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Ergodic Properties of the Fröbenius–Perron Semigroup

Sergio Plaza *

Abstract. We consider the Fröbenius–Perron semigroup of linear operators associated to a semidynamical system defined in a topological space X endowed with a finite or a σ–finite regular measure. Assuming strong continuity for the Fröbenius–Perron semigroup of linear operators in the space $L^1_\mu(X)$ or in the space $L^p_\mu(X)$ for $1 < p < \infty$ ([**11**]). We study in this article ergodic properties of the Fröbenius–Perron semigroup of linear operators.

1. Introduction

An important problem in the study of the dynamics of nonsingular transformations is to know whether or not they admit an absolutely continuous invariant measure (a.c.i.m.). For interval maps, for example, we have a well known theorem of Lasota-Yorke [**10**], which roughly states that if the map is piecewise smooth (C^r, with $r \geq 2$) and expanding, then it admits an a.c.i.m., and that with some additional conditions, it is exact with respect to this a.c.i.m. (see [**10**] for more details); extensions of this result have been obtained for the n–dimensional case (see [**2**]). When we deal with a continuous semi-dynamical or a dynamical system, that is, with a semi–flow or a flow, the problem is more complicated.

A useful tool to study the problem of the existence of an a.c.i.m. is the Fröbenius–Perron operator (see [**1**], [**9**] for more details). Let X be a topological space and let μ be a regular measure defined on the Borel σ–algebra of X (see below for definition). Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that a transformation $\tau : X \to X$ is *nonsingular* if for all $A \in \mathcal{A}$ such that $\mu(A) = 0$, we have $\mu(\tau^{-1}(A)) = 0$. If a transformation $\tau : X \to X$ is nonsingular then associated to it there exists a linear operator $P_\tau = P_{\tau,\mu} : L^1_\mu(X) \to L^1_\mu(X)$, called *Fröbenius–Perron operator* which is

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characterized by the relation:

$$\int_A P_\tau(f)d\mu = \int_{\tau^{-1}(A)} f d\mu \tag{F–P}$$

for all $f \in L^1_\mu(X)$ and all $A \in \mathcal{A}$. In fact, P_τ depends also on μ, and some times the notation $P_{\tau,\mu}$ is used in order to indicate such dependence on the measure. It is well known that an invariant density, that is, a non negative measurable function of unit norm and fixed for the Fröbenius–Perron operator corresponds to a density of an a.c.i.m. for the transformation τ.

Let $\tau_t : X \to X$ be a semi-dynamical system. Denote by P_t the Fröbenius–Perron operator associated to the transformation τ_t. The family $\{P_t\}_{t\geq 0}$ satisfies

$$\begin{cases} P_0 &= Id, \\ P_{t+s} &= P_t \circ P_s, \qquad \text{for all} \quad t,s \geq 0. \end{cases}$$

Therefore, $\{P_t\}_{t\geq 0}$ is a semigroup of linear operators on $L^1_\mu(X)$.

For a semigroup of continuous linear operators defined in the space $L^1_\mu(X)$ a central problem is to know conditions for the semigroup to be strongly continuous, that is, the following condition holds:

$$\lim_{t\to 0} P_t(f) = f,$$

for all $f \in L^1_\mu(X)$. If this is the case, we may consider the infinitesimal generator of the semigroup, defined by $A(f) = \lim_{t\to 0} \frac{P_t(f)-f}{t}$, for elements $f \in L^1_\mu(X)$ for which the above limit exists (see [**3**], [**12**]). It is known that a function $f : X \to \mathbb{R}$ satisfies $P_t(f) = f$, for all $t \geq 0$, if and only if A is defined for f and the differential equation $A(f) = 0$ is satisfied. In this manner, the problem of finding or proving the existence of an a.c.i.m. for the semi-dynamical system is equivalent to the problem of finding or proving the existence of a nontrivial zero for the infinitesimal generator of the Fröbenius–Perron semigroup of linear operators associated, provided that this semigroup is strongly continuous.

Assuming the existence of a *faithful invariant measure*, that is, an a.c.i.m. with positive density for the semi-dynamical system, or by using the following geometric condition on the semi-dynamical system: *there exists* $T > 0$ *such that*

$$\frac{\mu(\tau_t^{-1}(A))}{\mu(A)} \leq M \tag{GC}$$

for all $t \leq T$ and for all $A \in \mathcal{A}$, we say that a semi-dynamical system is *strongly non-singular* if it satisfies condition (GC). In [**11**], the strong continuity of the Fröbenius–Perron semigroup of linear operators in the space $L^1_\mu(X)$ was proved. In order to ensure the strong continuity of the Fröbenius–Perron semigroup of linear operators on the space $L^p_\mu(X)$, we need that the semi-dynamical system must be strongly non-singular and satisfy the following condition: for each closed subset $K \subset X$ and for each $t > 0$, the set $\cup_{s\leq t}\tau_s^{-1}(K)$ is a closed set in X; in this case, we say that the semi-dynamical system is *proper*.

Following the results obtained by the authors in [**11**] (the strong continuity of the Fröbenius–Perron semigroup of linear operators on the space $L^1_\mu(X)$, and on the space $L^p_\mu(X)$ for $1 < p < \infty$): We study in this paper general ergodic properties of the Fröbenius–Perron semigroup of linear operators. For example, we study the local and the ratio ergodic theorems for this class of semigroups of linear operators and we deduce some features about the dynamics of the semi-dynamical systems $\{\tau_t\}_{t\geq 0}$, as illustrated in the following theorem:

Theorem *Let M be a connected finite–dimensional smooth manifold, endowed with a Riemannian metric $|\cdot|$, and let m be the corresponding Borel measure. Assume that $\{\tau_t\}_{t\geq 0}$ is a C^r ($r \geq 2$) expanding semi-dynamical system. Then, the associated Fröbenius–Perron semigroup of operators $\{P_t\}_{t\geq 0}$ is asymptotically stable in the space $L^1_m(M)$, that is, there exists a unique density f_* such that (i) $P_t f_* = f_*$ for all $t \geq 0$, and (ii) $\lim_{t\to\infty} ||P_t f - f_*||_1 = 0$ for all densities f in $L^1_m(M)$. Furthermore, the semi-dynamical system $\{\tau_t\}_{t\geq 0}$ with the measure*

$$m_*(A) = \int_A f_*(x)dm\,,$$

for all A in the Borel σ–algebra on M, is exact. Moreover, m_* is the unique absolutely continuous normalized measure invariant under $\{\tau_t\}_{t\geq 0}$.

In section 2 we set up the notation and recall some basic results from semigroup theory; the definition and some useful properties of the Fröbenius–Perron semigroups are also given.

In section 3 we establish and prove our results in the space $L^1_\mu(X)$.

In section 4 we give analogous results to those obtained in section 3, for the $L^p_\mu(X)$ $1 < p < \infty$ case.

2. Basic Results

In this section we give a survey of definitions, results and notations that are necessary for the sequel.

2.1. Semi-dynamical Systems. Let X be a topological space. A family $\{\tau_t\}_{t\geq 0}$ of continuous transformations $\tau_t : X \to X$ is a *semi-dynamical system* if the following conditions are satisfied:

i) $\tau_0 = Id$,
ii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \geq 0$, and
iii) the map $[0,\infty] \times X \to X$ given by $(t,x) \to \tau_t(x)$ is continuous.

If each transformation τ_t has a continuous inverse τ_{-t} then the family $\{\tau_t\}_{t\in\mathbb{R}}$ is a continuous flow, that is, a continuous dynamical system.

We say that a semi-dynamical system $\{\tau_t\}_{t\geq 0}$ is *proper* if, for each compact set $K \subset X$ and for each $t > 0$, the set $\cup_{s\leq t}\tau_s^{-1}(K)$ is compact.

2.2. Semigroups in Banach Spaces. Let V be a Banach space with respect to a norm $\|\cdot\|$. A family $\{T_t\}_{t\geq 0}$ of continuous linear operators $T_t : V \to V$ ($t \geq 0$) is a *semigroup of linear operators* if the following conditions are satisfied:

$$\begin{cases} T_0 & = & Id, \quad \text{and} \\ T_{t+s} & = & T_t \circ T_s, \;\text{for all } t, s \geq 0 \end{cases}$$

(see [**12**] or [**3**] for more details). We recall that a semigroup $\{T_t\}_{t\geq 0}$ of linear operators is *strongly continuous*, if

$$\lim_{t\to 0} \|T_t(f) - f\| = 0 \qquad \text{for all } \ f \in V.$$

If $\{T_t\}_{t\geq 0}$ is a semigroup defined on V, then the adjoint family $\{T_t^*\}_{t\geq 0}$ is a semigroup defined on the dual space V^*. By duality, we have that if $\{T_t\}_{t\geq 0}$ is strongly continuous and V is reflexive, then $\{T_t^*\}_{t\geq 0}$ is strongly continuous in V^* ([**12**], Corollary 10.6, page 41).

2.3. Fröbenius–Perron Operator. Let $(X, \mathcal{A}, \mu)$ be a measure space. A transformation $\tau : X \to X$ is said to be *nonsingular* if $\mu(\tau^{-1}(A)) = 0$ for all $A \in \mathcal{A}$ such that $\mu(A) = 0$. Associated to a nonsingular transformation $\tau : X \to X$ there exists a linear operator $P_\tau = P_{\tau,\mu} : L^1_\mu(X) \to L^1_\mu(X)$, known as *Fröbenius–Perron operator*, which is characterized by the condition:

$$\int_A P_\tau(f) d\mu = \int_{\tau^{-1}(A)} f d\mu \tag{4}$$

for all $f \in L^1_\mu(X)$ and all $A \in \mathcal{A}$.

It is well known ([**1**], [**9**]) that a probability measure μ on X is *τ–invariant* (that is, $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$) if and only if $P_\tau(1) = 1$. In general, τ preserves a measure $d\nu = f d\mu$, with $f \in L^1_\mu(X)$, if and only if $P_\tau(f) = f$. It is also well known that the Fröbenius–Perron operator is a linear contraction in $L^1_\mu(X)$ endowed with the L^1_μ–norm, that is, $||P_\tau||_{L^1_\mu} \leq 1$ ([**9**], [**2**]). Moreover, for $f \in L^1_\mu(X)$ and a.e. $x \in X$, we have $|P(f)(x)| \leq P(|f|)(x)$.

Another important property of the Fröbenius–Perron operator is given by the equality

$$\int_X P_\tau(f) \cdot g d\mu = \int_X f \cdot (g \circ \tau) d\mu, \tag{6}$$

valid for all $f \in L^1_\mu(X)$ and all $g \in L^\infty_\mu(X)$. Equation (6) permits us to define a linear operator $K_\tau : L^\infty_\mu(X) \to L^\infty_\mu(X)$ given by $K_\tau(g) = g \circ \tau$. The operator K_τ is well defined if τ is a nonsingular transformation. This operator is called the *Koopman operator.* For more details about these concepts see [**1**] or [**9**].

If we have a semi-dynamical system $\{\tau_t\}_{t\geq 0}$ such that each transformation τ_t is nonsingular, then we denote the family of Fröbenius–Perron operators associated to τ_t by $P_t = P_{\tau_t}$; this is a semigroup of continuous linear operators in the space $L^1_\mu(X)$ [**9**]. We will also use the notation K_t for the Koopman operator K_{τ_t}.

We note that the Fröbenius–Perron operator may also be defined and is bounded in other spaces of functions if the transformation τ has good behavior, for example, spaces $L^p_\mu(X)$ $(p > 1)$, or $BV(X)$, the space of functions of bounded variation on X. For example, in section 4 we consider this operator on the space $L^p_\mu(X)$ $(p > 1)$, and we prove that the geometric condition (CG) ensures continuity of each operator P_t (and also the strong continuity of the semigroup $\{P_t\}_{t\geq 0}$ on the space $L^p_\mu(X)$ for $p > 1$). We note that if P_t is continuous in $L^p_\mu(X)$, then the duality equation (6) is valid for all $f \in L^p_\mu(X)$ and for all $g \in L^q_\mu(X)$, with $\frac{1}{p} + \frac{1}{q} = 1$, that is, K_t is the adjoint operator of P_t.

3. Ergodic Properties of the Fröbenius–Perron Semigroup in L^1

Let X be a topological space and let $\mathcal{A}$ be its Borel σ–algebra. Let μ be a measure defined over $\mathcal{A}$. We recall that the measure μ is *regular* if for all $A \in \mathcal{A}$ we have

$$\mu(A) = \sup\{\mu(K) \ : \ K \subset A, \ K \text{ compact}\} = \inf\{\mu(C) \ : \ A \subset C, \ C \text{ open}\}.$$

We note that a probability measure defined on the Borel σ–algebra on a metric space X is regular. In general, if a measure μ is regular then the set of continuous functions with compact support is dense in the space $L^p_\mu(X)$, for all $1 \leq p < \infty$.

Under the above conditions we have the following theorems (see [**11**]).

Theorem 1. Suppose that μ is a regular probability measure defined on the Borel σ–algebra on X and that the semi-dynamical system $\{\tau_t\}_{t\geq 0}$ has a faithful a.c.i.m., that is, an a.c.i.m. with positive density. Then the semigroup $\{P_t\}_{t\geq 0}$ is strongly continuous in $L^1_\mu(X)$.

Theorem 2. Let X be a topological space endowed with a regular probability measure μ and let $\{\tau_t\}_{t\geq 0}$ be a proper semi-dynamical system. If the semi-dynamical system is strongly nonsingular, then the associated Fröbenius–Perron semigroup of operators $\{P_t\}_{t\geq 0}$ is strongly continuous on the space $L^1_\mu(X)$.

As a consequence of the above results, we have the following theorem.

Theorem 3. Let X be a topological space endowed with a regular probability measure μ. Let $\{\tau_t\}_{t\geq 0}$ be a continuous semi-dynamical system defined on X. If the semi-dynamical system $\{\tau_t\}_{t\geq 0}$ has a faithful a.c.i.m.. or it is a proper semi-dynamical system, then

(I) $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_0^h P_t f(x)dt = f(x)$ for almost all $x \in X$ and for all $f \in L^1_\mu(X)$.

(II) $\displaystyle\lim_{h\to 0} \frac{\int_0^h P_t f(x)dt}{\int_0^h P_t g(x)dt} = \frac{f(x)}{g(x)}$ for all $f, g \in L^1_\mu(X)$, with $g \geq 0$, on the set $\{x \in X \ : \ g(x) \neq 0\}$.

(III) For all $f, g \in L^1_\mu(X)$ with $g \geq 0$, the limit $\displaystyle\lim_{\alpha\to\infty} \frac{\int_0^\alpha P_t f(x)dt}{\int_0^\alpha P_t g(x)dt}$ exists and it is finite for almost all x in the set $\{x \in X \ : \ \sup_{\alpha>0} \int_0^\alpha P_t g(x)dt > 0\}$.

Proof. Parts (I) and (II) of the theorem follow from the theorem in [**5**] or from the main theorem and its corollary in [**7**]. Part (III) follows directly from theorem 2.1 in [**4**].

Under the hypotheses of theorems 1 or 2, we have the following theorem.

Theorem 4. Suppose that the Fröbenius–Perron semigroup of linear operators $\{P_t\}_{t\geq 0}$ associated to a semi-dynamical systems $\{\tau_t\}_{t\geq 0}$ is strongly continuous. If $P_{t_0} f_0 = f_0$ for some $t_0 > 0$ and some density f_0, then

$$f_*(x) = \frac{1}{t_0}\int_0^{t_0} P_t f_0(x)dt$$

is an invariant density for the Fröbenius–Perron semigroup.

Proof. See [**9**] page 246, proposition 7.12.1.

Now let M be a finite–dimensional connected smooth manifold, endowed with a Riemannian metric $|\cdot|$, and let m be the corresponding Borel measure. We say that a C^r mapping ($r \geq 1$) $f: M \to M$ is *expanding* if there exists a constant $\lambda > 1$ such that for all $x \in M$ and for all $v \in T_x M$ we have that $|Df(x)v| \geq \lambda |v|$.

We say that a C^r, $r \geq 1$, semi-dynamical system $\{\tau_t\}_{t\geq 0}$ is *expanding* if for all $t > 0$ each mapping $\tau_t : M \to M$ is an expanding map.

Now we have the following theorem:

Theorem 5. Let M be a finite–dimensional connected smooth manifold, endowed with a Riemannian metric $|\cdot|$, and let m be the corresponding Borel measure. Assume that $\{\tau_t\}_{t\geq 0}$ is a C^r ($r \geq 2$) expanding semi-dynamical system. Then the associated Fröbenius–Perron semigroup of operators $\{P_t\}_{t\geq 0}$ is asymptotically stable in the space $L^1_m(M)$, that is, there exist a unique density f_* such that (i) $P_t f_* = f_*$ for all $t \geq 0$, and (ii) $\lim_{t\to\infty} ||P_t f - f_*||_1 = 0$ for all densities f in $L^1_m(M)$. Furthermore, the semi-dynamical system $\{\tau_t\}_{t\geq 0}$ with measure

$$m_*(A) = \int_A f_*(x)dm$$

for all A in the Borel σ–algebra on M, is exact. Moreover, m_* is the unique absolutely continuous normalized measure which is invariant under $\{\tau_t\}_{t\geq 0}$.

Proof. By hypothesis, for each $t > 0$, the map $\tau_t : M \to M$ is a C^r ($r \geq 2$) expanding map; thus it follows from the main theorem in [**6**] that the associated Fröbenius–Perron operator $P_t : L^1_m(M) \to L^1_m(M)$ is asymptotically stable. Hence, if we fix some $t_0 > 0$ we have that the linear operator $P_{t_0} : L^1_m(M) \to L^1_m(M)$ has a unique invariant density f_0 and the result follows from theorem 4 above.

Theorem 6. Let $\{\tau_t\}_{t\geq 0}$ be a nonsingular semi-dynamical system and let $\{P_t\}_{t\geq 0}$ be its corresponding Fröbenius–Perron semigroup of linear operators defined on the space $L^1_\mu(X)$. Suppose that, for some $t_0 > 0$, the linear operator $P = P_{t_0}$ has a unique invariant density f_* respect to which P is asymptotically stable, that is, $\lim_{n\to\infty} P^n f = f_*$ for all densities f. Then f_* is an invariant density for the Fröbenius–Perron semigroup of linear operators, for which $\{P_t\}_{t\geq 0}$ is asymptotically stable. In particular, the probability measure μ_* defined by $\mu_*(A) = \int_A f_* d\mu$ is an a.c.i.m. for the semi-dynamical system $\{\tau_t\}_{t\geq 0}$.

Proof. Since $P_{t_0} f_* = f_*$, it follows that $P_{nt_0} f_* = f_*$ for all $n \geq 1$. We fix an arbitrary value $s > 0$. Let $\tilde{f} = P_s f_*$. Then we have $||P_s f_* - f_*||_1 = ||P_s(P_{nt_0} f_*) - f_*||_1 = ||P_{nt_0}(P_s f_*) - f_*||_1 = ||P_{nt_0}\tilde{f} - f_*||_1$. Now, since the right–hand side of these equalities approaches 0 as $n \to \infty$, and the left–hand side does not depend on n, we deduce that $||P_s f_* - f_*||_1 = 0$, that is, $P_s f_* = f_*$. Now, since s is arbitrary it follows that $P_t f_* = f_*$ for all $t \geq 0$.

Now let f be an arbitrary density. The function $t \to ||P_t f - f_*||_1$ is a non–increasing function of the variable t. Letting $t_n = nt_0$, it follows that $\lim_{n\to\infty} ||P_{t_n} f -$

$f_*||_1 = 0$, that is, the function above is non–increasing and approaches 0 on a subsequence. Therefore, $\lim_{t\to\infty} ||P_t f - f_*||_1 = 0$, and this ends the proof of the theorem.

Under the hypotheses of theorem 4 or theorem 6, it follows that the measure

$$\mu_*(A) = \int_A f_* d\mu$$

for all A in the Borel σ–algebra is an a.c.i.m. for the semi-dynamical system $\{\tau_t\}_{t\geq 0}$. Thus, for an arbitrary integrable function $f : X \to \mathbb{R}$, the limit

$$f^*(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau_t(x))dt$$

exists for a.e. $x \in X$. This result is the well known Birkhoff's theorem.

4. Ergodic Properties in L^p, $1 < p < \infty$

In this section we consider a semi-dynamical system defined on a topological space X, provided with a regular measure μ defined on its Borel σ–algebra. In [**11**] the following result was proved.

THEOREM 7. Suppose the measure μ is regular on X. If $\{\tau_t\}_{t\geq 0}$ is a proper and strongly nonsingular semi-dynamical system, then for all $1 < p < \infty$, the associated Fröbenius–Perron semigroup $\{P_t\}_{t\geq 0}$ is a strongly continuous semigroup of linear bounded operators on the space $L^p_\mu(X)$.

From this result, jointly with [[**8**], theorem 1 and corollary] we obtain the following theorem.

THEOREM 8. Suppose the measure μ is regular on X. Let $\{\tau_t\}_{t\geq 0}$ be a proper and strongly nonsingular semi-dynamical system and let $\{P_t\}_{t\geq 0}$ be the corresponding Fröbenius–Perron semigroup of linear operators defined on $L^p_\mu(X)$ $(1 < p < \infty)$. Then

(A) $\lim_{h\to 0} \int_0^h P_t f(x)dt = f(x)$ for almost all $x \in X$ and for all $f \in L^p_\mu(X)$, and

(B) for all $f, g \in L^p_\mu(X)$ and for almost all x in the set $\{x : g(x) \neq 0\}$,

$$\lim_{h\to 0} \frac{\int_0^h P_t f(x)dt}{\int_0^h P_t g(x)dt} = \frac{f(x)}{g(x)}.$$

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Departamento de Matemática
Facultad de Ciencias,
Universidad de Santiago de Chile
Casilla 307, Correo 2. Santiago,
Chile.