SCIENTIA
Series A: *Mathematical Sciences*, Vol. 21 (2011), 63–73
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

An upper bound on the second fiber coefficient of the fiber cones

Guangjun Zhu

ABSTRACT. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$, I an $\mathfrak{m}$-primary ideal of R and K an ideal containing I. When depth $G(I) \geq d-1$ and $r(I|K) < \infty$, we present an upper bound on the second fiber coefficient $f_2(I, K)$ of the fiber cones $F_K(I)$, and also provide a characterization, in terms of $f_2(I, K)$, of the condition depth $F_K(I) \geq d-2$.

1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ having infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and K an ideal containing I. The fiber cone of I with respect to K is the standard graded algebra $F_K(I) = \bigoplus\limits_{n\geq 0} \frac{I^n}{KI^n}$. The graded algebra $F_K(I)$ for $K = \mathfrak{m}$ is called the fiber cones of I. For $K = I$, $F_K(I) = G(I)$, the associated graded ring of I. For every $n \geq 0$, we denoted by $H_K^0(I, n) := \lambda(\frac{I^n}{KI^n})$ the Hilbert function of $F_K(I)$, where λ denotes the length function. The higher iterated Hilbert function of $F_K(I)$ are defined as for every $n \geq 0$

$$H_K^i(I, n) = \begin{cases} H_K^0(I, n) & : i = 0 \\ \sum\limits_{j=0}^{n} H_K^{i-1}(I, j) & : i > 0 \end{cases}$$

It is well known [**2**, Corollary 4.1.8] that for every $i \geq 0$, $H_K^i(I, n)$ coincides with a polynomial for $n \gg 0$. Let the corresponding polynomial be denoted by

$$P_K^i(I, n) = \sum_{j=0}^{d+i-1} (-1)^j f_j(I, K) \binom{n+d+i-j-2}{d+i-j-1}.$$

We call $f_i(I, K)$ the i-th fiber coefficient of $F_K(I)$.

2000 *Mathematics Subject Classification.* 13C15, 13A15, 13A30, 13H15.

Key words and phrases. joint reduction, fiber coefficient, fiber cone, Rees-superficial sequence, depth.

This research was partially supported by the Natural Science Foundations for Colleges and Universities in Jiangsu Province (No. 10KJB110007, No. 09KJB110006) and by Pre-research Project of Soochow University (Q3107803).

The objective of this paper is to explore some connections between the fiber coefficients and depth of the fiber cone $F_K(I)$.

The relation between Hilbert coefficients and depth has been a subject of several papers in the context of the associated graded rings and Rees algebras of ideals. That conditions on Hilbert coefficients could force high depth for the associated graded rings was first observed by Sally in [**14**]. Since then numerous conditions have been proved for the Hilbert coefficients so that the associated graded ring of I, $G(I)$, is either Cohen-Macaulay or has almost maximal depth, i.e. the grade of the maximal homogeneous ideal of $G(I)$ is at least $d-1$.

Let J be a minimal reduction of I. Huckaba and Marley [**8**] gave a lower bound and an upper bound for the first Hilbert coefficient $e_1(I)$, and they also provided necessary and sufficient conditions on $e_1(I)$ so that $G(I)$ is Cohen-Macaulay and has almost maximal depth. They showed that:

(i) $e_1(I) \leq \sum\limits_{n=1}^{\infty} \lambda(I^n/JI^{n-1})$ with equality if and only if depth $G(I) \geq d-1$;

(ii) $e_1(I) \geq \sum\limits_{n=1}^{\infty} \lambda(I^n + J/J)$ with equality if and only if $G(I)$ is Cohen-Macaulay.

Corso, Polini and Rossi [**4**] gave an upper bound on $e_2(I)$ which is reminiscent of a similar bound on $e_1(I)$ due to Huckaba and Marley [**8**]. They showed that

$$e_2(I) \leq \sum_{n=2}^{\infty} (n-1)\lambda(I^n/JI^{n-1})$$

for any minimal reduction J of I. Furthermore, equality holds for some minimal reduction J of I if and only if depth $G(I) \geq d-1$.

Little is known about that translating information from the fiber coefficients into good depth properties of fiber cone $F_K(I)$. When $d \geq 2$, $a_1, \ldots, a_d$ is a joint reduction of $(I^{[d-1]}|K)$ such that $a_1^*, \ldots, a_{d-1}^*$ is a $G(I)$-regular sequence and $r_L(I|K) < \infty$ where $L = (a_1, \ldots, a_d)$. Zhu ([**15**],[**16**],[**17**]) proved that

(i) $f_1(I,K) \leq \sum\limits_{n=1}^{\infty} \lambda(\frac{KI^n}{(a_1,\ldots,a_{d-1})KI^{n-1}+a_dI^n}) - \lambda(\frac{R}{K})$; if equality holds, then depth $F_K(I) \geq d-2$.

(ii) $f_1(I,K) \geq \sum\limits_{n=1}^{\infty} \lambda(\frac{KI^n+L}{L}) - \lambda(\frac{R}{K})$. if equality occurs, then depth $F_K(I) \geq d-1$.

(iii) If $K = \bigcup\limits_{k\geq 1}(KI^k : J^k)$, then $f_2(I,K) \geq \sum\limits_{n\geq 2}(n-1)\lambda(\frac{KI^n+L}{L}) + \lambda(\frac{R}{K})$. When $d \geq 3$, $K = \bigcup\limits_{k\geq 1}[(KI^k + (a_1, \ldots, a_{d-3})) : J^k]$, $KI + (a_1, \ldots, a_{d-3}) = \bigcup\limits_{k\geq 1}[(KI^{k+1} + (a_1, \ldots, a_{d-3})) : J^k]$ and the above equality holds, then depth $F_K(I) \geq d-1$.

It is natural to consider whether the second fiber coefficient $f_2(I,K)$ has a similar upper bound on the first fiber coefficient $f_1(I,K)$ due to Zhu ([**15**],[**16**]). Under the condition that depth $G(I) \geq d-1$ and $r(I|K) < \infty$, we find a formula for $f_2(I,K)$, which generalize the bound of $e_2(I)$ obtained by Corso. el [**4**]. Namely, we show that: Let $a_1, \ldots, a_{d-1} \in I$, $a_d \in K$ be a Rees-superficial sequence for I and K. Set $J = (a_1, \ldots, a_{d-1})$, $L = (a_1, \ldots, a_d)$. If depth $G(I) \geq d-1$, and $r_L(I|K) < \infty$. Then $f_2(I, K) \leq \sum\limits_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K})$. Furthermore, the upper is attained if and only if depth $F_K(I) \geq d-2$.

2. Preliminaries

We firstly recall some basic facts about reductions from [**9**]. An ideal $J \subseteq I$ is called a reduction of I if there exists a positive integer n such that $I^{n+1} = JI^n$. A multiset of ideals consisting of m copies of an ideal I and n copies of an ideal K is denoted by $(I^{[m]}|K^{[n]})$. A sequence of elements $a_1, \ldots, a_{d-1} \in I$, $a_d \in K$ is called a joint reduction of $(I^{[d-1]}|K)$ if the ideal $(a_1, \ldots, a_{d-1})K + a_dI$ is a reduction of IK.

It is well known (cf. [**1**]) that for large values of r and s, the function $\lambda(R/K^rI^s)$ coincides with a polynomial $P(r,s)$ of total degree d in r and s. We write such a polynomial $P(r,s)$ as

$$P(r,s) = \sum_{i+j \leq d} e_{ij}(I|K)\binom{r+i}{i}\binom{s+j}{j},$$

where $e_{ij}(I|K)$ are certain integers. When $i+j=d$, we set $e_{ij}(I|K) = e_j(I|K)$ for $j = 0, \ldots, d$, these integers are called the mixed multiplicities of I and K.

An element $a \in I$ is called Rees-superficial for I and K if there exists a positive integer r_0 such that for all $r \geq r_0$ and all $s \geq 0$, $aR \cap I^rK^s = aI^{r-1}K^s$. A sequence of elements $a_1, \ldots, a_{d-1} \in I$, $a_d \in K$ is called a Rees-superficial sequence for I and K if for all $i = 1, \ldots, d$, $\overline{a_i}$ is Rees-superficial for $\overline{I}$ and $\overline{K}$, where " $^-$ " denotes residue classes in $R/(a_1, \ldots, a_{i-1})$. In this case, $a_1, \ldots, a_d$ is a joint reduction of $(I^{[d-1]}|K)$ and $e_{d-1}(I|K) = \lambda(R/(a_1, \ldots, a_d))$ by [**13**].

Let $HS_K^0(I,z) = \sum\limits_{n\geq 0} H_K^0(I,n)z^n$ be the Hilbert series of $F_K(I)$. For every $i > 0$, the iterated Hilbert series of $F_K(I)$ is defined as

$$HS_K^i(I,z) := \sum_{n\geq 0} H_K^i(I,n)z^n.$$

Since $H_K^i(I,n) - H_K^i(I,n-1) = H_K^{i-1}(I,n)$, it is easy to see that for every $i \geq 0$,

$$HS_K^i(I,z) = (1-z)HS_K^{i+1}(I,z).$$

As I is an $\mathfrak{m}$-primary ideal, there exists a unique polynomial $h(z) \in \mathbb{Z}[z]$ with $h(1) \neq 0$ such that

$$HS_K^i(I,z) = \frac{h(z)}{(1-z)^{d+i}}.$$

Clearly if $h(z) = h_0 + h_1z + \ldots + h_sz^s$ with $h_i \in \mathbb{Z}$, then

$$h_0 = \lambda(\frac{R}{K}) \quad and \quad h_1 = \lambda(\frac{I}{KI}) - d\lambda(\frac{R}{K}).$$

Write

$$P_K^i(I,n)=\sum_{j=0}^{d+i-1}(-1)^j f'_j(I,K)\binom{n+d+i-j-1}{d+i-j-1}.$$

Then, comparing with the earlier notation, we get that $f'_0(I,K)=f_0(I,K)$ and $f'_i(I,K)=f_i(I,K)+f_{i-1}(I,K)$ for all $i\geq 1$.

If we denoted by $h^{(j)}(z)$ the j-th formal derivative of $h(z)$, then

$$f'_i(I,K)=\frac{h^{(j)}(1)}{j!}=\sum_{k=j}^{s}\binom{k}{j}h_k$$

and $f'_0(I,K)=h(1)$.

If $g:\mathbb{Z}\to\mathbb{Z}$ is a function, let $\Delta[g(n)]:=g(n)-g(n-1)$, and Δ^i defined by $\Delta^i[g(n)]:=\Delta^{i-1}[\Delta[g(n)]]$. By convention, $\Delta^0[g(n)]=g(n)$. If $a\in I$ is a Rees-superficial element for I and K, then for large n, $H^0_{\overline{K}}(\overline{I},n)=\Delta[H^0_K(I,n)]$. In particular, $f_i(\overline{I},\overline{K})=f_i(I,K)$ for $i=1,\ldots,d-2$, where " $^-$ " denote the image modulo (a).

For $a\in I\backslash KI$, let a^* denote its initial form in the associated graded ring $G(I)$, and a^0 denote its initial form in the fiber cone of $F_K(I)$.

Proposition 2.1. *([**16**, Proposition 2.2]) There exist $a_1,\ldots,a_{d-1}\in I$, $a_d\in K$ such that $a_1,\ldots,a_d$ is a Rees-superficial sequence for I and K. Suppose that depth $G(I)\geq d-1$, we can choose the above $a_1,\ldots,a_d$ such that $a_1^*,\ldots,a_{d-1}^*$ is a $G(I)$-regular sequence.*

Throughout the paper, if depth $G(I)\geq d-1$, then we put any Rees-superficial sequence $a_1,\ldots,a_{d-1}\in I$, $a_d\in K$ for I and K such that $a_1^*,\ldots,a_{d-1}^*$ is a $G(I)$-regular sequence, $J=(a_1,\ldots,a_{d-1})$ and $L=(a_1,\ldots,a_d)$. Set $\nu_n=\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n})$.

Definition 2.2. *([**6**, Definition 1.2]) Let $a_1,\ldots,a_d$ be a joint reduction of $(I^{[d-1]}|K)$. If there exists an integer n such that $KI^n=JKI^{n-1}+a_dI^n$, define $r_L(I|K)$ to be the smallest such n, otherwise, $r_L(I|K)=\infty$. The smallest of all $r_L(I|K)$, where L is varying over joint reductions of $(I^{[d-1]}|K)$, is denoted by $r(I|K)$.*

Using the same arguments as in [**10**, Theorem 5.1] and [**9**, Theorem 2.9], we have

Proposition 2.3. *Let $a_1,\ldots,a_{d-1}\in I$, $a_d\in K$ be a Rees-superficial sequence for I and K, k a positive integer such that $a_1^*,\ldots,a_k^*$ is a $G(I)$-regular sequence. Then depth $_{(a_1^0,\ldots,a_k^0)}F_K(I)=k$ if and only if $KI^n\cap(a_1,\ldots,a_k)=(a_1,\ldots,a_k)KI^{n-1}$ for all $n\geq 1$.*

Proposition 2.4. *Let $d=1$, $a_1\in K$ be a Rees-superficial element for I and K such that $r_L(I|K)<\infty$. Then*

$$f_2(I,K)=\sum_{n\geq 2}(n-1)\lambda(\frac{KI^n}{a_1I^n})+\lambda(\frac{R}{K}).$$

Proof. Note that for all $n \geq 0$,

$$\lambda(\frac{I^n}{KI^n}) = \lambda(\frac{R}{a_1R}) + \lambda(\frac{a_1R}{a_1I^n}) - \lambda(\frac{KI^n}{a_1I^n}) - \lambda(\frac{R}{I^n}) = \lambda(\frac{R}{a_1R}) - \nu_n.$$

Put $HS_K^0(I,z) = \frac{\sum_{n\geq 0} h_n z^n}{1-z}$. Then we get that $h_0 = \lambda(\frac{R}{K})$, $h_1 = \lambda(\frac{I}{KI}) - \lambda(\frac{R}{K}) = \nu_0 - \nu_1$, $h_n = \lambda(\frac{I^n}{KI^n}) - \lambda(\frac{I^{n-1}}{KI^{n-1}}) = \nu_{n-1} - \nu_n$, $n = 2, 3, \ldots$, and $h_n = 0$ for $n \gg 0$.

Let $h(z) = \sum_{n\geq 0} h_n z^n$. Then $h(z) = \lambda(\frac{R}{K}) + \sum_{n\geq 1} [\nu_{n-1} - \nu_n] z^n$. Differentiating the above equality twice with respect to z, we get that $h'(z) = \sum_{n\geq 1} n[\nu_{n-1} - \nu_n] z^{n-1}$ and $h''(z) = \sum_{n\geq 2} n(n-1)[\nu_{n-1} - \nu_n] z^{n-2}$.

Thus

$$\begin{aligned}
f_2(I,K) &= \frac{h''(1)}{2!} - h'(1) + h(1) \\
&= \sum_{n\geq 2} \frac{n(n-1)}{2!}[\nu_{n-1} - \nu_n] - \sum_{n\geq 1} n[\nu_{n-1} - \nu_n] + \lambda(\frac{R}{K}) + \sum_{n\geq 1}[\nu_{n-1} - \nu_n] \\
&= \sum_{n\geq 3} (\frac{n(n-1)}{2!} - n + 1)[\nu_{n-1} - \nu_n] + \lambda(\frac{R}{K}) \\
&= \sum_{n\geq 3} \binom{n-1}{2} [\nu_{n-1} - \nu_n] + \lambda(\frac{R}{K}) \\
&= \sum_{n\geq 2} (n-1)\lambda(\frac{KI^n}{a_1I^n}) + \lambda(\frac{R}{K}).
\end{aligned}$$

$\square$

Proposition 2.5. *Let $d = 2$, $a_1 \in I$, $a_2 \in K$ an Rees-superficial sequence for I and K such that a_1^* is a $G(I)$-regular element and $r_L(I|K) < \infty$. Then*

$$f_2(I,K) = \sum_{n\geq 2} (n-1)\lambda(\frac{KI^n}{a_1KI^{n-1} + a_2I^n}) + \lambda(\frac{R}{K}).$$

Proof. Consider the exact sequence

$$0 \to \frac{R}{(I^n : a_1) \cap (KI^{n-1} : a_2)} \xrightarrow{\psi} \frac{R}{I^n} \oplus \frac{R}{KI^{n-1}} \xrightarrow{\phi} \frac{(a_2, a_1)}{a_2I^n + a_1KI^{n-1}} \to 0$$

where $\psi(r') = ((-ra_1)', (ra_2)')$, $\phi(x', y') = (xa_2 + ya_1)'$ and here primes denote the residue classes. We have

$$\lambda(\frac{(a_2, a_1)}{a_2I^n + a_1KI^{n-1}}) - \lambda(\frac{R}{I^n}) - \lambda(\frac{R}{KI^{n-1}}) + \lambda(\frac{R}{(I^n : a_1) \cap (KI^{n-1} : a_2)}) = 0.$$

Hence

$$\lambda(\frac{I^n}{KI^n}) - \lambda(\frac{I^{n-1}}{KI^{n-1}}) = e_1(I|K) - \lambda(\frac{KI^n}{a_2I^n + a_1KI^{n-1}}) + \lambda(\frac{(I^n : a_1) \cap (KI^{n-1} : a_2)}{I^{n-1}}).$$

As a_1^* is a $G(I)$-regular element, we have $I^n : a_1 = I^{n-1}$ for $n \geq 1$. Thus

$$\lambda(\frac{I^n}{KI^n}) - \lambda(\frac{I^{n-1}}{KI^{n-1}}) = e_1(I|K) - \lambda(\frac{KI^n}{a_2I^n + a_1KI^{n-1}}).$$

Set $HS_K^0(I,z) = \frac{\sum_{n\geq 0} h_n z^n}{(1-z)^2}$. Then we get that $h_0 = \lambda(\frac{R}{K})$, $h_1 = \lambda(\frac{I}{KI}) - 2\lambda(\frac{R}{K})$, $h_n = \lambda(\frac{I^n}{KI^n}) - 2\lambda(\frac{I^{n-1}}{KI^{n-1}}) + \lambda(\frac{I^{n-2}}{KI^{n-2}}) = \nu_{n-1} - \nu_n$ for $n \geq 2$, and $h_n = 0$ for $n \gg 0$.

Let $h(z) = \sum\limits_{n\geq 0} h_n z^n$. Then $h(z) = \lambda(\frac{R}{K}) + [\lambda(\frac{I}{KI}) - 2\lambda(\frac{R}{K})]z + \sum\limits_{n\geq 2}[\nu_{n-1} - \nu_n]z^n$. Differentiating the above equality twice with respect to z, we get that $h'(z) = [\lambda(\frac{I}{KI}) - 2\lambda(\frac{R}{K})] + \sum\limits_{n\geq 2} n[\nu_{n-1} - \nu_n]z^{n-1}$, $h''(z) = \sum\limits_{n\geq 2} n(n-1)[\nu_{n-1} - \nu_n]z^{n-2}$. Then

$$\begin{aligned}
f_2(I,K) &= \frac{h''(1)}{2!} - h'(1) + h(1) \\
&= \sum_{n\geq 2} \frac{n(n-1)}{2!}[\nu_{n-1} - \nu_n] - [\lambda(\frac{I}{KI}) - 2\lambda(\frac{R}{K})] - \sum_{n\geq 2} n[\nu_{n-1} - \nu_n] \\
&+ \lambda(\frac{R}{K}) + [\lambda(\frac{I}{KI}) - 2\lambda(\frac{R}{K})] + \sum_{n\geq 2}[\nu_{n-1} - \nu_n] \\
&= \sum_{n\geq 3}(\frac{n(n-1)}{2!} - n + 1)[\nu_{n-1} - \nu_n] + \lambda(\frac{R}{K}) \\
&= \sum_{n\geq 3}\binom{n-1}{2}[\nu_{n-1} - \nu_n] + \lambda(\frac{R}{K}) \\
&= \sum_{n\geq 2}(n-1)\lambda(\frac{KI^n}{a_1KI^{n-1} + a_2I^n}) + \lambda(\frac{R}{K}).
\end{aligned}$$

□

The following example shows that the assumption in Proposition 2.5 that depth $G(I) \geq 1$ cannot be dropped.

Example 2.6. *Let $R = k[x,y]_{\mathfrak{m}}$, where k is an infinite field, x and y are indeterminate and $\mathfrak{m} = (x,y)$. Let $K = \mathfrak{m}$, $I = (x^4, x^3y, xy^3, y^4)$. Then it can be seen that x^4, y is an Rees-superficial sequence for I and $\mathfrak{m}$, $r_L(I|\mathfrak{m}) = 2$ where $L = (x^4, y)$ and $F_{\mathfrak{m}}(I) = k[x^4, x^3y, xy^3, y^4]$. Note that for all $n \geq 2$, $I^n = \mathfrak{m}^{4n}$, we have that*

$$HS_K^0(I,z) = 1 + 4z + \sum_{n=2}^{\infty}(4n+1)z^n = \frac{1 + 2z + 2z^2 - z^3}{(1-z)^2}.$$

Then $f_2(I,\mathfrak{m}) = 0$, $\sum_{n=2}^{\infty}(n-1)\lambda(\frac{\mathfrak{m}I^n}{x^4\mathfrak{m}I^{n-1} + yI^n}) + \lambda(\frac{R}{\mathfrak{m}}) = 1$. This forces that $f_2(I,\mathfrak{m}) = \sum\limits_{n\geq 2}(n-1)\lambda(\frac{\mathfrak{m}I^n}{x^4\mathfrak{m}I^{n-1} + yI^n}) + \lambda(\frac{R}{\mathfrak{m}})$ is not true. Since $G(I)_+ \subseteq ann(x^2y^2)^*$, depth $G(I) = 0$.

3. Main Results and Examples

THEOREM 3.1. *Let $a \in I$ be an Rees-superficial element for I and K such that a^* is a $G(I)$-regular element. Then*

$$(-1)^{d-1} f_{d-1}(I,K) = (-1)^{d-1} f_{d-1}(\overline{I},\overline{K}) - \sum_{j\geq 0} \lambda(\frac{(KI^{j+1}:a)\cap I^j}{KI^j})$$

where " $-$ " denotes the image modulo (a). Furthermore the following facts are equivalent:

1. $f_{d-1}(I,K) = f_{d-1}(\overline{I},\overline{K})$,
2. $f_i(I,K) = f_i(\overline{I},\overline{K})$ *for all* $0 \leq i \leq d-1$,
3. a^0 *is an* $F_K(I)$*-regular element,*
4. $HS^i_{\overline{K}}(\overline{I}, z) = (1-z) HS^i_K(I,z)$.

Proof. Since a^* is a $G(I)$-regular element, we have $I^n \cap (aR) = aI^{n-1}$ for all $n \geq 1$. For $n \geq 1$, consider the exact sequence

$$0 \to \frac{(KI^n:a)\cap I^{n-1}}{KI^{n-1}} \to \frac{I^{n-1}}{KI^{n-1}} \xrightarrow{\mu_a} \frac{I^n}{KI^n} \to \frac{\overline{I}^n}{\overline{KI}^n} \to 0$$

where $\mu_a(x+KI^{n-1}) = ax + KI^n$. It follows that

$$H^0_{\overline{K}}(\overline{I},n) = H^0_K(I,n) - H^0_K(I,n-1) + \lambda(\frac{(KI^n:a)\cap I^{n-1}}{KI^{n-1}}).$$

It is easy to see that for all nonnegative integers n, i

$$H^{i+1}_{\overline{K}}(\overline{I},n) = H^i_K(I,n) + \sum_{j=0}^{n-1} \binom{n-j+i-1}{i} \lambda(\frac{(KI^{j+1}:a)\cap I^j}{KI^j}). \qquad (*)$$

Hence for $n \gg 0$,

$$\begin{aligned}\sum_{j=0}^{d+i-1} (-1)^j f_j(\overline{I},\overline{K}) \binom{n+d+i-j-2}{d+i-j-1} &= \sum_{j=0}^{d+i-1} (-1)^j f_j(I,K) \binom{n+d+i-j-2}{d+i-j-1} \\ &+ \sum_{j\geq 0} \binom{n+i-j-1}{i} \lambda(\frac{(KI^{j+1}:a)\cap I^j}{KI^j}).\end{aligned}$$

Since $f_j(\overline{I},\overline{K}) = f_j(I,K)$ for $j = 0,\ldots,d-2$, we get

$$\begin{aligned}\sum_{j=d-1}^{d+i-1} (-1)^j f_j(\overline{I},\overline{K}) \binom{n+d+i-j-2}{d+i-j-1} &= \sum_{j=d-1}^{d+i-1} (-1)^j f_j(I,K) \binom{n+d+i-j-2}{d+i-j-1} \\ &+ \sum_{j\geq 0} \binom{n+i-j-1}{i} \lambda(\frac{(KI^{j+1}:a)\cap I^j}{KI^j}).\end{aligned}$$

If $i = 0$, one has

$$(-1)^{d-1} f_{d-1}(I,K) = (-1)^{d-1} f_{d-1}(\overline{I},\overline{K}) - \sum_{j\geq 0} \lambda(\frac{(KI^{j+1}:a)\cap I^j}{KI^j}).$$

From the above formulas we can obtain the following equivalent facts. First of all notice that $f_{d-1}(I,K) = f_{d-1}(\overline{I},\overline{K})$ if and only if $(KI^{j+1} : a) \cap I^j = KI^j$ for all $j \geq 0$, or equivalently, if and only if a^0 is an $F_K(I)$-regular element. From the above equalities it is simple to see that $(KI^{j+1} : a) \cap I^j = KI^j$ for all $j \geq 0$ is also equivalent to $f_i(I,K) = f_i(\overline{I},\overline{K})$ for all $0 \leq i \leq d-1$. Thus (1),(2), and (3) are all equivalent. Moreover, From $(*)$, $(KI^{j+1} : a) \cap I^j = KI^j$ for all $j \geq 0$ if and only if $H^{i+1}_{\overline{K}}(\overline{I},n) = H^i_K(I,n)$ for all nonnegative integers n, i, or equivalently, if and only if $HS^i_{\overline{K}}(\overline{I},z) = (1-z)HS^i_K(I,z)$ for all $i \geq 0$. Then (3) and (4) are equivalent. □

THEOREM 3.2. *Let $a_1,\ldots,a_{d-1} \in I$, $a_d \in K$ be an Rees-superficial sequence for I and K such that $r_L(I|K) < \infty$. If depth $G(I) \geq d-1$, then*

$$f_2(I,K) \leq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K}).$$

The equality occurs if and only if depth $F_K(I) \geq d-2$.

Proof. If $d = 1,2$, the result follows from Proposition 2.4 and Proposition 2.5.

If $d = 3$, then $dim(\overline{R}) = 2$ where " $^-$ " denotes the image modulo (a_1). By Theorem 3.1, we have

$$\begin{aligned} f_2(I,K) &= f_2(\overline{I},\overline{K}) - \sum_{j\geq 0}\lambda(\frac{(KI^{j+1}:a_1)\cap I^j}{KI^j}) \\ &\leq f_2(\overline{I},\overline{K}) \\ &= \sum_{n=2}^{\infty}(n-1)\lambda(\frac{\overline{KI}^n}{\overline{JKI}^{n-1}+\overline{a}_3\overline{I}^n}) + \lambda(\overline{R}/\overline{K}) \\ &\leq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_3I^n}) + \lambda(\frac{R}{K}). \end{aligned}$$

where the second equality holds by Proposition 2.5. The equality occurs if and only if $(KI^{j+1} : a_1) \cap I^j = KI^j$ for all $j \geq 0$ and $KI^n \cap (a_1R) = a_1KI^{n-1}$ for all $n \geq 2$, if and only if a_1^0 is an $F_K(I)$-regular element by Proposition 2.3, or equivalently, if and only if depth $F_K(I) \geq 1$.

Now suppose that $d > 3$ and " $^-$ " denote the image modulo (a_1), then $dim(\overline{R}) = d-1$, $r_{\overline{L}}(\overline{I}|\overline{K}) < \infty$, $\lambda(\overline{R}/\overline{K}) = \lambda(\frac{R}{K})$, and $\lambda(\frac{\overline{KI}^n}{\overline{JKI}^{n-1}+\overline{a}_d\overline{I}^n}) \leq \lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n})$ for all $n \geq 1$. By inductive hypotheses, we have

$$\begin{aligned} f_2(I,K) &= f_2(\overline{I},\overline{K}) \\ &\leq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{\overline{KI}^n}{\overline{JKI}^{n-1}+\overline{a}_d\overline{I}^n}) + \lambda(\overline{R}/\overline{K}) \\ &\leq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K}). \end{aligned}$$

The equality occurs if and only if depth $F_{\overline{K}}(\overline{I}) \geq d-3$ if and only if depth $F_K(I) \geq d-2$ by [**10**, Lemma 2.7]. □

COROLLARY 3.3. *Let $a_1, \ldots, a_{d-1} \in I$, $a_d \in K$ be an Rees-superficial sequence for I and K such that $r_L(I|K) < \infty$. If depth $G(I) \geq d-1$ and*

$$f_2(I,K) \geq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K}) - 2,$$

then depth $F_K(I) \geq d-3$.

Proof. Let " $^-$ " and " $^\sim$ " denote the image modulo $(a_1, \ldots, a_{d-3})$ and $(a_1, \ldots, a_{d-2})$ respectively. Then $dim(\overline{R}) = 3$, $dim(\widetilde{R}) = 2$ and $\lambda(\overline{R}/\overline{K}) = \lambda(\widetilde{R}/\widetilde{K}) = \lambda(R/K)$. By Theorem 3.1, we have $f_2(\overline{I}, \overline{K}) \leq f_2(\widetilde{I}, \widetilde{K})$ and

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1)\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n)) &\leq \sum_{n=2}^{\infty}(n-1)\lambda(\overline{KI}^n/(\overline{JKI}^{n-1}+\overline{a}_d\overline{I}^n))\\ &\leq \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}).\end{aligned}$$

If $f_2(\overline{I}, \overline{K}) = f_2(\widetilde{I}, \widetilde{K})$, then depth $F_{\overline{K}}(\overline{I}) \geq 1$ by Theorem 3.1. Hence depth $F_K(I) \geq d-2$ by [**10**, Lemma 2.7]. If

$$\sum_{n=2}^{\infty}(n-1)\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n)) = \sum_{n=2}^{\infty}(n-1)\lambda(\overline{KI}^n/(\overline{JKI}^{n-1}+\overline{a}_d\overline{I}^n)),$$

then $f_2(\overline{I}, \overline{K}) = f_2(\widetilde{I}, \widetilde{K})$, and as before, depth $F_K(I) \geq d-2$. If

$$\sum_{n=2}^{\infty}(n-1)\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n)) \neq \sum_{n=2}^{\infty}(n-1)\lambda(\overline{KI}^n/(\overline{JKI}^{n-1}+\overline{a}_d\overline{I}^n)),$$

then

$$f_2(\overline{I}, \overline{K}) \leq f_2(\widetilde{I}, \widetilde{K}) - 1.$$

It follows that

$$\begin{aligned}&\sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K}) - 2\\ \leq\ & f_2(I,K) = f_2(\overline{I}, \overline{K})\\ \leq\ & f_2(\widetilde{I}, \widetilde{K}) - 1\\ =\ & \sum_{n=2}^{\infty}(n-1)\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n)) + \lambda(\widetilde{R}/\widetilde{K}) - 1\\ \leq\ & \sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n}) + \lambda(\frac{R}{K}) - 2.\end{aligned}$$

Hence we obtain that

$$\sum_{n=2}^{\infty}(n-1)\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n))=\sum_{n=2}^{\infty}(n-1)\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n})-1,$$

which implies that $\lambda(\widetilde{K}\widetilde{I}^2/(\widetilde{J}\widetilde{K}\widetilde{I}+\widetilde{a}_d\widetilde{I}^2))=\lambda(\frac{KI^2}{JKI+a_dI^2})-1$ and $\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n})=\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n))$ for all $n\geq 3$. Hence $f_1(I,K)=f_1(\widetilde{I},\widetilde{K})=\sum\limits_{n=1}^{\infty}\lambda(\widetilde{K}\widetilde{I}^n/(\widetilde{J}\widetilde{K}\widetilde{I}^{n-1}+\widetilde{a}_d\widetilde{I}^n))-\lambda(\widetilde{R}/\widetilde{K})=\sum\limits_{n=1}^{\infty}\lambda(\frac{KI^n}{JKI^{n-1}+a_dI^n})-\lambda(\frac{R}{K})-1$, from which it follows that depth $F_K(I)\geq d-3$. □

The following example provide an instance that the bound of $f_2(I,K)$ in Theorem 3.2 can be attained.

EXAMPLE 3.4. *Let k be a field and $R=k[[x,y,z]]$ the power series ring. Put $I=\mathfrak{m}^3$ where $\mathfrak{m}$ is the unique maximal ideal of R. It is easily seen that both the fiber cone $F_\mathfrak{m}(I)$ and the associated graded ring $G(I)$ are Cohen-Macaulay. It is easy to see that $x^3,y^3\in I$, $z\in\mathfrak{m}$ is an Rees-superficial sequence for I and $\mathfrak{m}$ such that $r_L(I|\mathfrak{m})=2$ where $L=(x^3,y^3,z)$. The Hilbert series of $F_\mathfrak{m}(I)$ is*

$$HS^0_\mathfrak{m}(I,z)=\sum_{n\geq 0}\binom{3n+2}{2}z^n=\sum_{n\geq 0}[9\binom{n+2}{2}-9\binom{n+1}{1}+1]z^n=\frac{1+7z+z^2}{(1-z)^3}.$$

We also have that $f_2(I,\mathfrak{m})=\sum\limits_{n=2}^{\infty}(n-1)\lambda(\frac{\mathfrak{m}I^n}{(x^3,y^3)\mathfrak{m}I^{n-1}+zI^n})+\lambda(\frac{R}{\mathfrak{m}})=1$.

We observe that in Theorem 3.2 the assumption on depth $G(I)\geq d-1$ cannot be weakened. The following example shows that depth $F_K(I)\geq 1$ does not imply the upper bound of $f_2(I,K)$ can be attained.

EXAMPLE 3.5. *Let R be the three-dimensional local Cohen-Macaulay ring*

$$k[[X,Y,Z,U,V,W]]/(Z^2,ZU,ZV,UV,YZ-U^3,XZ-V^3),$$

with k a field and X,Y,Z,U,V,W indeterminates. Let x,y,z,u,v,w denote the corresponding images of X,Y,Z,U,V,W in R, $I=\mathfrak{m}=(x,y,z,u,v,w)$ and $K=\mathfrak{m}$. Then $F_\mathfrak{m}(I)=G(\mathfrak{m})$. One has that depth $G(I)=1$. Indeed, we checked that the Hilbert series of $F_\mathfrak{m}(I)$ is

$$HS^0_\mathfrak{m}(I,z)=\frac{1+3z+3z^3-z^4}{(1-z)^3}.$$

Thus $f_2(I,\mathfrak{m})=1$. Let $J=(x,y,w)$, we can obtain $\lambda(\mathfrak{m}^2/J\mathfrak{m})=2$, $\lambda(\mathfrak{m}^3/J\mathfrak{m}^2)=2$ and $\mathfrak{m}^4=J\mathfrak{m}^3$. Thus $\sum\limits_{n=2}^{\infty}(n-1)\lambda(\mathfrak{m}^n/J\mathfrak{m}^{n-1})+\lambda(R/\mathfrak{m})=3$.

References

[1] P. B. Bhattacharya, The Hilbert function of two ideals, Math. Proc. Cambridge Philos. Soc. 53 (1957), 568-575.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Revised Edition, Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1998.

[3] C. D' Cruz, T. J. Puthenpurakal, The Hilbert coefficients of the fiber cone and the a-invariant of the associated graded ring, Canadian Journal of Mathematics 61 (4), (2009), 762-778.

[4] C. Corso, C. Polini and M. E. Rossi, Depth of associated graded rings via Hilbert coefficients of ideals, *J. Pure Appl. Algebra* 201(2005), 126-141.

[5] C. D' Cruz, K. N. Raghavan and J. K. Verma, Cohen-Macaulay fiber cones, Commutative algebra, algebraic geometry and computational methods (Hanoi, 1996), 233-246.

[6] C. D' Cruz, J. K. Verma, Hilbert series of fiber cones of ideals with almost minimal mixed multiplicity, *J. Algebra* 251 (2002), 98-109.

[7] T. Cortadellas, S. Zarzuela, On the depth of fiber cone of filtrations, *J. Algebra* 198(1997), 428-445.

[8] S.Huckaba and T.Marley, Hilbert coefficients and the depths of associated graded ring, *J. London Math. Soc.* 56 (1997), 64-76.

[9] A. V. Jayanthan, T. J. Puthenpurakal and J.K. Verma, On fiber cones of $\mathfrak{m}$-primary ideals, *Canad. J. Math.* Vol. 59, No 1, (2007), 109-126.

[10] A. V. Jayanthan, J.K. Verma, Hilbert coefficients and depths of fiber cones, *J. Pure Appl. Algebra* 201(2005), 97-115.

[11] A. V. Jayanthan, J.K. Verma, Fiber cones of ideals with almost minimal multiplicity, *Nagoya Math. J.* 177(2005), 155-179.

[12] A. V. Jayanthan, B. Singh, J.K. Verma, Hilbert coefficients and depths of form rings, *Comm. Algebra* 32(2004), 1445-1452.

[13] D. Rees, Generalizations of reductions and mixed multiplicities, *J. London Math. Soc.* (2) 29(1984), 397-414.

[14] J. D. Sally, On the associated graded ring of a local Cohen-Macaulay ring, *J. Math. Kyoto Univ.* 17(1977), 19-21.

[15] G. Zhu, On Fiber Coefficients of Fiber Cones, *Comm. Algebra* 36 (11)(2008), 4120-4131.

[16] G. Zhu, Fiber Coefficients and Depth of Fiber Cones, *Comm. Algebra*, 37 (10) (2009), 3686-3696.

[17] G. Zhu and Z Tang, A lower bound on the second fiber coefficient of the fiber cones, *Comm. Algebra*, 38 (5) (2010), 1919-1929.

Received 08 08 2010, revised 18 03 2011

School of Mathematics Science,
Soochow University,
Suzhou 215006,
P.R. China.
E-mail address: zhuguangjun@suda.edu.cn