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On value distribution of differential polynomial of algebroidal functions

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Abstract. In this paper, we generalized some results of the paper [1] in algebroidal functions, and get some interesting results.

1. Definitions and Symbols

In this paper, let $w = w(z)$ be an algebroidal function with $\gamma - branches$ determined by an irreducible equation

$$A_\gamma(z)w^\gamma + A_{\gamma-1}(z)w^{\gamma-1} + \ldots + A_1(z)w + A_0(z) = 0.$$

Where $A_j(z)$ are holomorphic functions in $|z| < +\infty$, and satisfy that $m(r, A_0(z)) = o(T(r,w))$ and $A_j(z)$ do not simultaneously equal zero at a point $(j = 0, 1, \ldots, \gamma)$. In particularly, $w(z)$ is a meromorphic function when $\gamma = 1$.

$N_1(r, \frac{1}{w})$ denotes the function of the number of zero of order 1 of w, $N_2(r, \frac{1}{w})$ denotes the function of the number of zero of order $\geqslant 2$. $S(r,w)$ denotes a quantity satisfying $o(T(r,w))(r \to \infty, r \notin E)$, E denotes the set of finite linear measures.

Definition 1.1. A meromorphic function $a(z)$ is called a small function of $w(z)$ if $T(r, a(z)) = S(r,w)$.

$a, a_0, \ldots, a_n$ denote small functions of w, $c, c_0, \ldots, c_n$ denote complex constants.

Definition 1.2. Let $n_0, n_1, \ldots, n_k$ be nonnegative integers.

$$M(w) = w^{n_0}(w')^{n_1} \ldots (w^{(k)})^{n_k}$$

is called a differential monomial of w, $r_M = n_0 + n_1 + \ldots + n_k$ is called degree of $M(w)$, $\Gamma_M = n_0 + 2n_1 + \ldots + (k+1)n_k$ is called weight number of $M(w)$.

Definition 1.3. Let $M_j(w)$ be a monomial of w, $a_j(z)(j = 1, 2, \ldots, n)$ be small functions of w, then $\Omega(w) = a_1M_1(w) + \ldots + a_nM_n(w)$ is called a differential polynomial of w. The integer $r_\Omega = max\{r_{M_j} : 1 \leqslant j \leqslant n\}$ is called degree of $\Omega(w)$, $\Gamma_\Omega = max\{\Gamma_{M_j} : 1 \leqslant j \leqslant n\}$ is called weight number of $\Omega(w)$.

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In the studying existence of defective value of algebroidal functions, the value distribution of differential polynomials is an important application of Nevenlinna theory, it is more complex and more interesting than the corresponding case of meromorphic functions.

2. Main Results

THEOREM 2.1. *Let w(z) be an algebroidal function with γ-branches, $\Omega(w)(\not\equiv 0)$ be a $(n-1)$-degree differential polynomial, supposed $\psi = w^n w' + \Omega(w)$, then*

$$T(r,w) \leqslant \gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{\psi}) + \gamma[(\alpha+1)\gamma^2+2(\alpha+2)\gamma+2(\alpha+2)]\bar{N}(r,w)$$
$$+[(\sigma_\Omega+1)\gamma^2+2(\sigma_\Omega+2)\gamma+2(\sigma_\Omega+1)]N_x(r,w)+S(r,w).$$

Where $\alpha = \Gamma_\Omega-(n-1)$, $\sigma_\Omega = max\{\sigma_i\}$, $\sigma_i = i_1+3i_2+\ldots+(2n-1)i_n$.

If $\gamma = 1$, we get following result.

COROLLARY 2.1. *([**1**], Theorem 3) Let f be a non-rational meromorphic function, $Q(f)(\not\equiv 0)$ is $(n-1)$-degree differential polynomial of f, if $\psi = f^n f' + Q(f)$, then*

$$T(r,w) \leqslant 8\bar{N}(r,\frac{1}{\psi}) + (5\alpha+9)\bar{N}(r,w)+S(r,w).$$

Here $\alpha = \Gamma_Q-(n-1)$.

THEOREM 2.2. *Let $w(z)$ be an algebroidal function with $\gamma-$branches, $\Omega(w)(\not\equiv 0)$ be $(n-1)$-degree differential polynomial, if $\bar{N}(r,w) = S(r,w), N_x(r,w) = S(r,w)$, and $\psi = w^n w' + \Omega(w)$, then*

$$T(r,w) \leqslant \gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{\psi}) + S(r,w).$$

THEOREM 2.3. *Let w be an algebroidal function with γ-branches, $P(w) = a_n w^n + a_{n-1}w^{n-1}+\ldots+a_0$. Where $a_n(\not\equiv 0), a_{n-1},\ldots,a_0$ be small functions of w, and $\frac{a_{n-1}}{a_n}$ be a constant. Supposed $\Omega(w)$ be a differential polynomial of w with degree $\leqslant n-1$, and $\psi = P(w)w'+\Omega(w)$, then*

$$\psi = (w+\frac{a_{n-1}}{a_n})^n w', or$$
$$T(r,w) \leqslant \gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{\psi}) + \gamma[(\beta+1)\gamma^2+2(\beta+2)\gamma$$
$$+2(\beta+2)]\bar{N}(r,w)+[(\sigma_\Omega+1)\gamma^2+2(\sigma_\Omega+2)\gamma+2(\sigma_\Omega+1)]N_x(r,w)+S(r,w).$$

Where $\beta = max\{1, \Gamma_\Omega - r_\Omega)\}$.

COROLLARY 2.2. *In Theorem 2.3, if $\bar{N}(r,w) = S(r,w)$, $N_x(r,w) = S(r,w)$, the other conditionns is same as in Theorem 2.3, then $\psi = a_n(w+\frac{a_{n-1}}{a_n})^n w'$, or $T(r,w) \leqslant \gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{\psi})+S(r,w)$.*

COROLLARY 2.3. *Let w and $P(w)$ be same as in Theorem 2.3, $a(z)(\not\equiv 0)$ be a small function of w, supposed $\bar{N}(r,w) = S(r,w), N_x(r,w) = S(r,w)$, then $T(r,w) \leqslant \gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{P(w)w'-a})+S(r,w)$.*

Corollary 2.4. *If a γ-values algebroidal function w satisfying $N_x(r,w) = S(r,w)$, $\Theta(\infty,w) > 1-\frac{1}{\gamma(\gamma+2)^2}$, let $a(z)(\not\equiv 0)$ be a small function of w, then $ww'-a$ has infinity of zero points.*

Corollary 2.5. *Let w be a γ-values algebroidal function satisfying $N_x(r,w) = S(r,w)$, $P(w)$ be same as in Theorem 2.3, if $\Theta(\infty,w) > 1 - \frac{1}{2\gamma(\gamma^2+3\gamma+3)}$, then $P(w)w' - a$ has infinity of zero points.*

If setting $\gamma = 1$ in the above Corollaries, we get same results as the Theorems in [**1**].

3. Some Lemmas

Lemma 3.1. [**2**] *Let w be a γ-values algebroidal function,*

$$\Omega_1(w) = \sum a_{(i)} w^{i_0} (w')^{i_1} \dots (w^{(n)})^{i_n} (\not\equiv 0)((i) = i_0, \dots, i_n),$$

$$\Omega_2(w) = \sum b_{(j)} w^{j_0} (w')^{j_1} \dots (w^{(m)})^{j_m} (\not\equiv 0)((j) = j_0, \dots, j_n),$$

be differential polynomial of w, if $w^n\Omega_1(w) = \Omega_2(w)$ and $n \geqslant r_{\Omega_2}$, then we have $m(r,\Omega_1(w)) = S(r,w)$. Where r_{Ω_2} is the degree of Ω_2, $w(z)$ is a allowable function for the coefficients $\{a_{(i)}(z)\}$ and $\{b_j(z)\}$ of $\Omega_1(w)$ and $\Omega_2(w)$(i.e. $w(z)$ satisfying $\sum T(r,a_{(i)}) + \sum T(r,b_j) = o(T(r,w))$).

Lemma 3.2. [**3**] *Let w be a γ-branches algebroidal function, $\Omega_1^*(w)$ and $\Omega_2^*(w)$ be quasi-differential polynomials of w, w be a allowable function for the coefficients of $\Omega_1^*(w)$ and $\Omega_2^*(w)$, and satisfy $w^n\Omega_1^*(w) = \Omega_2^*(w), n \geqslant r_{\Omega_2^*}$, then $m(r,\Omega_1^*(w)) = S(r,w)$.*

Lemma 3.3. *Let w be a γ-branches algebroidal function, $\{a_{(i)}(z)\}$ be a meromorphic function of z, and satisfy*

$$T(r,a_{(i)}) = S(r,w), i = 0,1,\dots,\Omega(z) = \sum_i a_{(i)}(z) w^{i_0}(w')^{i_1} \dots (w^{(n)})^{i_n}$$

be a differential polynomial of algebroidal function of w, if the pole of w with order of $\tau(\infty,w)$ is not a zero point and pole of the coefficient of $a_{(i)}$, and supposed that the non-pole branching points of $w(z)$ which produce poles of the derivatives of $w(z)$ are not zero points of $\{a_{(i)}(z)\}$, then the order of poles of $\Omega(w)$ is the most

$$r_\Omega \tau(\infty,w) + (\Gamma_\Omega - r_\Omega)\gamma + \sigma_\Omega(\lambda - 1),$$

and

$$N(r,\Omega(w)) \leqslant r_\Omega N(r,w) + (\Gamma_\Omega - r_\Omega)\gamma \bar{N}(r,w) + \sigma_\Omega N_x(r,w) + S(r,w).$$

Where $\sigma_\Omega = max\{[i_1 + 3i_2 + \dots + (2n-1)i_n](\lambda - 1)\}$.

Lemma 3.4. [**3**] *Let $w(z)$ be a γ-branches algebroidal function, $\Omega_1(w)(\not\equiv 0)$ and $\Omega_2(w)(\not\equiv 0)$ be differential polynomials of w, if $w^n\Omega_1(w) = \Omega_2(w)$, then*

$$(n - r_\Omega)T(r,w) \leqslant nN_x(r,\frac{1}{w}) + (\Gamma_{\Omega_2} - r_{\Omega_2})\gamma \bar{N}(r,w) + \sigma_{\Omega_2} N_x(r,w) + S(r,w).$$

Where $N_x(r, \frac{1}{w})$ *denotes the function of the number of zero points of* w *which are non-pole branch points of* $w(z)$ *in* $\chi(r)$. $\sigma_{\Omega_2} = max\{j_1 + 3j_2 + \ldots + (2m-1)j_m\}$.

LEMMA 3.5. [**2**] *Let* $w = w(z)$ *be a* γ*-branches algebroidal function,* $\Omega(w)$ *be a differential polynomials of* w*, if* $w(z)$ *is a allowable function for the coefficients* $\{a_{(i)}(z)\}$ *of* $\Omega(w)$*, then* $m(r, \Omega(w)) \leqslant r_\Omega m(r, w) + S(r, w)$.

LEMMA 3.6. [**3**] *Let* $w(z)$ *be a* γ*-branches algebroidal function,* $\Omega^*(w)$ *be a quasi-differential polynomials of* w*, then* $m(r, \Omega^*(w)) \leqslant r_{\Omega_*} m(r, w) + S(r, w)$.

4. The Proof Of Theorem 2.1

PROOF. By

$$\psi = w^n w' + \Omega(w) \tag{4.1}$$

and its derivative $\psi\prime = \frac{\psi\prime}{\psi} w^n w' + \frac{\psi\prime}{\psi}\Omega(w)$, and $\psi\prime = nw^{n-1}(w')^2 + w^n w'' + (\Omega(w))'$, we get

$$w^{n-1}F = \Omega(w)(\frac{\psi'}{\psi} - \frac{(\Omega(w))'}{\Omega(w)}). \tag{4.2}$$

Where

$$F = n(w')^2 + ww'' - \frac{\psi\prime}{\psi} ww''. \tag{4.3}$$

By using a similar method to the Theorem 3 in [**1**], when $F \equiv 0$, the proof follows from the Annotation 1 in the appendix of this paper.

In the following supposed $F \not\equiv 0$, by (4.2) and Lemma 3.3 , we get

$$m(r, F) = S(r, w). \tag{4.4}$$

Now we estimate $N(r, F)$. It easily follows from (4.3) that the only possible poles of F are from poles of w or the non-pole branching points of w which generate poles of derivatives of w, or poles of the coefficients of $\Omega(w)$, or zero points of ψ which are not zero points of ww'.

Let z_0 be pole of $w(z)$ with order $\tau(\infty, w)$, then it is pole of $(n-1)\tau(\infty, w)$. As some non-pole branching points of $w(z)$ producing pole of derivatives of $w(z)$, by (4.2) and Lemma 3.3, we get that z_0 is a pole of $\Omega(w)(\frac{\psi'}{\psi} - \frac{(\Omega(w))'}{\Omega(w)})$, its order is at most $(n-1)\tau(\infty, w) + (\alpha+1)\gamma + \sigma_\Omega(r-1)$, and z_0 is a pole of F, its order is at most $(\alpha+1)\gamma + \sigma_\Omega(r-1)$, therefore we have

$$N(r, F) \leqslant \gamma \bar{N}^*(r, \frac{1}{\psi}) + (\alpha+1)\gamma \bar{N}(r, w) + \sigma_\Omega N_x(r, w) + S(r, w). \tag{4.5}$$

Where $\bar{N}^*(r, \frac{1}{\psi})$ denotes the reducing function of the number of zero points of ψ which are not zero points of ww'(i.e. not counting the order of a zero point of ψ).

We use $\bar{N}^{**}(r, \frac{1}{\psi})$ denoting the reducing function of the number of zero points of ψ which are zero points of ww', then we have

$$\bar{N}(r,\frac{1}{\psi}) = \bar{N}^*(r,\frac{1}{\psi}) + \bar{N}^{**}(r,\frac{1}{\psi}). \tag{4.6}$$

By (4.4) and (4.5), we get

$$T(r,F) \leqslant \gamma\bar{N}^*(r,\frac{1}{\psi}) + (\alpha+1)\gamma\bar{N}(r,w) + \sigma_\Omega N_x(r,w) + S(r,w). \tag{4.7}$$

Let z_0 be a zero point of w with order $\tau(z_0,\frac{1}{w})$, $w(z)$ has β-branches such that $w=0$ at $z_0(1\leqslant\beta\leqslant\gamma)$, then we have $w(z)=(z-z_0)^{\frac{r(z_0,\frac{1}{w})}{\beta}}g(z)$, $g(z_0)\neq 0,\infty$ in a neighborhood of z_0, hence $w^{(\alpha)}(z)=(z-z_0)^{\frac{r(z_0,\frac{1}{w})-\alpha\beta}{\beta}}g_\alpha(z)$, $g_\alpha(z_0)\neq 0,\infty(\alpha=1,2,\ldots)$. When $\tau(z_0,\frac{1}{w})-\alpha\beta>0$, z_0 is a zero point of $w^{(\alpha)}(z)$. Hence we get that $\frac{\psi\prime}{\psi}$ only has β-order poles. Associating with (4.3), we infer that

$$N_2(r,\frac{1}{w}) + \frac{1}{2}N_2(r,\frac{1}{w'}) \leqslant N(r,\frac{1}{F}), \tag{4.8}$$

and

$$\bar{N}(r,\frac{1}{F}) \leqslant N(r,\frac{1}{F}) - \frac{1}{2}N_2(r,\frac{1}{w}), \tag{4.9}$$

which are same as the results in meromophic functions of the paper [**1**]. By (4.7) and (4.8) we get

$$\begin{aligned} 2N_2(r,\frac{1}{w}) + N_2(r,\frac{1}{w'}) &\leqslant 2\gamma\bar{N}^*(r,\frac{1}{\psi}) \\ &+2(\alpha+1)\gamma\bar{N}(r,w) + 2\sigma_\Omega N_x(r,w) + S(r,w). \end{aligned} \tag{4.10}$$

As $\frac{F}{w^2} = n(\frac{w'}{w})^2 + \frac{w''}{w} - \frac{\psi\prime}{\psi}\frac{w'}{w}$, it is obvious that $m(r,\frac{F}{w^2}) = S(r,w)$, therefore $2m(r,\frac{1}{w}) \leqslant m(r,\frac{F}{w^2}) + m(r,\frac{1}{F}) \leqslant T(r,F) - N(r,\frac{1}{F}) + S(r,w)$. By using (4.7), we get

$$\begin{aligned} m(r,\frac{1}{w}) &\leqslant \frac{1}{2}2\gamma\bar{N}^*(r,\frac{1}{\psi}) + \frac{1}{2}(\alpha+1)\gamma\bar{N}(r,w) \\ &+\frac{1}{2}\sigma_\Omega N_x(r,w) - \frac{1}{2}N(r,\frac{1}{F}) + S(r,w). \end{aligned} \tag{4.11}$$

Set

$$G_1 = -\frac{1}{2n+1}\frac{\psi\prime}{\psi} - \frac{n}{2n+1}\frac{F'}{F}. \tag{4.12}$$

It is east to see that

$$m(r,G) = S(r,w). \tag{4.13}$$

It easily follows from (4.12) that all poles of G are β-order poles$(1\leqslant\beta\leqslant\gamma)$, and the poles of G are from zero points of ψ and F, or poles of w and coefficients of $\Omega(w)$, or some non-pole branching points of w which generate poles of derivatives of $w(z)$. Therefore we have

$$N(r,G) \leqslant \gamma\{\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,\frac{1}{F}) + \bar{N}(r,w)\} + N_x(r,w) + S(r,w).$$

Associating with (4.9), we get

$$N(r,G) \leqslant \gamma\{\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,w) + N(r,\frac{1}{F}) - \frac{1}{2}N_2(r,\frac{1}{w})\} + N_x(r,w) + S(r,w). \tag{4.14}$$

By above, we have

$$T(r,G) \leqslant \gamma\{\bar{N}(r,\frac{1}{\psi}) + \bar{N}(r,w) + N(r,\frac{1}{F}) - \frac{1}{2}N_2(r,\frac{1}{w})\} + N_x(r,w) + S(r,w). \tag{4.15}$$

In the following, we estimate $N(r,\frac{1}{w})$. let z_1 be a 1-order zero point of w, but not a zero point of ψ and not a pole of coefficients of $\Omega(w)$, it follows from (4.3) that

$$F(z_1) = n(w'(z_1))^2. \tag{4.16}$$

By using a similar method to the proof of the the theorem 3 [**1**], we have

$$\frac{F'(z_1)}{F(z_1)} = \frac{2n+1}{n}\frac{w''(z_1)}{w'(z_1)} - \frac{1}{n}\frac{\psi\prime(z_1)}{\psi(z_1)}. \tag{4.17}$$

Set

$$H = w'' + Gw'. \tag{4.18}$$

If $H(z) \equiv 0$, then the proof follows from the annotation 3 in the appendix of this paper. In the following supposed $H(z) \not\equiv 0$, by (4.18), we have

$$\frac{1}{w'} = \frac{\frac{w''}{w'} + G}{H}.$$

Combining it with (4.13), we get

$$m(r,\frac{1}{w'}) \leqslant m(r,\frac{1}{H}) + S(r,w). \tag{4.19}$$

It follows from (4.12), (4.17) and (4.18) that

$$H(z_1) = 0. \tag{4.20}$$

In the following, we estimate $T(r,H)$. By (4.13) and (4.18), we get

$$m(r,H) \leqslant m(r,w') + S(r,w). \tag{4.21}$$

It follows from (4.12) and (4.18) that the poles of $H(z)$ are from zero points of ψ and F, or poles of coefficients of w and $\Omega(w)$, or some non-pole branching points of $w(z)$ which generate poles of $w^{(\alpha)}(z), \alpha = 1, 2, \ldots, n$. Hence

$$\begin{aligned}
&N(r,H)\leqslant \gamma[\bar{N}(r,\frac{1}{\psi})+\bar{N}(r,\frac{1}{F})]+N(r,w)+(\gamma+1)\bar{N}(r,w)\\
&+(\gamma+2)N_x(r,w)+S(r,w)\leqslant \gamma\bar{N}(r,\frac{1}{\psi})+N(r,w)+(\gamma+1)\bar{N}(r,w)\\
&+(\gamma+2)N_x(r,w)+\gamma N(r,\frac{1}{F})-\frac{1}{2}\gamma N_2(r,\frac{1}{w})+S(r,w).
\end{aligned}\tag{4.22}$$

For γ-branches algebroidal functions, we have $m(r,w')+N(r,w)+\bar{N}(r,w)\leqslant T(r,w')$, by (4.21) and (4.22), we get

$$\begin{aligned}
T(r,H)\leqslant T(r,w')+\gamma\bar{N}(r,\frac{1}{\psi})+\gamma\bar{N}(r,w)+(\gamma+2)N_x(r,w)\\
+\gamma N(r,\frac{1}{F})-\frac{1}{2}\gamma N_2(r,\frac{1}{w})+S(r,w).
\end{aligned}\tag{4.23}$$

We use $N_1^*(r,\frac{1}{w})$ to denote the function of the numbers of 1-order zero points of w which are not zero points of ψ and not poles of the coefficients of $\Omega(w)$, $N_1^{**}(r,\frac{1}{w})$ to denote the function of the numbers of the other 1-order zero points of w. It follows from (4.19), (4.20) and (4.23) that

$$N_1^*(r,\frac{1}{w})\leqslant N(r,\frac{1}{H})\leqslant T(r,H)-m(r,\frac{1}{H})+O(1)\leqslant T(r,w')+\gamma\bar{N}(r,\frac{1}{\psi})$$

$$+\gamma\bar{N}(r,w)+(\gamma+2)N_x(r,w)+\gamma N(r,\frac{1}{F})-\frac{1}{2}\gamma N_2(r,\frac{1}{w})-m(r,\frac{1}{w'})+S(r,w).$$

Obviously,

$$N_1^{**}(r,\frac{1}{w})\leqslant \bar{N}^{**}(r,\frac{1}{\psi})+S(r,w).\tag{4.24}$$

By above two expressions, we have

$$\begin{aligned}
N_1(r,\frac{1}{w})\leqslant T(r,w')+\gamma\bar{N}(r,\frac{1}{\psi})+N^{**}(r,\frac{1}{\psi})+(\gamma+2)N_x(r,w)\\
+\gamma\bar{N}(r,w)+\gamma N(r,\frac{1}{F})-\frac{1}{2}\gamma N_2(r,\frac{1}{w})-m(r,\frac{1}{w'})+S(r,w).
\end{aligned}\tag{4.25}$$

Because of $N(r,\frac{1}{w})=N_1(r,\frac{1}{w})+N_2(r,\frac{1}{w})$, by (4.25) we get

$$N(r,\frac{1}{w})\leqslant T(r,w')+\gamma\bar{N}(r,\frac{1}{\psi})+\bar{N}^{**}(r,\frac{1}{\psi})+\gamma\bar{N}(r,w)+(\gamma+2)N_x(r,w)$$

$$+\gamma N(r,\frac{1}{F})+(1-\frac{1}{2}\gamma)N_2(r,\frac{1}{w})-m(r,\frac{1}{w'})+S(r,w).$$

By above the expression and (4.11), we get

$$\begin{aligned}T(r,w)\leqslant T(r,w')+\gamma\bar{N}(r,\frac{1}{\psi})+\frac{1}{2}\gamma\bar{N}^{*}(r,\frac{1}{\psi})+\bar{N}^{**}(r,\frac{1}{\psi})\\+(\frac{\alpha}{2}+\frac{3}{2})\gamma\bar{N}(r,w)+(\gamma-\frac{1}{2})N(r,\frac{1}{F})+(1-\frac{1}{2}\gamma)N_2(r,\frac{1}{w})\\-m(r,\frac{1}{w'})+(\gamma+\frac{1}{2}\sigma_\Omega+2)N_x(r,w)+S(r,w).\end{aligned}\tag{4.26}$$

As $T(r,w')-m(r,\frac{1}{w'})=N(r,\frac{1}{w'})+O(1)$, by (4.6), (4.7) and (4.26), we get

$$\begin{aligned}T(r,w)\leqslant N(r,\frac{1}{w'})+\gamma(\gamma+1)\bar{N}(r,\frac{1}{\psi})+[(\alpha+1)\gamma^2+\gamma]\bar{N}(r,w)\\+(\sigma_\Omega\gamma+\gamma+2)N_x(r,w)+(1-\frac{1}{2}\gamma)N_2(r,\frac{1}{w})+S(r,w).\end{aligned}\tag{4.27}$$

To estimate $N(r\frac{1}{w'})$, let

$$U=\frac{H}{w}=\frac{w''+Gw'}{w}.\tag{4.28}$$

We first estimate $T(r,U)$. Since $H\not\equiv 0$, $U\not\equiv 0$. It follows from (4.13) and (4.28) that

$$m(r,U)=S(r,w).\tag{4.29}$$

It is obvious that the poles of U are from zero points of ψ, F and w, or poles of coefficients of w and $\Omega(w)$, or some non-pole branching points of $w(z)$ which generate poles of $w^{(\alpha)},\alpha=1,2,\ldots,n$. By (4.20) and (4.28), we know that the 1-order zero points of w which are not zero points of ψ and not poles of the coefficients of $\Omega(w)$ are not poles of U. Therefore

$$\begin{aligned}N(r,U)\leqslant\gamma\bar{N}(r,\frac{1}{\psi})+\gamma\bar{N}(r,\frac{1}{F})+N_1^{**}(r,\frac{1}{w})\\+N_2(r,\frac{1}{w})+2\gamma\bar{N}(r,w)+(\gamma+2)N_x(r,w)+S(r,w).\end{aligned}$$

By the above expression and (4.6), (4.7), (4.9), (4.24) and (4.29), we get

$$\begin{aligned}T(r,U)\leqslant\gamma(\gamma+1)\bar{N}(r,\frac{1}{\psi})+\gamma[(\alpha+1)\gamma+2]\bar{N}(r,w)\\+(1-\frac{1}{2}\gamma)N_2(r,\frac{1}{w})+[\gamma(\sigma_\Omega+1)+2]N_x(r,w)+S(r,w).\end{aligned}\tag{4.30}$$

Let z_2 be a 1-order zero point of w' which is not a zero point of w and ψ, and not a pole of the coefficients of $\Omega(w)$, i.e. $w'(z)$ has β-branches such that $w'(z_2)=0$, at $z_2(1\leqslant\beta\leqslant\gamma)$. We have $w'(z)=(z-z_2)^{\frac{1}{\beta}}\hat{w_1}(z)$, $\hat{w_1}(z_2)\neq 0,\infty$ in a neighborhood of z_2. It follows from (4.3) that $F(z_2)=w(z_2)w''(z_2)$. By a similar method to the (35) in [**1**], for algebroidal functions, we have

$$\frac{U'(z_2)}{U(z_2)}-\frac{F'(z_2)}{F(z_2)}-\frac{\psi\prime(z_2)}{\psi(z_2)}-G(z_2)=0.\tag{4.31}$$

Set

$$V=\frac{U'}{U}-\frac{F'}{F}-\frac{\psi\prime}{\psi}-G, \tag{4.32}$$

by (4.31), we get

$$V(z_2)=0. \tag{4.33}$$

If $V\equiv 0$, the proof follows from the annotation 4 in the appendix of this paper. Supposed $V\not\equiv 0$, by (4.3) and (4.32), we get

$$m(r,w)=S(r,w). \tag{4.34}$$

It is obvious that the poles of V are from zero points of ψ, U and F, or poles of coefficients of U, w and $\Omega(w)$, or some non-pole branching points of $w(z)$ which generate poles of $w^{(\alpha(z))}, \alpha=1,2,\dots,n$. Therefore

$$N(r,V)\leqslant \gamma\bar{N}(r,\frac{1}{\psi})+\gamma\bar{N}(r,\frac{1}{U})+\gamma\bar{N}(r,\frac{1}{F})+N_1^{**}(r,\frac{1}{w})$$

$$+\gamma[\bar{N}_2(r,\frac{1}{w})+\bar{N}(r,w)+N_x(r,w)]+S(r,w).$$

By the above expression, (4.6), (4.7), (4.9), (4.24), (4.30) and (4.34), we get

$$\begin{aligned}T(r,V)\leqslant \gamma(\gamma+1)^2\bar{N}(r,\frac{1}{\psi})+\frac{1}{2}\gamma(3-\gamma)N_2(r,\frac{1}{w})+\gamma[(\alpha+1)\gamma^2\\ +(\alpha+3)\gamma+1]\bar{N}(r,w)+[(\sigma_\Omega+1)\gamma^2+(\sigma_\Omega+3)\gamma]N_x(r,w)+S(r,w).\end{aligned} \tag{4.35}$$

Set $N_1^*(r,\frac{1}{w'})$ to denote the function of the number of 1-order zero points of w' which are not zero points of w and ψ and not poles of the coefficients of the $\Omega(w)$; $N_1^{**}(r,\frac{1}{w'})$ to denote the function of the number of 2-order zero points of w; $N_1^{***}(r,\frac{1}{w'})$ to denote the function of the number of the others 1-order zero points of w'. It follows from (4.33), (4.35) that

$$N_1^*(r,\frac{1}{w'})\leqslant N(r,\frac{1}{V})\leqslant \gamma(\gamma+1)^2\bar{N}(r,\frac{1}{\psi})+\frac{1}{2}\gamma(3-\gamma)N_2(r,\frac{1}{w})+\gamma[(\alpha+1)\gamma^2$$

$$+(\alpha+3)\gamma+1]\bar{N}(r,w)+[(\sigma_\Omega+1)\gamma^2+(\sigma_\Omega+3)\gamma]N_x(r,w)+S(r,w).$$

As $N_1^{**}(r,\frac{1}{w'})\leqslant\frac{1}{2}N_2(r,\frac{1}{w})$ and $N_1^{***}(r,\frac{1}{w'})\leqslant\bar{N}_1^{**}(r,\frac{1}{\psi})+S(r,w)$, then

$$N_1(r,\frac{1}{w'})\leqslant \gamma(\gamma+1)^2\bar{N}(r,\frac{1}{\psi})+\frac{1}{2}\gamma(3\gamma-\gamma^2+1)N_2(r,\frac{1}{w})+\gamma[(\alpha+1)\gamma^2$$

$$+(\alpha+3)\gamma+1]\bar{N}(r,w)+\bar{N}^{**}(r,\frac{1}{\psi})+[(\sigma_\Omega+1)\gamma^2+(\sigma_\Omega+3)\gamma]N_x(r,w)+S(r,w).$$

Hence, we have

$$\begin{aligned}N(r,\frac{1}{w'})&=N_1(r,\frac{1}{w'})+N_2(r,\frac{1}{w'})\leqslant\gamma(\gamma+1)^2\bar{N}(r,\frac{1}{\psi})+\bar{N}^{**}(r,\frac{1}{\psi})\\&+\frac{1}{2}\gamma(3\gamma-\gamma^2+1)N_2(r,\frac{1}{w})+\gamma[(\alpha+1)\gamma^2+(\alpha+3)\gamma+1]\bar{N}(r,w)\\&+[(\sigma_\Omega+1)\gamma^2+(\sigma_\Omega+3)\gamma]N_x(r,w)+N_2(r,\frac{1}{w'})+S(r,w).\end{aligned}\tag{4.36}$$

Noting $\gamma-\frac{1}{2}\gamma^2+\frac{3}{2}=-\frac{1}{2}(\gamma-1)^2+2\leqslant 2$, by (4.36) and (4.27), we get

$$\begin{aligned}T(r,w)&\leqslant\gamma(\gamma+1)(\gamma+2)\bar{N}(r,\frac{1}{\psi})+2N_2(r,\frac{1}{w})\\&+\bar{N}^{**}(r,\frac{1}{\psi})+N_2(r,\frac{1}{w'})+[(\alpha+1)\gamma^2+2(\alpha+2)\gamma+2]\gamma\bar{N}(r,w)\\&+[(\sigma_\Omega+1)\gamma^2+2(\sigma_\Omega+2)\gamma+2)]N_x(r,w)+S(r,w).+\bar{N}^{**}(r,\frac{1}{\psi})\end{aligned}\tag{4.37}$$

By (4.37) and (4.10), we have

$$\begin{aligned}T(r,w)&\leqslant\gamma(\gamma^2+3\gamma+4)\bar{N}(r,\frac{1}{\psi})\\&+\gamma[(\alpha+1)\gamma^2+2(\alpha+2)\gamma+2(\alpha+2)]\bar{N}(r,w)\\&+[(\sigma_\Omega+1)\gamma^2+2(\sigma_\Omega+2)\gamma+2(\sigma_\Omega+1)]N_x(r,w)+S(r,w).\end{aligned}\tag{4.38}$$

The proof is completed. □

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