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Sharp Function Inequality for Multilinear Commutator of Singular Integral Operators with Non-Smooth Kernels

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Abstract. In this paper, we establish a sharp function estimate for some multilinear commutator of the singular integral operators with non-smooth kernels. As the application, we obtain the $L^p(1 < p < \infty)$ norm inequality for the multilinear commutator.

1. Introduction and Results

As the development of singular integral operators,their commutators have been well studied (see [1-2][4][6-7][9-12]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss states that the commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [2][4][7][9-12], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. In [3] and [8], the boundedness of the singular integral operators with non-smooth kernels and their commutators are obtained. The main purpose of this paper is to study the sharp function inequality for some multilinear commutator of the singular integral operators with non-smooth kernels. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

Definition 1. A family of operators $D_t, t > 0$ is said to be an "approximations to the identity" if, for every $t > 0$, D_t can be represented by the kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y)f(y)dy$$

for every $f \in L^p(R^n)$ with $p \geqslant 1$, and $a_t(x, y)$ satisfies:

$$|a_t(x, y)| \leqslant h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t),$$

where s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0$$

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for some $\epsilon > 0$.

Definition 2. A linear operator T is called the singular integral operators with non-smooth kernels if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x,y)$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an "approximations to the identity" $\{B_t, t > 0\}$ such that TB_t has associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1 t^{1/2}} |K(x,y) - k_t(x,y)|dx \leqslant c_2 \quad \text{for all} \quad y \in R^n.$$

(2) There exists an "approximations to the identity" $\{A_t, t > 0\}$ such that A_tT has associated kernel $K_t(x,y)$ which satisfies

$$|K_t(x,y)| \leqslant c_4 t^{-n/2} \quad \text{if} \quad |x-y| \leqslant c_3 t^{1/2},$$

and

$$|K(x,y) - K_t(x,y)| \leqslant c_4 t^{\delta/2}|x-y|^{-n-\delta} \quad \text{if} \quad |x-y| \geqslant c_3 t^{1/2},$$

for some $c_3, c_4 > 0, \delta > 0$.

Given some locally integrable functions b_j $(j = 1, \cdots, m)$. The multilinear operator associated to T is defined by

$$T_b(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m}(b_j(x) - b_j(y))\right] K(x,y)f(y)dy.$$

The main purpose of this paper is to prove a sharp function inequality for the multilinear commutators of the singular integral operator with non-smooth kernel when $b_j \in BMO(R^n)$. As the application, we obtain the $L^p(p > 1)$ norm inequality for the multilinear commutators.

First, let us introduce some notations. Throughout this paper, $Q = Q(x,d)$ will denote a cube of R^n with sides parallel to the axes, whose center is x and side length is d. For a locally integrable function b, the sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |b(y) - b_Q|dy,$$

where, and in what follows, $b_Q = |Q|^{-1}\int_Q b(x)dx$. It is well-known that (see [6])

$$b^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|}\int_Q |b(y) - c|dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that(see [6])

$$||b - b_{2^kQ}||_{BMO} \leqslant Ck||b||_{BMO} \text{ for } k \geqslant 1.$$

For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{BMO} = \prod_{j=1}^{m} ||b_j||_{BMO}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdot\cdot\cdot, \sigma(j)\}$ of $\{1, \cdot\cdot\cdot, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdot\cdot\cdot, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdot\cdot\cdot, b_m)$ and $\sigma = \{\sigma(1), \cdot\cdot\cdot, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdot\cdot\cdot, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. The sharp maximal function $M_A(f)$ associated with the "approximations to the identity" $\{A_t, t > 0\}$ is defined by

$$M_A^{\#}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

We shall prove the following theorems.

Theorem 1. Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $\tilde{x} \in R^n$,

$$M_A^{\#}(T_b(f))(\tilde{x}) \leqslant C \left(M_r(f)(\tilde{x}) + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_r(T_{b_{\sigma^c}}(f))(\tilde{x}) \right).$$

Theorem 2. Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then T_b is bounded on $L^p(R^n)$ for any $1 < p < \infty$, that is

$$||T_b(f)||_{L^p} \leqslant C||f||_{L^p}.$$

2. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1.([3][8]) Let T be the singular integral operators with non-smooth kernels as **Definition 2**. Then, for every $f \in L^p(R^n), 1 < p < \infty$,

$$||T(f)||_{L^p} \leqslant C||f||_{L^p}.$$

Lemma 2. Let $\{A_t, t > 0\}$ be an "approximations to the identity"and $b \in BMO(R^n)$. Then, for every $f \in L^p(R^n), p > 1$, $1 \leqslant u < r < \infty$ and $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \left(\frac{1}{|Q|} \int_Q |A_{t_Q}((b - b_Q)f)(y)|^u dy \right)^{1/u} \leqslant C||b||_{BMO} M_r(f)(\tilde{x}),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Proof. We fix $f \in L^p(R^n), p > 1$, $x_0 \in R^n$ and $x_0 \in Q$ for some cube Q with $\tilde{x} \in Q$. Then

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |A_{t_Q}((b-b_Q)f)(y)|^u dy\right)^{1/u} \\
\leqslant\ &\left(\frac{1}{|Q|}\int_Q\int_{R^n} h_{t_Q}(x,y)^u|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
\leqslant\ &\left(\frac{1}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)^u|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
&+\left(\sum_{k=1}^{\infty}\frac{1}{|Q|}\int_Q\int_{2^{k+1}Q\backslash 2^kQ} h_{t_Q}(x,y)^u|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
=\ &I_1+I_2.
\end{aligned}$$

We have, by the Hölder's inequality,

$$\begin{aligned}
I_1 &\leqslant \left(C\frac{1}{|Q||2Q|}\int_Q\int_{2Q}|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
&\leqslant C\left(\frac{1}{|2Q|}\int_{2Q}|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
&\leqslant C\frac{1}{|2Q|^{1/u}}\left[\left(\int_{2Q}|f(y)|^r dy\right)^{u/r}\left(\int_{2Q}|(b(y)-b_Q)|^{\frac{ur}{r-u}}\right)^{\frac{r-u}{r}} dy\right]^{1/u} \\
&\leqslant C\left(\frac{1}{|2Q|}\int_{2Q}|f(y)|^r dy\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|(b(y)-b_Q)|^{\frac{ur}{r-u}}dy\right)^{\frac{r-u}{ru}} \\
&\leqslant C||b||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For I_2, notice for $x \in Q$ and $y \in 2^{k+1}Q\backslash 2^kQ$, then $|x-y| \geqslant 2^{k-1}t_Q$ and $h_{t_Q}(x,y) \leqslant C\frac{s(2^{2(k-1)})}{|Q|}$. Thus

$$\begin{aligned}
I_2 &\leqslant \left(C\sum_{k=1}^{\infty}s(2^{2(k-1)})\frac{1}{|Q|^2}\int_Q\int_{2^{k+1}Q}|(b(y)-b_Q)f(y)|^u dydx\right)^{1/u} \\
&\leqslant C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b(y)-b_Q)f(y)|^u dy\right)^{1/u} \\
&\leqslant C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b(y)-b_{2^{k+1}Q})f(y)|^u dy\right)^{1/u} \\
&\quad +C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})|b_Q-b_{2^{k+1}Q}|\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^u dy\right)^{1/u}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|^{\frac{ur}{r-u}}dy\right)^{\frac{r-u}{ru}}\\
&\quad\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^{r}dy\right)^{1/r}\\
&\quad+C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})|b_Q-b_{2^{k+1}Q}|\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^{r}dy\right)^{1/r}\\
&\leqslant C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})||b||_{BMO}M_r(f)(\tilde{x})\\
&\quad+C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})k||b||_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C\sum_{k=1}^{\infty}2^{(k-1)n}s(2^{2(k-1)})(k+1)||b||_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C||b||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}
$$

where the last inequality follows from

$$\sum_{k=2}^{\infty}2^{(k-1)n}s(2^{2(k-1)})(k+1)\leqslant C\sum_{k=2}^{\infty}2^{-(k-1)\varepsilon}(k+1)<\infty$$

for some $\epsilon>0$. This completes the proof.

Lemma 3.([3][8]) For any $\gamma>0$, there exists a constant $C>0$ independent of γ such that

$$|\{x\in R^n : M(f)(x)>D\lambda, M_A^{\#}(f)(x)\leqslant\gamma\lambda\}|\leqslant C\gamma|\{x\in R^n : M(f)(x)>\lambda\}|$$

for $\lambda>0$, where D is a fixed constant which only depends on n. So that

$$||M(f)||_{L^p}\leqslant C||M_A^{\#}(f)||_{L^p}$$

for every $f\in L^p(R^n), 1<p<\infty$.

Lemma 4. Let $1<q<\infty$, $b_j\in BMO(R^n)$ for $j=1,\cdots,k$. Then

$$\frac{1}{|Q|}\int_Q\prod_{j=1}^{k}|b_j(y)-(b_j)_Q|dy\leqslant C\prod_{j=1}^{k}||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q\prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^{q}dy\right)^{1/q}\leqslant C\prod_{j=1}^{k}||b_j||_{BMO}.$$

Proof. Choose $1<p_j<\infty$ $j=1,\cdots,k$ such that $1/p_1+\cdots+1/p_k=1$, we obtain, by the Hölder's inequality,

$$\frac{1}{|Q|}\int_Q\prod_{j=1}^{k}|b_j(y)-(b_j)_Q|dy\leqslant\prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q|b_j(y)-(b_j)_Q|^{p_j}dy\right)^{1/p_j}\leqslant C\prod_{j=1}^{k}||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k} |b_j(y)-(b_j)_Q|^q dy\right)^{1/q} \leqslant \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y)-(b_j)_Q|^{p_j q} dy\right)^{1/p_j q} \leqslant C\prod_{j=1}^{k} ||b_j||_{BMO}.$$

The lemma follows.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T_b(f)(x)-A_{t_Q}T_b(f)(x)\right| dx \leqslant C\left(M_r(f)(\tilde{x})+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_r(T_{b_{\sigma^c}}(f))(\tilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. When $m = 1$ see [10]. Consider now the case $m \geqslant 2$. We write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$,

$$\begin{aligned} T_b(f)(x) \quad = \quad & \int_{R^n}\left[\prod_{j=1}^{m}(b_j(x)-b_j(y))\right]K(x,y)f(y)dy \\ &= \int_{R^n}\prod_{j=1}^{m}\left[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})\right]K(x,y)f(y)dy \\ &= \prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\int_{R^n}K(x,y)f(y)dy \\ &+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K(x,y)f(y)dy \\ &+(-1)^m\int_{R^n}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})K(x,y)f(y)dy \end{aligned}$$

and

$$\begin{aligned} A_{t_Q}T_b(f)(x) \quad = \quad & \int_{R^n}\left[\prod_{j=1}^{m}(b_j(x)-b_j(y))\right]K_t(x,y)f(y)dy \\ &= \int_{R^n}\prod_{j=1}^{m}\left[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})\right]K_t(x,y)f(y)dy \\ &= \prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\int_{R^n}K_t(x,y)f(y)dy \\ &+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K_t(x,y)f(y)dy \\ &+(-1)^m\int_{R^n}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})K_t(x,y)f(y)dy, \end{aligned}$$

then

$$
\begin{aligned}
&\left|T_b(f)(x)-A_{t_Q}T_b(f)(x)\right| \leqslant \left|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\int_{R^n}K(x,y)f(y)dy\right| \\
&+\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K(x,y)f(y)dy\right| \\
&+\left|\int_{R^n}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})K(x,y)f_1(y)dy\right| \\
&+\left|\prod_{j=1}^{m}(b_j(x)-(b_j)_{2Q})\int_{R^n}K_t(x,y)f(y)dy\right| \\
&+\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_{\sigma^c}K_t(x,y)f(y)dy\right| \\
&+\left|\int_{R^n}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})K_t(x,y)f_1(y)dy\right| \\
&+\left|\int_{R^n}\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})(K(x,y))-K_t(x,y))f_2(y)dy\right| \\
&= I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x)+I_6(x)+I_7(x),
\end{aligned}
$$

thus

$$
\begin{aligned}
& \frac{1}{|Q|}\int_Q\left|T_b(f)(x)-A_{t_Q}T_b(f)(x)\right|dx \\
\leqslant\ & \frac{1}{|Q|}\int_Q I_1(x)dx+\frac{1}{|Q|}\int_Q I_2(x)dx+\frac{1}{|Q|}\int_Q I_3(x)dx+\frac{1}{|Q|}\int_Q I_4(x)dx+\frac{1}{|Q|}\int_Q I_5(x)dx \\
& +\frac{1}{|Q|}\int_Q I_6(x)dx+\frac{1}{|Q|}\int_Q I_7(x)dx \\
=\ & I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4, I_5, I_6 and I_7. For I_1, by the Hölder's inequality with exponent $1/p_1+\cdots+1/p_m+1/r=1$, $1<p_j<\infty$, $j=1,\cdots,m$, and Lemma

4, we get

$$\begin{aligned}
I_1 \leqslant\ & \frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||T(f)(x)|dx \\
\leqslant\ & \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{1/p_1}\cdots\left(\frac{1}{|Q|}\int_Q |b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m} \\
& \times\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
\leqslant\ & C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For I_2, by the Minkowski's inequality and Lemma 4, we get

$$\begin{aligned}
I_2 \leqslant\ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||T((b-(b)_{2Q})_{\sigma^c}f)(x)|dx \\
\leqslant\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T_{b_{\sigma^c}}(f)(x)|^r dx\right)^{1/r} \\
\leqslant\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(T_{b_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For I_3, choose $1<p<r,\ 1<q_j<\infty,\ j=1,\cdots,m$ such that $1/q_1+\cdots+1/q_m+p/r=1$, by the boundedness of T on $L^p(R^n)$(see Lemma 1) and Hölder's inequality, we get

$$\begin{aligned}
I_3 =\ & \frac{1}{|Q|}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f\chi_{2Q})(x)|dx \\
\leqslant\ & \left(\frac{1}{|Q|}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p} \\
\leqslant\ & C\left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^p\cdots|b_m(x)-(b_m)_{2Q}|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p} \\
\leqslant\ & C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r} \\
& \times\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{pq_1}dx\right)^{1/pq_1}\cdots\left(\frac{1}{|2Q|}\int_{2Q}|b_m(x)-(b_m)_{2Q}|^{pq_m}dx\right)^{1/pq_m} \\
\leqslant\ & C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For I_4, I_5, I_6, choose $1<u<r,\ 1<p_j<\infty,\ j=1,\cdots,m$, such that $1/p_1+\cdots+1/p_m+1/u=1$, by Lemma 2 and similar to the proof of I_1, I_2, I_3, we get

$$\begin{aligned}
I_4 &\leqslant \frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})A_{t_Q}(f)(x)|dx\\
&\leqslant \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{1/p_1}\cdots\left(\frac{1}{|Q|}\int_Q |b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q |A_{t_Q}(f)(x)|^u dx\right)^{1/u}\\
&\leqslant C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).\\
I_5 &\leqslant \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||A_{t_Q}((b-(b)_{2Q})_{\sigma^c}f)(x)|dx\\
&\leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{u'}dx\right)^{1/u'}\left(\frac{1}{|Q|}\int_Q |A_{t_Q}((b-(b)_{2Q})_{\sigma^c}f)(x)|^u dx\right)^{1/u}\\
&\leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).\\
I_6 &\leqslant \frac{1}{|Q|}\int_Q |A_{t_Q}(\prod_{j=1}^m (b_j-(b_j)_{2Q})f\chi_{2Q})(x)|dx\\
&\leqslant \left(\frac{1}{|Q|}\int_Q |A_{t_Q}(\prod_{j=1}^m (b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^u dx\right)^{1/u}\\
&\leqslant C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For I_7, note that $|x-y|\geqslant d=t^{1/2}$, taking $1<p_j<\infty\ j=1,\cdots,m$ such that $1/p_1+\cdots+1/p_m+1/r=1$, then

$$\begin{aligned}
I_7 &= \frac{1}{|Q|}\int_Q\left|\int_{R^n}\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})(K(x,y))-K_t(x,y))f_2(y)dy\right|dx\\
&\leqslant \int_{R^n}|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f_2(y)|\left(\frac{1}{|Q|}\int_Q |K(x,y)-K_t(x,y)|dx\right)dy\\
&\leqslant C\int_{(2Q)^c}|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)|\left(\frac{1}{|Q|}\int_Q \frac{d^\delta}{|x_0-y|^{n+\delta}}dx\right)dy\\
&\leqslant C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)|\frac{d^\delta}{|x_0-y|^{n+\delta}}dy
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\frac{d^{\delta}}{(2^k d)^{n+\delta}}|2^{k+1}Q|\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}\\
&\quad\times\prod_{j=1}^{m}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j}\\
&\leqslant C\sum_{k=1}^{\infty}2^{-k\delta}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r}\\
&\quad\times\prod_{j=1}^{m}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j}\\
&\leqslant C\sum_{k=1}^{\infty}k^m 2^{-k\delta}\prod_{j=1}^{m}||b_j||_{BMO}M_r(f)(\tilde{x})\\
&\leqslant C||\vec{b}||_{BMO}M_r(f)(\tilde{x}),
\end{aligned}
$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Choose $1<r<p$ in Theorem 1 and by using Lemma 3. We first consider the case $m=1$, we have

$$
\begin{aligned}
||T_b(f)||_{L^p} &\leqslant ||M(T_b(f))||_{L^p}\leqslant C||M_A^{\#}(T_b(f))||_{L^p}\\
&\leqslant C||M_r(f)+M_r(T(f))||_{L^p}\leqslant C||f||_{L^p}+||T(f)||_{L^p}\\
&\leqslant C||f||_{L^p}.
\end{aligned}
$$

When $m\geqslant 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof.

3. Applications

In this section we shall apply Theorem 1 and 2 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [5][8]). Given $0\leqslant\theta<\pi$. Define

$$S_\theta=\{z\in C:|\arg(z)|\leqslant\theta\}\bigcup\{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta=S_\theta\setminus\{0\}$. An closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L)\subset S_\theta$ and for every $\nu\in(\theta,\pi]$, there exists a constant C_ν such that

$$|\eta|||(\eta I-L)^{-1}||\leqslant C_\nu,\quad \eta\notin\tilde{S}_\theta.$$

For $\nu\in(0,\pi]$, let

$$H_\infty(S_\mu^0)=\{f:S_\theta^0\to C:\ f\ \text{ is holomorphic and } ||f||_{L^\infty}<\infty\},$$

where $||f||_{L^\infty}=\sup\{|f(z)|:z\in S_\mu^0\}$. Set

$$\Psi(S_\mu^0)=\left\{g\in H_\infty(S_\mu^0):\exists s>0,\exists c>0\text{ such that }|g(z)|\leqslant c\frac{|z|^s}{1+|z|^{2s}}\right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geqslant 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$||g(L)|| \leqslant N||g||_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leqslant |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [5] and Theorem 2, we get

Theorem 3. Assume the following conditions are satisfied:

(i) The holomorphic semigroup e^{-zL}, $0 \leqslant |\arg(z)| < \pi/2 - \theta$ is represented by the kernels $a_z(x, y)$ which satisfy, for all $\nu > \theta$, an upper bound

$$|a_z(x, y)| \leqslant c_\nu h_{|z|}(x, y)$$

for $x, y \in R^n$, and $0 \leqslant |\arg(z)| < \pi/2 - \theta$, where $h_t(x, y) = Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ satisfies

$$||g(L)(f)||_{L^2} \leqslant c_\nu ||g||_{L^\infty} ||f||_{L^2}.$$

Then, for $b_j \in BMO(R^n)$ with $j = 1, \cdots, m$, the multilinear commutator $g(L)_b$ associated to $g(L)$ and b_j satisfies:

(a) For $1 < r < \infty$ and $\tilde{x} \in R^n$,

$$M_A^\#(g(L)_b(f))(\tilde{x}) \leqslant CM_r(f)(\tilde{x});$$

(b) For any $1 < p < \infty$, $g(L)_b$ is bounded on $L^p(R^n)$, that is

$$||g(L)_b(f)||_{L^p} \leqslant C||f||_{L^p}.$$

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