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# Approximate Amenability of Matrix Algebras

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Abstract. In this paper, we study approximate amenability of matrix algebras. We show that every derivation from $M_n(\mathcal{A})$ into $M_n(E^{(m)})$ is the sum of an inner derivation and a derivation induced by a derivation from $\mathcal{A}$ into $E^{(m)}$, where $\mathcal{A}$ is a Banach algebra and $E$ is a Banach $\mathcal{A}$-bimodule. By using this, we provide many results in approximate and permanent weak amenability of these algebras.

## 1. Introduction

The notion of an approximate amenability of Banach algebras was defined by Ghahramani and Loy in [**7**] and developed in [**3, 4, 5, 8, 9**]. Let $\mathcal{A}$ be a Banach algebra, and suppose that $X$ is a Banach $\mathcal{A}$-bimodule such that the following statements hold

$$\|a.x\| \leq \|a\|\|x\| \qquad \text{and} \qquad \|x.a\| \leq \|a\|\|x\|$$

for all $a \in \mathcal{A}$, and $x \in X$. We can define right and left actions of $\mathcal{A}$ on dual space $X^*$ of $X$ via,

$$\langle x, \lambda.a\rangle = \langle a.x, \lambda\rangle \qquad \text{and} \qquad \langle x, a.\lambda\rangle = \langle x.a, \lambda\rangle,$$

for all $a \in \mathcal{A}$, $x \in X$ and $\lambda \in X^*$.

Similarly, the second dual $X^{**}$ of $X$ become a Banach $\mathcal{A}$-bimodule under the following actions

$$\langle \lambda, a.\Lambda\rangle = \langle \lambda.a, \Lambda\rangle \qquad \text{and} \qquad \langle \lambda, \Lambda.a\rangle = \langle b.\lambda, \Lambda\rangle,$$

for all $a \in \mathcal{A}$, $x \in X$, $\lambda \in X^*$, and $\Lambda \in X^{**}$.

Throughout, if $\mathcal{A}$ is a Banach algebra we write $\mathcal{A}^\sharp$ for the unitization of $\mathcal{A}$. Suppose that $X$ is a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to X$ is a linear map which satisfies $D(ab) = a.D(b) + D(a).b$ for all $a, b \in \mathcal{A}$. The derivation $\delta$ is said to be inner if there exist $x \in X$ such that $\delta(a) = \delta_x(a) = a.x - x.a$ for all $a \in \mathcal{A}$. Denoting the linear space of bounded derivations from $\mathcal{A}$ into $X$ by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations by $N^1(\mathcal{A}, X)$, we consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the first Hochschild cohomology group of $\mathcal{A}$ with coefficients in $X$. The Banach algebra $\mathcal{A}$ is said to be amenable if $H^1(\mathcal{A}, X^*) = \{0\}$ for all Banach $\mathcal{A}$-bimodule $X$. Banach algebra $\mathcal{A}$ is said to be

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n-weakly amenable if every continuous derivation from $\mathcal{A}$ into the $n-$th dual space $\mathcal{A}^{(n)}$ is inner. The Banach algebra $\mathcal{A}$ is said to be permanently weakly amenable if it is $n-$weakly amenable for all $n \in \mathbb{N}$.

Let $\mathcal{A}$ be a Banach algebra. The Banach algebra $\mathcal{A}$ is called approximately amenable, if for every Banach $\mathcal{A}$-bimodule $X$ and every bounded derivation $D : \mathcal{A} \to X^*$ there exists a net $(D_\alpha)$ of inner derivations such that $\lim_\alpha D_\alpha(a) = D(a)$ for all $a \in \mathcal{A}$.

Let $\mathcal{A}$ be a Banach algebra. An approximate diagonal for $\mathcal{A}$ is a net $(M_i)$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that, for each $a \in \mathcal{A}$,

$$a.M_i - M_i.a \to 0 \qquad \text{and} \qquad a\pi(M_i) \to a.$$

Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is pseudo-amenable if it has an approximate diagonal.

In this paper, at first we study approximate and weak amenability of $M_n(\mathcal{A})$ and after that we consider approximate amenability of Munn-algebra $\mathfrak{LM}(\mathfrak{A}, P)$.

## 2. Approximate Amenability of Matrix Algebras

### 2.1. Approximate and Weak Amenability of $M_n(\mathcal{A})$.

In Example 6.2 of [**7**], approximate amenability of algebra $M_n$ of $n \times n$ matrices studied. In this section we investigate approximate amenability of algebra $M_n(\mathcal{A})$, which $\mathcal{A}$ is a Banach algebra and all elements of $M_n(\mathcal{A})$ are in $\mathcal{A}$. It is clear that $M_n(\mathcal{A})$ is operator space.

Let $E$ be a unital Banach $\mathcal{A}$-bimodule. $M_n(E)$ will be Banach $M_n(\mathcal{A})$-bimodule with the following actions

$$(a.x)_{ij} = \sum_{s=1}^{n} a_{is}.x_{sj} \quad \text{and} \quad (x.a)_{ij} = \sum_{s=1}^{n} x_{is}.a_{sj},$$

for each $a = (a_{ij}) \in M_n(\mathcal{A})$ and $x = (x_{ij}) \in M_n(E)$. We identify the dual of $M_n(E)$ with $M_n(E^*)$ and we have

$$(a.\Lambda)_{ij} = \sum_{s=1}^{n} a_{js}.\lambda_{is}, \quad (\Lambda.a)_{ij} = \sum_{s=1}^{n} \lambda_{sj}.a_{si} \quad \text{and} \quad \langle a, \Lambda\rangle = \sum_{i,j=1}^{n} a_{ij}.\lambda_{ij},$$

for each $a = (a_{ij}) \in M_n(\mathcal{A})$ and $\Lambda = (\lambda_{ij}) \in M_n(E^*)$. Let $D : \mathcal{A} \longrightarrow E^*$ be a continuous derivation, then $D$ induces a derivation as a $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(E^*)$ by $\mathfrak{D}(a_{ij}) = (D(a_{ij}))$ or $\mathfrak{D}(a_{ij}) = (D(a_{ji}))$. $E_{ij}$ is a $n \times n$ matrix, such that whose $(i, j)^{th}$ entry is 1 and other entries are 0. For each $a \in \mathcal{A}$, the matrix $a \otimes E_{ij}$ is a matrix whose $(i, j)^{th}$ entry is $a$ and other entries are 0. The proof of the following Lemma is similar to the Theorem of [**1**].

Lemma 2.1. *Let $\mathcal{A}$ be a unital Banach algebra, and $E$ be a Banach $\mathcal{A}$-bimodule. Then every derivation from $M_n(\mathcal{A})$ into $M_n(E^{(m)})$ ($E^{(m)}$ is the $m-$th dual of $E$) is the sum of an inner derivation and a derivation induced by a derivation from $\mathcal{A}$ into $E^{(m)}$.*

PROOF. By above argument, we have

$$((\lambda_{ij})(a_{ij}))_{kl} = \sum_{s=1}^{n} \lambda_{sl}.a_{sk} \quad \textit{and} \quad ((a_{ij})(\lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ls}\lambda_{ks},$$

for each $a = (a_{ij}) \in M_n(\mathcal{A})$ and $(\lambda_{ij}) \in M_n(E^*)$. Then by use of the above relations for each $\lambda \in E^*, (\Lambda_{ij}) \in M_n(E^{**})$, $(a_{ij}) \in M_n(\mathcal{A})$, and $k$ and $l$, we have

$$\begin{aligned}\langle \lambda \otimes E_{kl}, (\Lambda_{ij}).(a_{ij})\rangle &= \langle (a_{ij}).(\lambda \otimes E_{kl}), (\Lambda_{ij})\rangle = \langle \sum_{s=1}^{n} (a_{sl}.\lambda \otimes E_{kl}), (\Lambda_{ij})\rangle \\ &= \sum_{s=1}^{n} \langle a_{sl}.\lambda, \Lambda_{ks}\rangle = \langle \lambda, \sum_{s=1}^{n} \Lambda_{ks}.a_{sl}\rangle.\end{aligned}$$

Thus for every $\lambda \in E^*$,

$$((\Lambda_{ij}).(a_{ij}))_{kl} = \sum_{s=1}^{n} \Lambda_{ks}.a_{sl},$$

and similarly for every $\lambda \in E^*$,

$$((a_{ij}).(\Lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ks}.\Lambda_{sl}.$$

Similarly, the following inequalities hold for each positive integer number $m$, when $m$ is odd and is even, respectively, for each $(a_{ij}) \in M_n(\mathcal{A})$ and $(\lambda_{ij}) \in M_n(E^{(m)})$

$$\begin{gathered}((\lambda_{ij})(a_{ij}))_{kl} = \sum_{s=1}^{n} \lambda_{sl}.a_{sk} \quad \text{and} \quad ((a_{ij})(\lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ls}\lambda_{ks} \\ \text{and} \qquad (2.1) \\ ((\lambda_{ij})(a_{ij}))_{kl} = \sum_{s=1}^{n} \lambda_{ks}.a_{sl} \quad \text{and} \quad ((a_{ij})(\lambda_{ij}))_{kl} = \sum_{s=1}^{n} a_{ks}\lambda_{sl}.\end{gathered}$$

Suppose that $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(E^{(m)})$ is a continuous derivation. For each $i, j$ and $k, l$, define $D_{ij}^{kl} : \mathcal{A} \longrightarrow E^{(m)}$ by $D_{ij}^{kl}(a) := (\mathfrak{D}(a \otimes E_{ij}))_{kl}$, for each $a \in \mathcal{A}$. Let $m$ be an odd positive number, for each $a, b \in \mathcal{A}$ and each $1 \leq t \leq n$, with easy calculations we have

$$\begin{aligned}(\mathfrak{D}(a \otimes E_{it})(b \otimes E_{tj}))_{kl} &= \sum_{s=1}^{n} (\mathfrak{D}(a \otimes E_{it}))_{sl}(b \otimes E_{tj})_{sk} \\ &= \sum_{s=1}^{n} D_{it}^{sl}(a).b\delta_{ts}\delta_{jk} = D_{it}^{tl}(a).b\delta_{jk},\end{aligned}$$

and

$$\begin{aligned}((a \otimes E_{it})\mathfrak{D}(b \otimes E_{tj}))_{kl} &= \sum_{s=1}^{n} (a \otimes E_{it})_{ls}(\mathfrak{D}(b \otimes E_{tj}))_{ks} \\ &= \sum_{s=1}^{n} a\delta_{il}\delta_{ts}.D_{tj}^{ks}(b) = a\delta_{il}.D_{tj}^{kt}(b).\end{aligned}$$

Then

$$D^{kl}_{ij}(ab) = a\delta_{il}.D^{kt}_{tj}(b)+D^{tl}_{it}(a).b\delta_{jk}. \tag{2.2}$$

Thus, $D^{kl}_{ij}$ is a derivation from $\mathcal{A}$ into $E^{(m)}$. From 2.1 and 2.2, the following equalities hold

$$D^{jl}_{ij}(a) = D^{il}_{ii}(1).a \quad (i \neq l), \qquad D^{ki}_{ij}(a) = a.D^{kj}_{jj}(1) \quad (j \neq k),$$

and by suitable choices of $t$ in each case,

$$\begin{aligned} D^{jj}_{jj}(a) &= D^{ij}_{ji}(1).a + D^{ji}_{ij}(a) = D^{ij}_{ji}(1).a + D^{li}_{il}(1).a + D^{jl}_{lj}(a) \\ &= D^{ij}_{ji}(1).a + D^{li}_{il}(1).a + D^{ll}_{ll}(a) + a.D^{jl}_{lj}(1), \end{aligned}$$

and so

$$D^{ij}_{ji}(a) = a.D^{ij}_{ji}(1) + D^{jj}_{jj}(a).$$

Therefore $D^{ll}_{ll}(1) = 0$, and hence $D^{ij}_{ji}(1) = -D^{ij}_{ji}(1)$ and consequently

$$D^{ji}_{ij}(a) = D^{li}_{il}(1).a - a.D^{lj}_{jl}(1) + D^{ll}_{ll}(a).$$

By relations 2.1 and 2.2, for each $(a_{rs}) \in M_n(\mathcal{A})$ we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D^{ij}_{kl}(a_{kl}) = \sum_{k=1}^{n} D^{ij}_{ki}(1)a_{ki} + \sum_{k=1}^{n} a_{jk}D^{ij}_{jk}(1) + D^{ji}_{ij}(a_{ji}) \\ &= \sum_{k=1}^{n} D^{kj}_{kk}(1)a_{ki} + \sum_{k=1}^{n} a_{jk}D^{ik}_{kk}(1) + D^{ji}_{ij}(a_{ji}), \end{aligned}$$

and

$$(\mathfrak{D}(E_{kk}E_{ii}))_{ik} = \sum_{k=1}^{n} D^{sk}_{kk}(1)\delta_{si} + \sum_{k=1}^{n} \delta_{ks}D^{is}_{ii}(1) = D^{ik}_{kk}(1) + D^{ik}_{ii}(1) = 0.$$

This shows that $D^{ik}_{kk}(1) = -D^{ik}_{ii}(1)$. In this term we define $D_{kj} = D^{kj}_{kk}$. By above result we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D_{kj}(1)a_{kj} - \sum_{k=1}^{n} a_{jk}D_{ik}(1) + D^{ji}_{ij}(a_{ji}) \\ &= ((D_{rs}(1))(a_{rs}) - (a_{rs})(D_{rs}(1)))_{ij} + D^{ji}_{ij}(a_{ji}). \end{aligned}$$

Therefore proof is complete in the case that $m$ is odd.

Now let $m$ be an even positive number

$$\begin{aligned} (\mathfrak{D}(a \otimes E_{it})(b \otimes E_{tj}))_{kl} &= \sum_{s=1}^{n} (\mathfrak{D}(a \otimes E_{it}))_{ks}(b \otimes E_{tj})_{sl} \\ &= \sum_{s=1}^{n} D^{ks}_{it}(a).b\delta_{ts}\delta_{jl} = D^{kt}_{it}(a).b\delta_{jl} \end{aligned}$$

and

$$((a \otimes E_{it})\mathfrak{D}(b \otimes E_{tj}))_{kl} = \sum_{s=1}^{n} (a \otimes E_{it})_{ks}(\mathfrak{D}(b \otimes E_{tj}))_{sl}$$

$$= \sum_{s=1}^{n} a\delta_{ik}\delta_{ts}.D_{tj}^{sl}(b) = a\delta_{ik}.D_{tj}^{tl}(b),$$

for every $a, b \in \mathcal{A}$. Then

$$D_{ij}^{kl}(ab) = a\delta_{ik}.D_{tj}^{tl}(b) + D_{it}^{kt}(a).b\delta_{jl}. \tag{2.3}$$

Thus, $D_{ij}^{kl}$ is a derivation from $\mathcal{A}$ into $E^{(m)}$. By 2.1 and 2.3, the following equalities hold

$$D_{ij}^{kj}(a) = D_{ii}^{ki}(1).a \quad (k \neq i), \qquad D_{ij}^{il}(a) = a.D_{jj}^{jl}(1) \quad (j \neq l),$$

and

$$\begin{aligned} D_{ii}^{ii}(a) &= D_{ji}^{ji}(a) + D_{ij}^{ij}(1).a = D_{ij}^{ij}(1).a + D_{jl}^{jl}(1).a + D_{li}^{li}(a) \\ &= D_{ij}^{ij}(1).a + D_{jl}^{jl}(1).a + D_{ll}^{ll}(a) + a.D_{li}^{li}(1), \end{aligned}$$

and so

$$D_{ji}^{ji}(a) = D_{ii}^{ii}(a) + D_{ji}^{ji}(1).a.$$

Therefore $D_{ij}^{ij}(1) = -D_{ji}^{ji}(1)$ and consequently

$$D_{ji}^{ji}(a) = D_{jl}^{jl}(1).a - a.D_{il}^{il}(1) + D_{ll}^{ll}(a).$$

By using of the relations 2.1 and 2.3, for $(a_{rs}) \in M_n(\mathcal{A})$, we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D_{kl}^{ij}(a_{kl}) = \sum_{k=1}^{n} D_{kj}^{ij}(1)a_{kj} + \sum_{k=1}^{n} a_{ik}D_{ik}^{ij}(1) + D_{ji}^{ij}(a_{ij}) \\ &= \sum_{k=1}^{n} D_{kk}^{ik}(1)a_{ki} + \sum_{k=1}^{n} a_{jk}D_{kk}^{kj}(1) + D_{ji}^{ij}(a_{ij}) \end{aligned}$$

and

$$(\mathfrak{D}(E_{kk}E_{ii}))_{ik} = \sum_{k=1}^{n} D_{kk}^{ks}(1)\delta_{is} + \sum_{k=1}^{n} \delta_{is}D_{ii}^{sk}(1) = D_{kk}^{ki}(1) + D_{ii}^{ik}(1) = 0.$$

This shows that $D_{kk}^{ki}(1) = -D_{ii}^{ik}(1)$. In this term we define $D_{kj} = D_{kk}^{kj}$. With above result we have

$$\begin{aligned} (\mathfrak{D}(a_{rs}))_{ij} &= \sum_{k,l=1}^{n} D_{ik}(1)a_{ki} - \sum_{k=1}^{n} a_{jk}D_{jk}(1) + D_{ji}^{ij}(a_{ij}) \\ &= ((D_{rs}(1))(a_{rs}) - (a_{rs})(D_{rs}(1)))_{ij} + D_{ji}^{ij}(a_{ij}). \end{aligned}$$

Therefore proof is complete. □

LEMMA 2.2. *Let $D : \mathcal{A} \longrightarrow E^{(m)}$ be a continuous derivation. The induced derivation $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(E^{(m)})$ from $D$ is inner (approximately inner) when $D$ is inner (approximately inner).*

REMARK 2.3. If $\mathcal{A}$ have not unit, we replace it by the forced unitization of $\mathcal{A}$ and we show it with $\mathcal{A}^\sharp$. From Proposition 2.4 of [**8**], $\mathcal{A}$ is approximately amenable if and only if $\mathcal{A}^\sharp$ approximately amenable. For proof of the following Theorem we use this Proposition.

THEOREM 2.4. *Let $\mathcal{A}$ be a Banach algebra and $M_n$ be a $n \times n$ matrix. $M_n(\mathcal{A})$ is approximately amenable if and only if $\mathcal{A}$ is approximately amenable.*

PROOF. Let $M_n(\mathcal{A})$ be approximately amenable and $E$ be a Banach $\mathcal{A}$-bimodule. Let $D : \mathcal{A} \longrightarrow E^*$ be a continuous derivation. According above argument $M_n(E^*)$ is a Banach $M_n(\mathcal{A})$-bimodule. Now suppose that $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(E^*)$is a continuous derivation. Since $M_n(\mathcal{A})$ is approximately amenable, thus there exists a net $(\xi_\alpha) \subseteq M_n(E^*)$ such that

$$\mathfrak{D}(a) = \lim_\alpha (a.\xi_\alpha - \xi_\alpha.a) \qquad (a \in M_n(\mathcal{A})).$$

Let $a \in \mathcal{A}$, and $\mathfrak{a}$ be a matrix in $M_n(\mathcal{A})$ such that entire of first row and first column is $a$ and others entire is zero. Then $(\xi_\alpha)_{1,1} \in E^*$, and we have

$$D(a) = \mathfrak{D}(\mathfrak{a}) = \mathfrak{D}(a)_{1,1} = \lim_\alpha (a.\xi_\alpha - \xi_\alpha.a)_{1,1} = \lim_\alpha (a.(\xi_\alpha)_{1,1} - (\xi_\alpha)_{1,1}.a).$$

This show that $\mathcal{A}$ is approximately amenable.

Conversely, Let $\mathcal{A}$ be an approximately Banach algebra (see remark 2.3). Let $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(E^*)$ be a continuous derivation. From Lemmas 2.1 and 2.2, $\mathfrak{D}$ is the sum of an inner derivation and a derivation induced by a derivation from $\mathcal{A}$ into $E^*$. Therefore there are derivation $D : \mathcal{A} \longrightarrow E^*$ and inner derivation $\delta$ such that $\mathfrak{D} = D + \delta$. Since $\mathcal{A}$ is approximately amenable, thus $\mathfrak{D}$ is an approximately inner. □

THEOREM 2.5. *Let $\mathcal{A}$ be pseudo-amenable, then $M_n(\mathcal{A})$ is pseudo-amenable and if $\mathcal{A}$ is pseudo-contractible, then $M_n(\mathcal{A})$ is too.*

PROOF. Let $(m_\alpha)$ be an approximate diagonal for $\mathcal{A}$. For each $\alpha$, define

$$M_\alpha = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \otimes m_\alpha \in M_n(\mathcal{A}) \widehat{\otimes} M_n(\mathcal{A})$$

With the above definition $(M_\alpha)$ is an approximate diagonal for $M_n(\mathcal{A})$. □

THEOREM 2.6. *Let $M_n(\mathcal{A})$ be pseudo-amenable and have a b.a.i., then $\mathcal{A}$ is approximately amenable.*

PROOF. Let $M_n(\mathcal{A})$ be pseudo-amenable and have a b.a.i., then $M_n(\mathcal{A})$ is approximately amenable by Theorem 2.3 of [**9**], and by Theorem 2.4, $\mathcal{A}$ is approximately amenable. □

COROLLARY 2.7. *Let $M_n(\mathcal{A})^{**}$ be approximately amenable. Then $M_n(\mathcal{A})$ is approximately amenable.*

THEOREM 2.8. *Let $\mathcal{A}$ be a permanently weakly amenable Banach algebra. Then $M_n(\mathcal{A})$ is a permanently weakly amenable.*

PROOF. Let $\mathfrak{D} : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{A}^{(m)})$ be a continuous derivation for some $m \in \mathbb{N}$. According to Lemma 2.1, $\mathfrak{D}$ is sum of the two derivations of which one is an inner derivation and the other is an induced derivation from $\mathcal{A}$ into $\mathcal{A}^{(m)}$. Since $\mathcal{A}$ is permanently weakly amenable, then $M_n(\mathcal{A})$ is too. □

### 2.2. Approximate Amenability of $\ell^1$-Munn Algebras.

Definition 2.9. Let $\mathfrak{A}$ be a unital Banach algebra, $I$ and $J$ be arbitrary index sets and $P$ be a $J \times I$ nonzero matrix over $\mathfrak{A}$ such that $\sup\{\|P_{ij}\| : i \in I, j \in J\} \leq 1$. Let $\mathfrak{LM}(\mathfrak{A}, P)$ be the vector space of all $I \times J$ matrices $\mathcal{A}$ over $\mathfrak{A}$ such that $\|\mathcal{A}\|_\infty = \sum_{i\in I, j\in J} \|\mathcal{A}_{ij}\| < \infty$. $\mathfrak{LM}(\mathfrak{A}, P)$ with the product $\mathcal{A} \circ \mathcal{B} = \mathcal{A}P\mathcal{B}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{LM}(\mathfrak{A}, P)$ and the $\ell^1 - norm$ is a Banach algebra that is named $\ell^1$-Munn $I \times J$ matrix algebra over $\mathfrak{A}$ with sandwich matrix $P$ or briefly $\ell^1$-Munn algebra (see [**6**]).

Theorem 2.10. *Let $\mathfrak{LM}(\mathfrak{A}, P)$ has a bounded approximate identity. Then $\mathfrak{LM}(\mathfrak{A}, P)$ is approximately amenable if and only if $\mathfrak{A}$ is approximately amenable, $I$ and $J$ are finite and $P$ is invertible.*

Proof. Let $\mathfrak{LM}(\mathfrak{A}, P)$ be approximately amenable, then by Lemma 3.7 of [**6**], $I$ and $J$ are finite and $P$ is invertible. Since $\mathfrak{LM}(\mathfrak{A}, P)$ is isometrically algebraic isomorphic to $\mathfrak{LM}_m \widehat{\otimes} \mathfrak{A}$ (Lemma 3.3 of [**6**]), where $m = |I| = |J|$. Thus $\mathfrak{LM}_m \widehat{\otimes} \mathfrak{A}$ is approximately amenable and therefore by Theorem 2.4, $\mathfrak{A}$ is approximately amenable. The converse of proof by Theorem 2.4 is clear. □

When $I = J$ and $P$ is the identity $J \times J$ matrix over $\mathfrak{A}$, $\mathfrak{LM}(\mathfrak{A}, P)$ denoted by $\mathcal{M}_J(\mathfrak{A})$. Similarly Theorem 2.5, if $\mathfrak{A}$ has an approximate diagonal then $\mathcal{M}_J(\mathfrak{A})$ has too. If $I$ and $J$ are finite and $\mathfrak{LM}(\mathfrak{A}, P)^{**}$ is approximately amenable, then $\mathfrak{A}$ is approximately amenable.

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