \sum_{n\geqslant 0} \frac{q^{n}}{(q^{n+1};q)_{n+1}} \tag{78}$$

where $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$.

Zwegers [23] has found following two identities for fifth order mock theta functions, $\chi_0(q)$ and $\chi_1(q)$, as

$$\chi_0(q) = 2 - \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geqslant 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2kl+2km+2lm+\frac{1}{2}(k+l+m)} \tag{79}$$

$$\chi_1(q) = \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geqslant 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2kl+2km+2lm+\frac{3}{2}(k+l+m)} \tag{80}$$

The mock theta conjectures related the functions $\chi_0(q)$ and $\chi_1(q)$ to differences of "rank generating functions"[1]

$$R_{b,c}(d;q) = \sum_{n\geqslant 0}[N(b,5,5n+d) - N(c,5,5n+d)]q^n \tag{81}$$

In equation (81), $N(b,t,r)$ denotes the number of partitions of r with rank congruent to b mod t. Now we describe rank, as

Rank [11]: The number of parts of the partition subtracted from the largest part of the partition. As example, the partition $1+1+1+3$ of 6 has rank equal to $3-4=-1$

Andrews and Garvan [13], establish that the 5th order identities naturally divide into two classes and prove that any given identity is true only when all identities are true for every number of its class. Hence the truth of all 5th order identities reduces to the form of the truth of two identities which are called "mock theta conjectures".

The mock theta conjectures: Folsom [1, p.4144], has stated as

$$First : \chi_0(q) - 1 = R_{1,0}(0;q) \tag{82}$$

$$Second : \chi_1(q) = R_{2,1}(3;q) + R_{2,0}(3;q) \tag{83}$$

Summary of proofs obtain for mock theta conjectures:

1. In 1986, Andrews [12], discovered Hecke-type identities, and in 1988 with help of these identities Hickerson [9], proved the mock theta conjectures. we state Heck modular form identities, as **Heck modular form identities[12,p.120, equation(5.1)]:** E.Hecke made an extensive study of double theta type series involving an indefinite

quadratic form, and showed that,

$$(q)_\infty^2 = \sum_{m=\infty}^{\infty} \sum_{n \geqslant 2|m|} (-1)^{n+m} q^{\left(\begin{array}{c} n+1 \\ 2 \end{array} \right) - \frac{m(3m-1)}{2}} \tag{84}$$

2. After proceeding in a systemic manner for more general mock theta-type identities, the families of modular forms have been constructed by Bringmann, Ono and Rhoades [22], and these authors states, as

....the results of Zwegers' thesis.......and the proof of [22, Theorem 1.1].... reduces the proof of the mock theta conjectures to the verification of two simple identities for classical weakly holomorphic modular forms.

Folsom [1], in response of above remark, provide a short proof of the mock theta conjectures by considering each side of the identities as the holomorphic projection of a harmonic weak Maass form and also by using following results. As described in [21 and 23], not much was known regarding the role of Ramanujan's mock theta functions within the context of modular forms. Zwegers [23], completed several of Ramanujan's mock theta functions to obtain weight 1/2 weak Maass forms, and Bringmann and Ono [21] have further generalize the results of Zwegers [23].

Open problem: Combinatorial proof for "mock theta conjectures".

Hints/ Possibility: Andrews is leading the twenty first century in providing combinatorial proofs of q-series identities. One can proceed with the help of research work obtained by him.

Acknowledgement: Author is thankful to the Department of Science and Technology, Government of India, New Delhi, for the financial assistance, and Centre for Mathematical Sciences for providing necessary facilities.

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Received 23 11 2011, revised 27 08 2012

International Scientific Research and
Welfare Organization,
New Delhi -110008, India
E-mail address: `mpchaudhary_2000@yahoo.com`