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A Kleinian group version of Torelli's Theorem

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Abstract. Each closed Riemann surface S of genus $g \geq 1$ has associated a principally polarized Abelian variety $J(S)$, called the Jacobian variety of S. Classical Torelli's theorem states that S is uniquely determined, up to conformal equivalence, by $J(S)$. On the other hand, if S is either a non-compact analytically finite Riemann surfaces or an analytically finite Riemann orbifold, then it seems that there is not a natural way to associate to it a principal polarized Abelian variety. We survey some results concerning a Torelli's type of theorem for the case of homology Riemann orbifolds and Kleinian groups.

1. Introduction

A *principally polarized Abelian variety* of dimension d is by definition a pair (T, H), where T is a complex torus of dimension d, that is, $T = \mathbb{C}^d/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2d}$ is a lattice in $\mathbb{C}^d$, and H is a positive Hermitian inner product H in $\mathbb{C}^d$ whose imaginary part E, when restricted to $\Lambda \times \Lambda$, has integral values and there is a basis of Λ over which E has the form

$$E = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We recall some basics facts necessary for this exposition; a good reference about principally polarized Abelian varieties is [**2**].

Two principally polarized Abelian varieties (of the same dimension), say (T_1, H_1) and (T_2, H_2), are said to be *equivalent* if there is a holomorphic isomorphism $h : T_1 \to T_2$ which is an isometry respect to the corresponding induced Hermitian inner products. Each principally polarized Abelian variety of dimension d is equivalent to one of the form (T, H_0), where H_0 is the canonical Hermitian inner product in $\mathbb{C}^d$

$$H(x, y) = x_1\overline{y_1} + \cdots + x_d\overline{y_d}$$

and Λ is generated by the canonical vectors e_1,..., e_d and other d vectors z_1,..., z_d, where the matrix Z, whose rows are z_1,..., z_d, is symmetric and whose imaginary part is positive definite. In this way, a parameter space of principally polarized Abelian

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varieties of dimension d is provided by the Siegel space $\mathcal{S}_d$ and the moduli space of principally polarized Abelian varieties is the quotient $\mathcal{A}_d = \mathcal{S}_d/Sp(d,\mathbb{Z})$, where $Sp(d,\mathbb{Z})$ is the group of integral symplectic matrices.

Let $\mathcal{F}$ be some collection of complex manifolds or complex orbifolds and assume that for each $V \in \mathcal{F}$ there is associated a d-dimensional principally polarized Abelian variety $J(V)$ so that, for $V_1, V_2 \in \mathcal{F}$ holomorphically equivalent one has that $J(V_1)$ and $J(V_2)$ are equivalent principally polarized Abelian varieties. A Torelli's theorem, for such a collection, will mean to have the property that $J(V)$ determines uniquely $V \in \mathcal{F}$, up to holomorphic equivalence.

Classical Torelli's theorem, firstly proved by Torelli in [**27**], takes care of the case when $\mathcal{F}$ is the category of closed Riemann surfaces of a fixed genus g. Let S be a closed Riemann surface, say of genus g, let $\mathrm{H}^{1,0}(S)$ be the g-dimensional complex vector space of its holomorphic 1-form, let $(\mathrm{H}^{1,0}(S))^*$ be the dual space of $\mathrm{H}^{1,0}(S)$ and let $H_1(S,\mathbb{Z})$ be its first homology group. Integration of 1-forms on 1-cycles permits to see $\mathrm{H}_1(S,\mathbb{Z})$ as a lattice in $(\mathrm{H}^{1,0}(S))^*$. The quotient $J(S) = (\mathrm{H}^{1,0}(S))^*/\mathrm{H}_1(S,\mathbb{Z})$ is a complex torus of dimension g, called the *Jacobian variety* of S. The Jacobian $J(S)$ comes with a natural Hermitian product induced by the intersection of cycles in $\mathrm{H}_1(S,\mathbb{Z})$. In this way, $J(S)$ together with such a Hemitian product turns out to be a principally polarized Abelian variety of dimension g. By choosing a point $p_0 \in S$, there is a natural holomorphic embedding [**6**] $\Phi_{p_0} : S \hookrightarrow J(S)$, defined by $\Phi_{p_0}(p) = [\int_{p_0}^{p}]$. If $\sum \alpha_j q_j$ is a divisor on the surface S, then define $\Phi_{p_0}(\sum \alpha_j q_j) = \sum \alpha_j \Phi_{p_0}(q_j)$. Let $W^d \subset J(S)$ be the image under Φ_{p_0} of the positive divisors of degree at most d. By the Abel's theorem [**6**] $\Phi_{p_0} : S \to W^1$ is a conformal homeomorphism. The polarization of $J(S)$ can be interpreted by W^{g-1}. Classical Torelli's theorem [**4, 6, 22, 27, 28**] asserts that W^1 is determined up to translations and a reflection by both $J(S)$ and W^{g-1}; that is, the principally polarized Abelian variety $J(S)$ determines S up to conformal equivalence. An extended version of Torelli's theorem was obtained by Martens in [**23**] (see also [**5**]) and a topological point of view of Torelli's theorem has been recently posted in [**29**].

In a recent paper [**1**] I.V. Artamkin proved a Torelli's theorem for the class of stable Riemann surfaces whose components are rational, that is, the complement of the nodes consists of punctured spheres. For the more general class of stable Riemann surfaces it seems to be a hard problem to associate a principally polarized Abelian variety in order to have a Torelli's theorem form them.

In this paper, we survey some kind of Torelli's theorem for homology Riemann orbifolds, that is, those Riemann orbifolds with the property that the derived subgroup of their orbifold fundamental groups uniformizes a closed Riemann surface, and also in terms of Kleinian groups.

2. A Torelli's version for homology Riemann orbifolds

A *Riemann orbifold* $\mathcal{O}$ is provided by a Riemann surface S, called the *underlying Riemann surface structure* of it, together a discrete collection of points $p_j \in S$, called its *cone points*, where each of these cone points p_j has associated an integer $n_j \geq 2$, called the *order* of p_j. A *conformal automorphism* of the Riemann orbifold $\mathcal{O}$ is a conformal automorphism of S which preserves the collection of cone points and their orders. If S is a closed Riemann surface of genus g, then $\mathcal{O}$ has a finite number of cone points, say $p_1, ..., p_r$. If n_j denotes the order of p_j, then the tuple $(g, r; n_1, ..., n_r)$ is called the *signature* of $\mathcal{O}$. The orbifold is say to be of *hyperbolic type* if $2g - 2 + r - \sum_{j=1}^{r} n_j^{-1} > 0$, equivalently by Köbe-Poincaré uniformization theorem [**18, 19, 26**], that there is a Fuchsian group Γ so that $\mathcal{O} = \mathbb{H}^2/\Gamma$; we say that Γ *uniformizes* $\mathcal{O}$. If the derived subgroup Γ' turns out to be a torsion free co-compact Fuchsian group (that is, $S = \mathbb{H}^2/\Gamma'$ is a closed Riemann surface), then we say that $\mathcal{O}$ is a *homology Riemann orbifold* and that S is a *homology closed Riemann surface*.

Not every closed Riemann orbifold is necessarily a homology one. A necessary and sufficient condition for $\mathcal{O}$, with signature $(g, r; n_1, ..., n_r)$, to be a homology Riemann orbifold is that

(i) $g = 0$, and
(ii) the following Maclachlan's conditions [**21**] are satisfied

$$\text{mcm}(k_1, ..., k_{j-1}, k_{j+1}, ..., k_n) = \text{mcm}(k_1, ..., k_n), \quad \forall j = 1, ..., n, \tag{1}$$

where mcm denotes the "minimum common multiple".

Now, if $\mathcal{O}$ is a homology Riemann orbifold and $\widetilde{S}$ is a homology closed Riemann surface of it, then we may define the Jacobian variety of $\mathcal{O}$ as $J(\mathcal{O}) := J(\widetilde{S})$. In this way, the Jacobian of a homology Riemann orbifold is uniquely defined, up to equivalence, by the orbifold $\mathcal{O}$.

To obtain a Torelli's theorem in the class of homology Riemann orbifolds is equivalent to prove that the orbifold is uniquely determined, up to conformal equivalence, by the homology closed Riemann surface. In this direction, some positive results have been obtained.

Theorem 1 (Torelli's theorem for homology orbifolds [**3, 7, 8, 15, 16**]). *Let $\mathcal{O}$ be either*

(1) *a hyperbolic Riemann orbifold of signature $(0, n; p, ..., p)$, where p is a prime; or*
(2) *a Riemann orbifold of signature $(0, 4; k, k, k, k)$, where $k \geq 3$; or*
(3) *a homology Riemann orbifold of signature $(0, 3; k_1, k_2, k_3)$;*

then $\mathcal{O}$ is, up to conformal equivalence, uniquely determined by its homology cover. In particular, $J(\mathcal{O})$ determines uniquely $\mathcal{O}$, up to conformal equivalence.

Conjecture 1. *If $\mathcal{O}$ is a homology Riemann orbifold, then it is uniquely determined, up to conformal equivalence, by its homology closed Riemann surface.*

3. A Kleinian group version of Torelli's theorem

Before to proceed with the Kleinian group version of Torelli's theorem, let us return to the definition of the classical Torelli's theorem. Let S be a closed Riemann surface of genus $g \geq 2$. Köbe-Poincaré uniformization theorem [**18, 19, 26**] asserts the existence of a Fuchsian group Γ, acting on the hyperbolic plane $\mathbb{H}^2$, so that $S = \mathbb{H}^2/\Gamma$. The homology cover of S is given by $\widetilde{S} = \mathbb{H}^2/\Gamma'$, where Γ' denotes the derived subgroup of Γ. Clearly, $S = \widetilde{S}/H$, where $H < Aut(\widetilde{S})$ is isomorphic to $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Let $\pi : (\mathrm{H}^{1,0}(S))^* \to J(S)$ be the universal covering, whose covering group is isomorphic to $\mathrm{H}_1(S,\mathbb{Z})$. Let $\widehat{S} \subset (\mathrm{H}^{1,0}(S))^*$ be the lift of $W^1 \cong_{\Phi_{p_0}} S$; which is a Riemann surface on which $\mathrm{H}_1(S,\mathbb{Z})$ acts as a group of conformal automorphisms and so that $\widehat{S}/\mathrm{H}_1(S,\mathbb{Z}) = W^1$. Clearly, $\widehat{S}$ is conformally equivalent to the homological cover $\widetilde{S}$ of S. In [**25**] B. Maskit proved that $\widetilde{S}$ determines uniquely, up to conformal equivalence, the surface S.

3.1. Torsion free Kleinian groups.

Theorem 2 (B. Maskit [**25**]). *If Γ is a torsion free, co-compact Fuchsian group, then Γ' determines Γ uniquely. In other words, the homological cover of a closed Riemann surface of genus $g \geq 2$ determines it uniquely up to conformal equivalence.*

The previous result may be seen as a kind of Kleinian groups version of the classical Torelli's theorem. Unfortunately, it is not known if the above commutator rigidity is equivalent to Torelli's theorem (at least for the author, it is not clear how to realize the polarization of $J(S)$). The above can be carry out with any Kleinian group (either with or without torsion) and we may state the following natural questions, which are natural generalizations of Theorem 2.

Let G be a finitely generated, non-elementary Kleinian group.

(1) Is G uniquely determined by its derived subgroup G'?
(2) Is G uniquely determined, up to conjugation, by its derived subgroup G'?

Partial answers to the above are provided in the following.

Theorem 3 ([**9, 10, 11, 12, 13, 14**]). *If F is any of the following type of groups, then its derived subgroup F' determines F uniquely.*

(1) *A finitely generated torsion free Fuchsian group of the first kind.*
(2) *A Schottky-type group of type (g,t), with $g+t \geq 2$.*
(3) *A Schottky group of genus $g \geq 2$.*
(4) *A non-elementary, torsion free, noded Fuchsian group.*
(5) *A non-elementary, torsion free, finitely generated noded function group.*
(6) *A non-elementary, torsion free, finitely generated function group.*
(7) *A torsion free, finitely generated extended Fuchsian group.*

Conjecture 2. *If G is a finitely generated, torsion free, non-elementary extended Kleinian group, then its derived subgroup G' determines it uniquely up to conjugation by a Möbius transformation.*

3.2. Kleinian groups with torsion. The commutator property in general fails under the presence of torsion in the Kleinian groups as can be seen in the following example [**10**].

3.2.1. **Example.** Set $J_0(z) = -z$, $T(z) = 1/z$ and $H(z) = J(T(z)) = -1/z$. Let J_1 and J_2 be elliptic transformations of order two so that $TJ_1 = J_1T$ and $HJ_2 = J_2H$. We assume $J_j \neq J_0, T, H$. Let us consider the following two groups

$$\Gamma = \langle J_0, J_1, J_2, T\rangle = ((\langle J_0, T\rangle *_{\langle J_1\rangle}) *_{\langle J_2\rangle} \cong (\mathbb{Z}_2^2 *_{\mathbb{Z}_2}) *_{\mathbb{Z}_2}$$
$$G = \langle J_0, J_1, J_2\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$
$$K = \langle J_0, TJ_1, HJ_2\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$$

We note that $\Gamma = \langle G, T\rangle = \langle K, T\rangle$, $[\Gamma : G] = [\Gamma : K] = 2$. Let Ω be the region of discontinuity of Γ (so the same region of discontinuity of G and K). The orbifold Ω/Γ has signature $(0; 5; 2, 2, 2, 2, 2)$, the orbifolds Ω/G and Ω/K both hav signature $(0, 6; 2, 2, 2, 2, 2, 2)$.

Inside G we have a Schottky group G_0 of genus 2 as index two subgroup; the one generated by J_0J_1 and J_0J_2. Similarly, inside K we have a Schottky group K_0 of genus 2 as index two subgroup; the one generated by J_0TJ_1 and J_0HJ_2.

As T and H do not belong to G, we have that $G \neq K$. The following equalities:

$$[J_0, TJ_1] = [J_0, J_1]$$
$$[J_0, HJ_2] = [J_0, J_2]$$
$$[TJ_1, HJ_2] = [J_1, J_0J_2][J_0, J_2]$$
$$[TJ_1, J_0HJ_2][J_0, HJ_2] = [J_1, J_2]$$

asserts that $G' = K'$.

The above example shows that, for non-elementary Kleinian groups with torsion, the commutator rigidity property does not hold in general. In order to provide positive answers in this case, we recall the following result of [**16**] and its simple proof.

Theorem 4 ([**16**]). *Let us fix non-negative integers γ, r and s so that $2\gamma-2+r > 0$ and $s \geq 1$. Then there is a prime integer $q(r, \gamma, s)$, depending only on r, γ and s, so that if $p \geq q(r, \gamma, s)$ is a prime integer and $H < Aut(S)$, where S is some closed Riemann surface of genus at least 2, so that $|H| = p^s$ and S/H is an orbifold of type $(\gamma; r)$, then H is unique, in particular, $H \lhd Aut(S)$.*

Proof. Let us assume we have a closed Riemann surface S (uniformized by the hyperbolic plane) admitting a group H as group of conformal automorphisms, where $p \geq 3$ is a prime, so that S/H is an orbifold $\mathcal{O}$ of type $(\gamma; r)$ and $|H| = p^s$. Koebe-Poincaré's uniformization theorem asserts the existence of a co-compact torsion free Fuchsian group Γ so that $S = \mathbb{H}^2/\Gamma$. If we denote by $N[\Gamma]$ the normalizer of Γ in the group of conformal automorphisms of $\mathbb{H}^2 \cong PSL(2, \mathbb{R})$, then we have that $N[\Gamma]$ is again a co-compact Fuchsian group [**24**] maybe with torsion so that $Aut(S) = N[\Gamma]/\Gamma$ and $S/Aut(S) = \mathbb{H}^2/N[\Gamma]$. We also have a group Γ_1 so that $\Gamma \lhd \Gamma_1 < N[\Gamma]$ so that $H = \Gamma_1/\Gamma$ and $\mathcal{O} = \mathbb{H}^2/\Gamma_1$.

Let us set the value $n_1(\gamma, r)$ as follows:

(i) $n_1(0;r) = n_1(1;r) = r+1$; and
(ii) $n_1(\gamma;r) = 2\gamma+2$, for $\gamma \geq 2$.

The choice of $n_1(\gamma;r)$ is to ensure that if $p \geq n_1(\gamma;r)$ is a prime, then no orbifold of type $(\gamma;r)$ admits an orbifold automorphism of order p.

Lemma 1. *If $p \geq n_1(\gamma,r)$ is a prime so that p^s divides $|Aut(S)|$, then $|\ Aut(S)\ |= ap^s$, where $a \in \mathbb{N}$ is relative prime to p.*

Proof. Assume $|\ Aut(S)\ |= ap^{s+1}$, where $a \in \mathbb{N}$. Sylow's theorem asserts the existence of a group $K_p < Aut(S)$ so that $|\ K_p\ |= p^{s+1}$ and $H \lhd K_p$. In particular, this asserts that on the orbifold S/H should be a (orbifold) automorphism of order $p \geq n_1(\gamma;r)$, a contradiction. $\square$

We now continue with our proof. As a consequence of lemma 1, we have that for $p \geq n_1(\gamma,r)$ the p-Sylows subgroups of $Aut(S)$ are groups of order p^s, all of them conjugate to H. In particular, the order of $Aut(S)$ is given by $|Aut(S)| = bp^s(1+kp)$, where $(p,b)=1$. We assume from now on that $p \geq n_1(\gamma,r)$.

If we are able to find a value $n_2(\gamma,r) \geq n_1(\gamma,r)$ so that for $p \geq n_2(\gamma,r)$ we have $k=0$, then we will be done with the proof.

As a consequence of the results in Keen [**17**], $N[\Gamma]$ has a canonical presentation of the form:

$$(*) \left\{ \begin{array}{lll} N[\Gamma] & = & \langle a_1,...,a_h,b_1,...,b_h,x_1,...,x_s : \\ & & x_1^{m_1} = \cdots = x_r^{m_s} = \prod_{i=1}^h a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^s x_j = 1\rangle, \\ \text{where} & & m_1,...,m_s \in \{2,3,4,...\}. \end{array} \right.$$

We note that the signature of a Fuchsian group G is exactly the signature of the orbifold $\mathbb{H}^2/G$, in fact, this is in part what Keen's result is telling us. Riemann-Hurwitz's formula asserts that:

$$|\ Aut(S)\ |= \frac{2(g(p)-1)}{M[N[\Gamma]]},$$

where

$$M[N[\Gamma]] = 2(h-1) + \sum_{j=1}^{s}(1-\frac{1}{m_j})$$

and

$$g(p) = p^s(\gamma-1)+1+\frac{\sum_{j=1}^r p^{s-l_j}(p^{l_j}-1)}{2}.$$

Let us observe that if we set $A(\gamma;r) = \gamma + r/2 - 1$, then $g(p)-1 \leq A(\gamma;r)p^s$ and, in particular,

$$(*) \quad bp^s(1+kp) = |Aut(S)| \leq \frac{2A(\gamma;r)p^s}{M[N[\Gamma]]}, \quad (p,b)=1.$$

On the other hand, the minimum value that $M[N[\Gamma]]$ may have is 1/42. It follows that $|Aut(S)| = bp^s(1+kp) \leq 84A(\gamma;r)p^s$. If $k > 0$, as $b \geq 1$, then $(*)$ obligates to have

$$p \leq 84A(\gamma;r) - 1.$$

It follows that if we choose $n(\gamma,r) = Max\{n_1(\gamma,r), 84A(\gamma;r)\}$, then for $p \geq n(\gamma,r)$ we have that $k = 0$ as desired and, in particular, $|Aut(S)| = bp^s$, where $(b,p) = 1$. This finish the proof of the theorem.

□

Now, returning to our commutator rigidity problem, Theorem 4 provides the following consequence.

Theorem 5. *Let $n \geq 2$ be a positive integer. Then, there is a prime p_n (depending only on n) so that for every prime $p \geq p_n$ the co-compact Fuchsian group*

$$\Gamma = \langle x_1, ..., x_n : x_1^p = \cdots = x_n^p = x_1 x_2 \cdots x_n = 1 \rangle < PSL(2,\mathbb{R})$$

is uniquely determined by its derived subgroup.

Theorem 6 ([**20**]). *If $n \in \{3,4\}$ and $r \in \{3,4,...\}$, then a Fuchsian group with presentation*

$$\Gamma = \langle x_1, ..., x_n : x_1^r = \cdots = x_n^r = x_1 x_2 \cdots x_n = 1 \rangle < PSL(2,\mathbb{R}).$$

is uniquely determined by its derived subgroup.

3.3. Torsion free Kleinian groups in space. The commutator rigidity property in general fails for finitely generated, torsion free, non-elementary Kleinian groups in higher dimensions as can be seen from the following example [**10**].

3.3.1. **Example.** Let us consider in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ the line $L = \{(x,y,z) : y = z = 0\}$ and one of its orthogonal planes, say $M = \{(x,y,z) : x = 0\}$. Let τ be a non-trivial rotation with axis of rotation being L, and let σ be the reflection about M. In the semi-space $M^+ = \{(x,y,z) : x > 0\}$ we consider $g \geq 2$ pairwise disjoint Euclidian spheres, say Σ_1,..., Σ_g, each one orthogonal to the line L, and all of them bounding a common domain. Let σ_j be the reflection on S_j and set $A_j = \sigma\sigma_j$, $B_j = \tau A_j$, for $j = 1, ..., g$. The two groups

$$G = \langle A_1, ..., A_g \rangle$$
$$K = \langle B_1, ..., B_g \rangle$$

turn out to be purely loxodromic Kleinian groups, isomorphic to a free group of rank g (that is, spatial Schottky groups of genus g). A common fundamental domain for these groups is given by the common domain bounded by the circles Σ_1,..., Σ_g, $\Sigma_1' = \sigma(\Sigma_1)$,..., $\Sigma_g' = \sigma(\Sigma_g)$. As $\tau \notin G$, we have that $G \neq K$, and as τ commutes with σ and σ_j, we have that $G' = K'$.

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