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On Some $QTAG$-Modules

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Abstract. In this paper we study totally projective $QTAG$-modules and the extensions of bounded $QTAG$-modules. In the first section we study totally projective modules M/N and M'/N' where N, N' are isomorphic nice submodules of M and M' respectively. In fact the height preserving isomorphism between nice submodules is extented to the isomorphism from M onto M' with the help of Ulm-Kaplansky invariants. In the second section extensions of the bounded $QTAG$-modules are studied. Here the invariants are automorphisms of bounded submodules of the extending module together with the cardinality of the minimal generating set of maximal summand of the extension module. The equivalence of epimorphisms is the main tool in this study.

1. Introduction.

All the rings R considered here are associative with unity. A module M over R is a $QTAG$-module if any finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Here modules are unital. If the lattice of the submodules of M is totally ordered, it is called a serial module and if the composition length of a serial module is finite it is a uniserial module. An $x \in M$ is uniform if xR is a nonzero uniform (hence uniserial) submodule of M. For any module M with a composition series, $d(M)$ denotes its length. For a uniform element $x \in M$, $e(x) = d(xR)$ is the exponent of x and $H(x)$ the height of x is $\sup\{d(U/xR)\}$ where U runs through all the uniserial modules containing x. For an integer $k \geqslant 0$, $H_k(M)$ is the submodule of M generated by the elements of height at least k and $M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ or $H_\omega M$. M is k-bounded if $H(x) \leqslant k$ for all $x \in M$ and it is h-divisible if M^1=M (or $H_1(M) = M$). A submodule N of M is h-pure in M if $H_k(N) = H_k(M) \cap N$ for all $k \in Z^+$ and it is h-neat in M if $H_1(M) \cap N = H_1(N)$. The isotype submodule $N \subset M$ is defined as $H_\sigma(N) = H_\sigma(M) \cap N$ by using transfinite induction where $H_\sigma(M)$= $\bigcap_{\rho<\sigma} H_\rho(M)$. For the next ordinal $\omega+1$, $H_{\omega+1}(M)$ is defined

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as $H_1(H_\omega(M))$ and $H_{\omega 2}(M)$ is $H_\omega(H_\omega(M))$. N is nice in M if for all ordinals σ, $H_\sigma(M/N) = (H_\sigma(M) + N)/N$.

M is reduced if it doesn't contain any h-divisible module or it is free from the elements of infinite height. For a reduced $QTAG$-module, there is a chain of submodules $M = M^0 \supset M^1 \supset M^2 \supset M^3 \supset ......M^\tau = 0$ for some ordinal τ, where $M^{\sigma+1} = (M^\sigma)^1$. If ρ is a limit ordinal then M^ρ is equal to $M^\rho = \bigcap\limits_{\sigma<\rho} M^\sigma$. Here M^σ is the σth Ulm submodule of M and $M_\sigma = M^\sigma/M^{\sigma+1}$ is the σth Ulm factor of M. $M_0, M_1, \cdots ,,$ is the Ulm sequence of M and τ is the length of M.

The cardinality of the minimal generating set of uniform elements of M is denoted by $g(M)$ and the σth Ulm-Kaplansky invariant of M, $f_M(\sigma)$ is $g(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M)))$.

2. Nice Submodules of Totally Projective *QTAG*-Modules

In a $QTAG$-module M, a submodule $N \subset M$ is nice if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height and x is said to be proper with respect to N if x is the element of maximum height in $x + N$ [3].

We start by the following result :

Proposition 2.1. In a $QTAG$-module M, $N \subset M$ is nice in M if and only if every coset of N contains an element of maximum height.

Proof. Suppose every coset of N has an element of maximum height. Assume on contrary that N is not nice. Let $x + N \in M/N$ such that $H_{M/N}(x + N) > H_M(x)$, $x + N$ be the coset of minimum height and x be the element of maximum height in $x + N$. Let $y + N \in M/N$ such that $d\Big(\dfrac{(y + N)R}{(x + N)R}\Big) = 1$ and $H(y + N) \geqslant H(x)$ where y is the element of maximum height in $y + N$. Since $x + N \in (y + N)R$, $x = yr + z$ for some $r \in R$, $z \in N$, implying that $H(y) < H(x)$. Also $H(y) = H(y + N) \geqslant H(x)$. This implies that N is nice in M. The converse is trivial.

From the above discussion the following consequences are immediate:

(i) Direct summands of a $QTAG$-module are nice.
(ii) All finitely generated submodules are nice.
(iii) For every ordinal α, $H_\alpha(M)$ is always nice in M.

We may recall that $x \in x + N$ is the element of maximum height in $x + N$, then x is proper with respect to N.

Proposition 2.2. Let N, K be submodules of a $QTAG$-module M such that K is nice in M and N/K is nice in M/K. Then N is nice in M.

Proof. Consider $x+N \in M/N$. Since M/N is nice in M/K, we have $H_{M/N}(x+N) = H(x+K+N/K) = H(x+y+z)$ for some suitable $y \in K$, $z \in N$ and the results follows.

To study nice modules and totally projective modules in the light of Ulm invariants we need the following definitions:

Definition 2.3. For a submodule N of a $QTAG$-module M and an ordinal α we define $N(\alpha) = \{x \mid x \in M,\ e(x) \leqslant 1,\ H(x) \geqslant \alpha$ and $H(x+y) > \alpha$ for some $y \in N\}$, i.e.

$$N(\alpha) = \mathrm{Soc}(H_\alpha(M)) \cap (N + H_{\alpha+1}(M)).$$

Definition 2.4. For a submodule N of $QTAG$-module M and an ordinal α, the α^{th} Ulm invariant of M relative to N is defined as

$$f_\alpha(M,N) \ = \ g\Big(\mathrm{Soc}H_\alpha(M)/(\mathrm{Soc}\,H_\alpha(M) \cap (H_{\alpha+1}(M)+N))\Big) \qquad [2.8, 4]$$

Remark 2.5. $\mathrm{Soc}(H_\alpha(M)) \cap \Big(H_{\alpha+1}(M)+N)\Big)$ is same as $N(\alpha)$ and if $N=0$, $f_\alpha(M,N) = f_M(\alpha)$.

Definition 2.6. Let N, N' be submodules of $QTAG$-modules M, M' respectively. A homomorphism $f: M \to M'$ is said to be height preserving if $H_M(x) = H_{M'}(f(x))$ for every uniform element $x \in N$.

Theorem 2.7. Let N, N' be nice submodules of $QTAG$-modules M and M' respectively such that M/N and M'/N' are reduced. Let $f: N \to N'$ be a height preserving isomorphism and $f_\alpha(M,N) = f_\alpha(M',N')$. For each ordinal α consider an isomorphism $\psi_\alpha : Soc(H_\alpha(M))/N(\alpha) \to Soc(H_\alpha(M'))/N'(\alpha)$. If $x \in M$, then there are nice submodules K and K' of M and M' respectively and a height preserving isomorphism $f_1 : K \to K'$ extending f such that

(a) $K = N + xR$
(b) K/N and K'/N' are finitely generated
(c) for each ordinal α, ψ_α induces an isomorphism $K(\alpha)/N(\alpha) \to K'(\alpha)/N'(\alpha)$
(d) $f_\alpha(M,K) = f_\alpha(M',K')$

Proof. Consider an element $x \in M$ which is proper with respect to N such that there exists an element $y \in N$ satisfying $d(xR/yR) \geqslant 1$.

Now $H(y) > H(x)+1$. Let $H(x) = \beta$. Now two cases arise.

Case 1. If $H(y) > \beta+1$, consider an element z with $H(z) > \beta$ and $yR = z'R$ where $d\Big(\dfrac{zR}{z'R}\Big) = 1$.

Then $e(x-z) = 1$, $H(x-z) = \beta$ and $x-z$ is proper with respect to N. Therefore if $\psi_\beta(x-z+N(\beta)) = u + N'(\beta)$ then $e(u) = 1$ and $H(u) = \beta$ and u is proper with

respect to N'. Consider an element w with height greater than β and $d\Big(\dfrac{wR}{vR}\Big) = 1$ for some v. We define $f(y) = v$ and extend f by sending x to $w + u$. We put $K = N + xR$ and $K' = N' + (w + u)R$. The submodules K and K' are nice because K/N, K'/N' are finitely generated and N, N' are nice. For the ordinals $\alpha(\neq \beta)$, $K(\alpha) = N(\alpha)$ and $K'(\alpha) = N'(\alpha)$ and ψ_α induces an isomorphism $K(\alpha)/N(\alpha) \to K'(\alpha)/N'(\alpha)$. If $\alpha = \beta$ then $x - z \in K(\alpha)$ and $\psi_\alpha(x - z + N(\alpha)) = u + N'(\alpha) \in K'(\alpha)/N'(\alpha)$. Now $d(K/N) = 1$ and there is a natural epimorphism $K/N \to K(\alpha)/N(\alpha)$. Therefore $g(K(\alpha)/N(\alpha)) = g(K'(\alpha)/N'(\alpha)) = 1$ and ψ_α induces an isomorphism, implying (d).
Case 2. If $H(y) = \beta + 1$. Again consider an element w such that $H(w) \geqslant \beta$ and an element v with $d\Big(\dfrac{wR}{vR}\Big) = 1$. Let $f(y) = v$. Then $H(w) = \beta$ because $H(v) = \beta + 1$. If $H(w + w') \geqslant \beta + 1$ where $w' \in N'$, then $H(w') = \beta$ and $w' = f(u)$ where $H(u) = \beta$ and $H(x + u) = \beta$. Therefore $x + u$ is proper with respect to N and $H(x') > \beta + 1$ where $d\Big(\dfrac{(x+u)R}{x'R}\Big) = 1$. If $w \in N'$ then $w = 0$ and $H(w) > H(x)$, thus following case (i) we may infer that w is proper with respect to N. We may extend f by defining $f(x) = w$. Let $K = N + xR$ and $K' = N' + wR$. If $\alpha > \beta$, then $x \in H_{\alpha+1}(M)$ and $u + x + N(\alpha) = 0$ therefore $K(\alpha)/N(\alpha) = 0 = K'(\alpha)/N'(\alpha)$. If $\alpha = \beta$ then $H(x + u) = \beta$ and $H(x') > \beta + 1$. Again by case (i) $K(\alpha)/N(\alpha) = 0 = K'(\alpha)/N'(\alpha)$ and the result follows. A p-group is totally projective if and only if it has a nice system [Th 82.3,1]. On the similar lines it can be proved for $QTAG$-modules. Thus totally projective $QTAG$-modules may be defined in terms of nice submodules as follows:

Definition 2.8. A reduced $QTAG$-module M is totally projective if there is a family $\mathcal{A}$ of nice systems of M such that

(i) $\{0\} \in \mathcal{A}$;
(ii) the sum of the submodules of any subset of $\mathcal{A}$ is in $\mathcal{A}$;
(iii) if $N, K \in \mathcal{A}$ and N/K is countably generated, then there exists $L \in \mathcal{A}$ with $L \supseteq N$ and L/N is countably generated.

Remark 2.9. Countably generated reduced $QTAG$-modules are totally projective.

Now we are able to prove the following theorem:

Theorem 2.10. Let M, M' be $QTAG$-modules and N, N' their nice submodules respectively. Let M/N and M'/N' be totally projective and $f_\alpha(M, N) = f_\alpha(M', N')$. Then every height preserving isomorphism $\psi : N \to N'$ extends to an isomorphism from M to M'.

Proof. Let $\psi_\alpha : \mathrm{Soc}(H_\alpha(M))/N(\alpha) \to \mathrm{Soc}(H_\alpha(M'))/N'(\alpha)$ be an isomorphism for every ordinal α. Since M/N and M'/N' are totally projective, there exist families $\mathcal{A}$, $\mathcal{A}'$ of nice submodules of M/N and M'/N' respectively [Definition 2.8]. Let $\mathcal{F}$ be the family of all height preserving isomorphisms $L \to L'$, which are the extensions of ψ such that $L/N \in \mathcal{A}$, $L'/N' \in \mathcal{A}'$ and for each α, ψ_α induces an isomorphism $L(\alpha)/N(\alpha) \to L'(\alpha)/N'(\alpha)$. Now $\mathcal{F}$ contains a maximal element $\psi_0 : L \to L'$ and by assumption the Ulm invariants $f_\alpha(M,L) = f_\alpha(M',L')$ and by Proposition 2.3 L and L' are nice submodules and for each α, ψ_α induces an isomorphism $\mathrm{Soc}(H_\alpha(M))/L(\alpha) \to \mathrm{Soc}(H_\alpha(M'))/L'(\alpha)$.

Let $M \neq L$ and $x \in M$, $x \notin L$. By Theorem 2.7 there are nice submodules $K = L + xR$ and $K' = L' + \psi_0(x)R$ and a height preserving isomorphism $K \to K'$ extending ψ_0 such that ψ_α induces an isomorphism $K(\alpha)/N(\alpha) \to K'(\alpha)/N'(\alpha)$ and hence an isomorphism $K(\alpha)/N(\alpha) \to K'(\alpha)/N'(\alpha)$. Now $K \subset L_1$ such that $L_1 = K + \sum\limits_{i=1}^{\infty} x_{1i}R$, $L_1/N \in \mathcal{A}$. Again there is a height preserving isomorphism mapping $K + x_{1i}R$ onto K_1' which is an extension of the isomorphism $K \to K'$. Now $K_1' \subset L_1' = K_1' + \sum\limits_{i=1}^{\infty} y_{2i}R$ with $L_1'/N' \in \mathcal{A}'$ and there is a height preserving isomorphism $K_1 \to K_1' + y_{21}R$ which extends the previous isomorphism. Now $K_1 \subset L_2 = K_1 + \sum\limits_{i=1}^{\infty} x_{2i}R$ with $L_2/N \in \mathcal{A}$.

This implies that there is a height preserving isomorphism $K_1 + x_{12}R + x_{21}R \to K_2'$ extending the previous isomorphism. Now $K_2' \subset L_2' = K_2' + \sum\limits_{i=1}^{\infty} y_{3i}R$ with $L_2'/N' \in \mathcal{A}'$. On repeating the process we get a height preserving isomorphism from $\cup L_i = \cup K_i \to \cup L_i'$, extending ψ_0 and satisfying the given conditions. But $(\cup L_i)/N \in \mathcal{A}$ and $(\cup L_i')/N' \in \mathcal{A}'$. This implies that $L = M$. Similarly $L' = M'$.

Remark 2.11. Two totally projective $QTAG$-modules are isomorphic if they have the same Ulm invariants.

3. Extensions of Bounded $QTAG$-Modules

Among the $QTAG$-modules without elements of infinite height the closed modules [2] and the direct sums of uniserial modules are very significant. For these two types of $QTAG$-modules, Ulm invariants play an important role to distinguish non-isomorphic modules. In the previous section we characterized totally projective modules of arbitrary length in terms of nice submodules. Here we study the modules M for which $H_1(M^1) = 0$ and M/M^1 is closed. These modules form a class of extensions of modules N for which $H_1(N) = 0$.

Let M be a $QTAG$-module. To study the extensions by M of 1-bounded $QTAG$-module, we consider the family of epimorphisms $f : N \to M$ such that $H_1(\mathrm{Ker}\, f) = 0$.

Definition 3.1. In the class of epimorphisms $f : N \to M$, $H_1(\mathrm{Ker}\, f) = 0$, two epimorphisms f_1 and f_2 are equivalent if there exists an isomorphism ϕ from the domain of f_1 to the domain of f_2 and an automorphism ψ of M such that $\psi f_1 = f_2\phi$. This is denoted by $(\psi, \phi) : f_1 \cong f_2$.

Remark 3.2. With $f : N \to M$ we associate a cardinal number $m(f)$ and a submodule $K_f \subseteq M$ such that

$$m(f) \;=\; g(\mathrm{Ker}\, f/H_1(N) \cap \mathrm{Ker}\, f)$$

and

$$K_f \;=\; f(\mathrm{Soc}(N)).$$

Definition 3.3. Two submodules L_1 and L_2 of a $QTAG$-module M are equivalent if $\exists$ an automorphism ψ of M such that $\psi(L_1) = L_2$. This is denoted by $\psi : L_1 \cong L_2$.
Remark 3.4. If $(\psi, \phi) : f_1 \cong f_2$ then $m(f_1) = m(f_2)$ and $\psi : K_{f_1} \to K_{f_2}$ i.e. m and K are invariants. We start with the following results:

Lemma 3.5. Let $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ be epimorphisms such that $\mathrm{Ker}\, f_i \subseteq \mathrm{Soc}(H_1(N_i))$, $i = 1, 2$. Let ψ be an epimorphism of M such that $\psi f_1(\mathrm{Soc}(N_1)) = f_2(\mathrm{Soc}(N_2))$ and ϕ be a map from a h-neat submodule L of N_1 to N_2 such that $f_2\phi = \psi f_1$ on L. Then ϕ may be extended to an epimorphism $\bar{\phi} : N_1 \to N_2$ such that $f_2\bar{\phi} = \psi f_1$.

Proof. Consider $x \in H_1(N_1)$. Now there exists $y \in N_1$ such that $d\Big(\dfrac{yR}{xR}\Big) = 1$. Consider $f_2^{-1}\psi f_1 y = y'$ (say). Now $y' \in N_2$ and there exists $x' \in N_2$ such that $d\Big(\dfrac{y'R}{x'R}\Big) = 1$. For $x \in H_1(N_1)$ we define $\phi^*(x) = x'$. In order to prove that ϕ^* is a homomorphism we have to show that $\phi^*(0) = 0$.

But $e(x) = 1$ for all $x \in \mathrm{Soc}(N_1)$, $\psi f_1(\mathrm{Soc}(N_1)) = f_2(\mathrm{Soc}(N_2))$, $f_2^{-1} f_2(\mathrm{Soc}(N_2)) = \mathrm{Soc}(N_2) + \mathrm{Ker}\, f_2 = \mathrm{Soc}(N_2)$ and $H_1(\mathrm{Soc}(N_2)) = 0$.

Similarly the map given by $x' \to x$, is a map from $H_1(N_2)$ to $H_1(N_1)$ which is an inverse for ϕ^*. Thus ϕ^* is an isomorphism from $H_1(N_1)$ to $H_1(N_2)$, satisfying $f_2\phi^* = \psi f_1$ on $H_1(N_1)$. Since L is h-neat and $f_2\phi = \psi f_1$, $\phi^* = \phi$ on $H_1(N_1) \cap L = H_1(L)$. By defining $\phi^*(z) = \phi(z)$ for all $z \in L$, we may extend the domain of ϕ^* to $L + H_1(N_1)$. Now $N_1/(L + H_1(N_1))$ is 1-bounded. Consider the minimal generating set $\{\bar{x}_i\}$ of $N_1/(L + H_1(N_1))$ where $x_i \in N_1$. Now the elements y_i may be selected from N_2 such that $\psi f_1(x_i) = f_2(y_i)$. We may define

$$\bar{\phi}(x + \Sigma x_i r_i) \;=\; \phi^*(z) + \Sigma y_i r_i$$

where $z \in L + H_1(N_1)$.

Since $f_2(y_i r_i) = \psi f_1(x_i r_i)$, $z + \Sigma x_i r_i = 0$ suggests that $\phi^*(x_i' r_i) = y_i' r_i$ where $d\Big(\dfrac{x_i'R}{x_iR}\Big) = d\Big(\dfrac{y_i'R}{y_iR}\Big) = 1$, this implies that

$$\bar{\phi}(z + \Sigma x_i r_i) = \phi^*(z) + \phi^*(\Sigma x_i r_i) \;=\; \phi^*(z + \Sigma x_i r_i) \;=\; \phi^*(0) \;=\; 0,$$

again implying that $f_2\bar{\phi} \;=\; \psi f_1$.

To prove that $\bar{\phi}$ is one to one consider $\bar{\phi}(x) = 0$.Now

$$f_2\bar{\phi}(x) \;=\; \psi f_1(x) \;=\; 0,$$

which implies that

$$x \in \mathrm{Ker}\, f_1 \subseteq H_1(N_1).$$

The image of $\bar{\phi}$ contains $H_1(N_2) \supseteq \operatorname{Ker} f_2$ and maps onto M under f_2, it must coincide with N_2.

This implies that $\bar{\phi}$ is an isomorphism.

Lemma 3.6. Let $N_1,\ N_2,\ M$ be the $QTAG$-modules and $f_1,\ f_2$ the homomorphisms from $N_1,\ N_2$ onto M such that $H_1(\operatorname{Ker} f_i) = 0,\ i = 1,2$ and $\operatorname{Ker} f_1/(H_1(N_1) \cap \operatorname{Ker} f_1) \cong \operatorname{Ker} f_2/(H_1(N_2) \cap \operatorname{Ker} f_2)$. Then if ψ is an automorphism of M such that $\psi f_1(Soc(N_1)) = f_2(Soc(N_2))$, there exists an isomorphism $\phi : N_1 \to N_2$ such that $f_2\phi = \psi f_1$.

Proof. Let $\operatorname{Ker} f_i = (H_1(N_i) \cap \operatorname{Ker} f_i) \oplus L_i$, $i = 1,2$. Now $L_1 \cong L_2$ and being direct summands L_i are pure in N_i. Therefore we have $N_i = L_i \oplus N_i'$. By Lemma 3.5 when $L = 0$ we may get a map from N_1' to N_2' which gives the required map $\phi : N_1 \to N_2$ when combined with any isomorphism from L_1 to L_2.

Lemma 3.7. Let M be a $QTAG$-module, m a cardinal number and K a submodule of $\operatorname{Soc}(M)$. Then $\exists$ an epimorphism $f : N \to M$ such that $H_1(\operatorname{Ker} f) = 0$, $m(f) = m$ and $K_f = K$.

Proof. Consider a $QTAG$-module M^0 such that $H_1(M^0) = M$ and $\operatorname{Soc}(M^0) = \operatorname{Soc}(M)$. This is possible because M^0 is an extension of M with uniserial modules of length one. Again consider a direct sum L of m uniserial modules of length one, put $N = L \oplus M^0/K$ and define $f : N \to M$ such that $f(x) = 0$ for $x \in L$ and $f(x_0 + K) = y_0$ where $x_0 \in M^0$ and $d\Big(\dfrac{x_0R}{y_0R}\Big) = 1$. Now $x + (x_0 + K) \in \operatorname{Soc}(N)$ if and only if $y_0 \in K$. This implies that $f(\operatorname{Soc}(N)) = K$.

Again $f(x + (x_0 + K)) = 0$ if and only if $y_0 = 0$ and $\operatorname{Soc}(M^0) = \operatorname{Soc}(M) \subseteq H_1(M^0)$,which implies that
$Ker\, f = L + (H_1(N) \cap \operatorname{Ker} f)$ and $m(f) = g(L) = m$.

An immediate consequence of the above results may be stated as follows:

Theorem 3.8. Let M be a $QTAG$-module and $\mathcal{A}$ the class of extensions by M of direct sum of uniserial modules of length 1, i.e. $\mathcal{A}$ be the class of epimorphisms $f : N \to M$ such that $H_1(\operatorname{Ker} f) = 0$. Let $\mathcal{A}'$ be the set of submodules of $\operatorname{Soc}(M)$, the equivalence between $f_1, f_2 \in \mathcal{A}$ be a pair (ψ, ϕ) of isomorphisms such that $\psi f_1 = f_2\phi$, and an equivalence between $K_1, K_2 \in \mathcal{A}'$ be an automorphism ψ of M such that $\psi(K_1) = K_2$. Then the map which associates to f, the pair $(m(f), K_f)$ gives a $1-1$ correspondence between equivalence classes of elements of $\mathcal{A}$ and pairs $(m(f), K_f)$ related to the elements of $\mathcal{A}'$. If $m(f_1) = m(f_2)$ and $\psi : K_{f_1} \cong K_{f_2}$, then ψ may be lifted to an isomorphism ϕ such that $(\psi, \phi) : f_1 \cong f_2$. If $m(f_1) = 0$ then ϕ can be chosen to agree with any partial lifting of ψ whose domain is h-neat (or neat) submodule of the domain of f_1.

There are many other interesting problems which are still open.

Problem 1. Let M be an uncountably generated $QTAG$-module with a countably generated basic submodules. Then how many non-isomorphic submodules N exist such that $H_1(N^1)$ is a uniserial module and $N/H_1(N^1)$ is isomorphic to M. For a $QTAG$

module M, the submodules $H_k(M)$, $k = 0, 1, 2, \cdots$ form a neighborhood system of zero. Thus a topology arises known as h-topology. Closed and dense submodules are already defined with respect to this topology[2]. bf Problem 2. Let M, M' be closed

$QTAG$-modules and N, N' dense submodules of $Soc(M)$ and $Soc(M')$ respectively. If N, N' are isomorphic, is it possible to extend this to an isomorphism from M onto M'?

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