SCIENTIA
Series A: *Mathematical Sciences*, Vol. 19 (2010), 13–23
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Center of a graph with respect to edges

A.P. Santhakumaran

Abstract. For any vertex v and any edge e in a non-trivial connected graph G, the *eccentricity* $e(v)$ of v is $e(v) = \max\{d(v,u) : u \in V\}$, the *vertex-to-edge eccentricity* $e_1(v)$ of v is $e_1(v) = \max\{d(v,e) : e \in E\}$, the *edge-to-vertex eccentricity* $e_2(e)$ of e is $e_2(e) = \max\{d(e,u) : u \in V\}$ and the *edge-to-edge eccentricity* $e_3(e)$ of e is $e_3(e) = \max\{d(e,f) : f \in E\}$. The set $C(G)$ of all vertices v for which $e(v)$ is minimum is the *center* of G; the set $C_1(G)$ of all vertices v for which $e_1(v)$ is minimum is the *vertex-to-edge center* of G; the set $C_2(G)$ of all edges e for which $e_2(e)$ is minimum is the *edge-to-vertex center* of G; and the set $C_3(G)$ of all edges e for which $e_3(e)$ is minimum is the *edge-to-edge center* of G. We determine these centers for some standard graphs. We prove that for any graph G, either $C(G) \subseteq C_1(G)$ or $C_1(G) \subseteq C(G)$; and either $C_2(G) \subseteq C_3(G)$ or $C_3(G) \subseteq C_2(G)$. we also prove that $C_1(G)$ is a subgraph of some block of G; and the vertex set of every graph with at least two vertices is the vertex-to-edge center of some connected graph. Block graphs G having exactly one cut vertex are characterized for (i) $C(G) = C_1(G)$; and (ii) $C_2(G) = C_3(G)$. It is proved that for any non-trivial tree T, (i) $C_1(T) = C(T)$; and (ii) $C_2(T) = C_3(T)$, where the subgraph induced by $C_2(T)$ is a star.

1. Introduction

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic definitions and terminologies, we refer to [1,3]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is an $u-v$ *geodesic* in G. It is known that the distance d is a metric on the vertex set V. The *eccentricity* $e(v)$ of a vertex v is $e(v) = \max\{d(v,u) : u \in V\}$ and the collection of vertices with minimum eccentricity is called the *center* of G and is denoted by $C(G)$. The *distance sum* of a vertex v is $d(v) = \sum_{u \in V} d(v,u)$, and the collection of vertices with minimum distance sum is called the *median* of G and is denoted by $M(G)$. If T is a tree, the *branch weight* $bw(v)$ of v is the largest number of vertices in a component of $T-v$. The *branch weight*

2000 *Mathematics Subject Classification.* Primary 05C12.
Key words and phrases. center, vertex-to-edge center, edge-to-vertex center, edge-to-edge center.

centroid or simply the *centroid* of T is the collection of vertices with minimum branch weight.

THEOREM 1.1. [**6**] *The center of a tree consists of either a single vertex or two adjacent vertices; the centroid of a tree consists of either a single vertex or two adjacent vertices.*

THEOREM 1.2. [**9**] *For any tree T, the median of T equals the branch weight centroid.*

Two areas in which the concept of centrality in graphs and networks is widely applied are facility location problems and social networks. Many problems of finding the " best" site for a facility in a graph or network are in one of the two categories: (i) minimax location problems and (ii) minisum location problems. For example, if one is locating an emergency response facility such as fire service station or police station, then the main problem is to minimize the distance from the location of the facility to the vertex farthest from it. On the other hand, if one is locating a service facility such as post office or electricity office, then the main problem is to minimize the sum of the distances from the location of the facility to all the vertices of the graph. The minimax location problem and the minisum location problem refer to the center and the median respectively of a graph. These problems are of the vertex-serves-vertex type, where both the "facility" and the "customer" will be located on vertices. The nature of facility (such as super high way or railway line) to be constructed could necessitate selecting a structure (such as path) rather than just a vertex at which to locate a facility. Similarly, the facility may be required to service structures or areas within the network, and not just vertices. In view of this Slater[7] extended this concept of vertex centrality to more structural situations and proposed that four classes of facility location problems should be considered: (i) vertex-serves-vertex, (ii) vertex-serves-structure, (iii) structure-serves-vertex, and (iv) structure-serves-structure. Further Slater[8] studied in detail the structure-serves-vertex problem by taking the structure to be a path, leading to the concepts of *path center*, *path median* and *path centroid* of a graph. In this paper, we study the last three types of problems by taking the structure to be an edge and investigate the relations among them and also with the first one.

For subsets $S, T \subseteq V$ and any vertex v, let $d(v, S) = \min\{d(v, u) : u \in S\}$ and $d(S, T) = \min\{d(x, y) : x \in S, y \in T\}$. In particular, if $f = xy$ and $g = wz$ are edges of G, then $d(v, f) = \min\{d(v, x), d(v, y)\}$ and $d(f, g) = \min\{d(x, w), d(x, z), d(y, w), d(y, z)\}$. To define and develop the general problem, Slater [7] introduced the definition, let $C = \{C_i : i \in I\}$ and $S = \{S_j : j \in J\}$, where each C_i and each S_j is a subset of V. Let $e_s(C_i) = \max\{d(C_i, S_j) : j \in J\}$; C_i is called a *(C, S)-center* if $e_s(C_i) \leqslant e_s(C_k)$ for all $k \in I$. Actually, depending upon the problem, one may wish to include other conditions. For example, one might also require the minimality condition that there does not exist $C_h \subseteq C_i$ with $C_h \neq C_i$ and $e_s(C_h) = e_s(C_i)$, as Slater [8] did for path centers. Thus, in view of this definition, our problem of studying the vertex-serves-structure, structure-serves-vertex, and structure-serves-structure situations by taking the structure to be an edge reduces to considering the (V, C)-center, (C, V)-center and (C, C)-center respectively, where C denotes the collection of all edges in G. Centrality concepts have interesting applications in social networks [4,5]. In a social network, an edge represents two individuals having "a common interest" and hence the study of center of a graph with respect to edges has interesting applications in social networks.

2. Vertex-to-edge center of a graph

DEFINITION 2.1. For any vertex v in a connected graph G, the vertex-to-edge eccentricity $e_1(v)$ of v is $e_1(v) = max\{d(v,e) : e \in E\}$. A vertex v for which $e_1(v)$ is minimum is called a vertex-to-edge central vertex of G and the set of all vertex-to-edge central vertices of G is the vertex-to-edge center $C_1(G)$ of G. Any edge e for which $e_1(v) = d(v,e)$ called an eccentric edge of v.

EXAMPLE 2.1. For the graph G given in Figure 2.1, $C(G) = \{v_3, v_4, v_5\}, C_1(G) = \{v_1, v_2, v_3, v_4, v_5, v_7\}$ and for the graph H given in Figure 2.1, $C(H) = \{v_1, v_2, v_3, v_4, v_5\}$ and $C_1(H) = \{v_2, v_4\}$. The eccentricities and the vertex-to-edge eccentricities of the vertices of G and H in Figure 2.1 are given in Table 2.1 and Table 2.2 respectively.

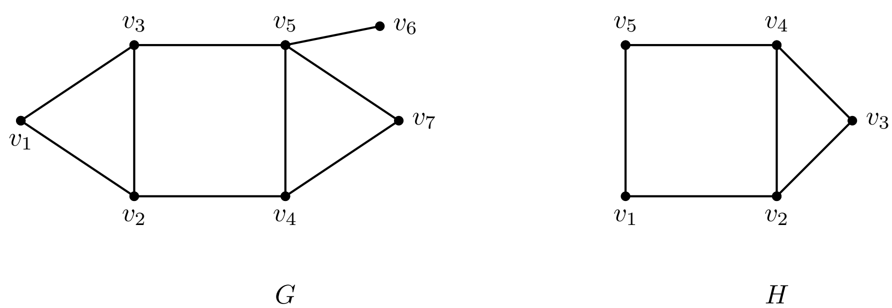

FIGURE 2.1.

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
$e(v)$	3	3	2	2	2	3	3
$e_1(v)$	2	2	2	2	2	3	2

Table 2.1.

v	v_1	v_2	v_3	v_4	v_5
$e(v)$	2	2	2	2	2
$e_1(v)$	2	1	2	1	2

Table 2.2.

REMARK 2.1. The subgraph induced by the center $C(G)$ or the subgraph induced by the vertex -to- edge center $C_1(G)$ of a connected graph G need not be connected. For the graph G given in Figure 2.2, $C(G) = C_1(G) = \{v_3, v_6\}$ and the subgraph induced is not connected .

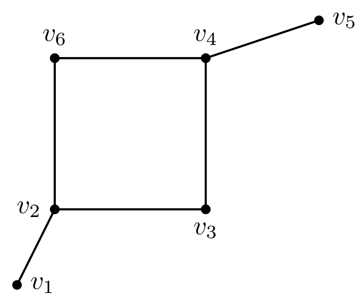

FIGURE 2.2. G

EXAMPLE 2.2. If G is a cycle or a complete graph or complete bipartite graph, then it is easily verified that $C(G) = C_1(G) = V$.

THEOREM 2.1. *For any vertex v in a graph $G, e(v) - 1 \leqslant e_1(v) \leqslant e(v)$. Further, $e_1(v) = e(v)$ if and only if both the vertices of any eccentric edge of v are eccentric vertices of v.*

PROOF. Let v be any vertex of G and let u be an eccentric vertex of v. Let $P : v = v_0, v_1, v_2, ..., v_k = u$ be a $v - u$ geodesic. Then $d(v, u) = k = e(v)$. Let $e = v_{k-1}v_k$. Since $d(v, e) = k - 1 = e(v) - 1$, it follows that $e_1(v) \geqslant e(v) - 1$. Also, since $e_1(v) =$min$\{d(v, e) : e \in E\}$, it follows that $e_1(v) \leqslant d(v, u) = e(v)$. Thus $e(v) - 1 \leqslant e_1(v) \leqslant e(v)$.

Now, suppose that $e_1(v) = e(v)$ for some vertex v in G. Then for any eccentric edge $e = xy$ of v, we have $d(v, e) = e_1(v) = e(v) \geqslant d(v, x)$. Thus $d(v, x) \leqslant d(v, e) \leqslant d(v, y)$. Similarly, we can prove that $d(v, y) \leqslant d(v, e) \leqslant d(v, x)$. It follows that $e(v) = d(v, x) = d(v, y)$ and so x and y are eccentric vertices of v.

Conversely, if both x and y are eccentric vertices of v, then $e(v) = d(v, x) = d(v, y)$. Hence $e_1(v) = e(v)$.

□

COROLLARY 2.1. *For any vertex v of a bipartite graph G, $e_1(v) = e(v) - 1$.*

PROOF. Let $e = xy$ be an eccentric edge of v. If both x and y are eccentric vertices of v, then $e(v) = d(v, x) = d(v, y)$. Let P and Q be $v - x$ and $v - y$ geodesics respectively. If u is the last vertex that is common to both P and Q, then the $u - x$ section of P followed by the edge xy and the $y - u$ section of Q^{-1} forms an odd cycle in G, which is a contradiction to G a bipartite graph. Hence it follows from Theorem 2.1 that $e_1(v) = e(v) - 1$. □

COROLLARY 2.2. *For any bipartite graph G, $C_1(G) = C(G)$.*

COROLLARY 2.3. *For any tree T, $C_1(T)$ consists of one vertex or two adjacent vertices.*

PROOF. This follows from Theorem 1.1 and Corollary 2.2. □

Corollary 2.4. *For any graph G, either $C(G) \subseteq C_1(G)$ or $C_1(G) \subseteq C(G)$.*

Proof. Suppose that the result is false. Then there exist vertices u and v such that $u \in C_1(G) - C(G)$ and $v \in C(G) - C_1(G)$. Then $e(v) < e(u)$... (1) and $e_1(u) < e_1(v)$... (2). If $e_1(u) = e(u)$, it follows from (1) and (2) that $e(v) < e_1(v)$, which is a contradiction to Theorem 2.1. Similarly, if $e_1(v) = e(v) - 1$, it follows from (1) and (2) that $e_1(u) < e(u) - 1$, which is again a contradiction to Theorem 2.1. Hence it follows from Theorem 2.1 that $e_1(u) = e(u) - 1$... (3) and $e_1(v) = e(v)$... (4). Now, it follows from (2), (3) and (4) that $e(u) - 1 < e(v)$. This gives $e(u) \leqslant e(v)$, which is a contradiction to (1). Hence the proof is complete. □

A *block* of a graph is a maximal connected subgraph having no cut-vertices. A graph G with all its blocks complete is called a *block graph.*

It is proved in [1] that the center $C(G)$ of a graph G is contained in a block of G. A similar result holds for $C_1(G)$ also.

Theorem 2.2. *The vertex -to- edge center $C_1(G)$ of every connected graph G is a subgraph of some block of G.*

Proof. Assume, to the contrary, that G is a connected graph whose vertex-to-edge center $C_1(G)$ is not a subgraph of a single block of G. Then there is a cut vertex v of G such that $G - v$ has two components G_1 and G_2, each of which contains vertices of $C_1(G)$. Let $e = xy$ be an eccentric edge of v. Then $e_1(v) = d(v, e) = d(v, x)$ (say). Let P be a $v - x$ geodesic. At least one of G_1 and G_2 contains no vertices of P, say G_2 contains no vertices of P. Let w be a vertex of $C_1(G)$ that belongs to G_2 and let Q be a $w - v$ geodesic in G. Now Q followed by P gives a $w - x$ geodesic whose length is greater than that of P. It follows that $e_1(w) > e(v)$, which contradicts the fact that $w \in C_1(G)$. □

It is proved in [2] that every graph G is the center of some graph H. We prove a similar result for the vertex-to- edge center also. However, the technique adopted in this case is different.

Theorem 2.3. *The vertex set of every graph G with at least two vertices is the vertex-to-edge center of some connected graph.*

Proof. Let G be a given graph. We show that G is the center of some connected graph H (i.e. the vertex set of G is the center of H). First, if G is not complete, make G a complete graph. Then add two new vertices u and v to G and join them to every vertex of G but not to each other. Next, we add two new vertices u' and v' and join u' to u and v' to v. The resulting graph H is given in Figure 2.3 Then it is clear that $e_1(u') = e_1(v') = 3, e_1(u) = e_1(v) = 2$ and $e_1(w) = 1$ for every vertex w in G. Hence the vertex set of G is the vertex-to-edge center of H. □

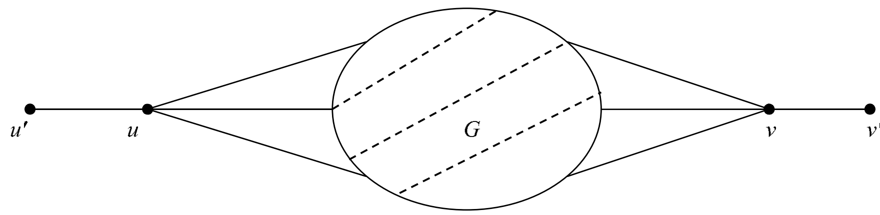

FIGURE 2.3. H

In the next two theorems, we give two classes of graphs G for which $C(G) = C_1(G)$.

THEOREM 2.4. *If G is a connected block graph with all its end-blocks K_2 or all its end-blocks $K_m (m \geqslant 3)$, then $C(G) = C_1(G)$.*

PROOF. If G is complete, then it is clear that $C(G) = C_1(G) = V$. If G is not complete, then G has at least two blocks. Let v be any vertex of G and let e be an eccentric edge of v. Then it is clear that e lies on an end-block B of G having exactly one cut vertex, say u of G and $e_1(v) = d(v, e) = d(v, u)$ or $d(v, u) + 1$ according as B contains exactly one edge or more than one edge. If B contains just one edge, then $e_1(v) = e(v) - 1$. Otherwise, it is clear that the eccentricity $e(v)$ of v is attained at both x and y of e and it follows from Theorem 2.1 that $e_1(v) = e(v)$. Hence $C(G) = C_1(G)$. □

REMARK 2.2. The converse of Theorem 2.4 is false. For the graph G given in Figure 2.4, $C(G) = C_1(G) = \{v_4\}$. However, all the end-blocks of G are neither K_2 nor all the end-blocks are $K_m (m \geqslant 3)$. Both types of eccentricities of the vertices of G are given in Table 2.3.

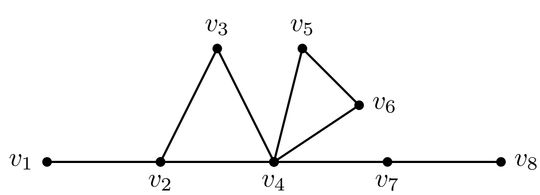

FIGURE 2.4. G

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$e(v)$	4	3	3	2	3	3	3	4
$e_1(v)$	3	2	2	1	2	2	2	3

Table 2.3.

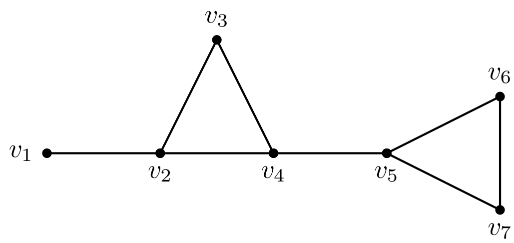

FIGURE 2.5. G

REMARK 2.3. If G is a connected block graph having end-blocks K_2 and complete graphs $K_m (m \geqslant 3)$, then $C(G)$ need not be equal to $C_1(G)$. For the graph G given in Figure 2.5, $C(G) = \{v_4\}$ and $C_1(G) = \{v_4, v_5\}$.

THEOREM 2.5. *Let $G = (K_{n_1} \cup Kn_2 ... \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geqslant 4, n_i \geqslant 2 (1 \leqslant i \leqslant r)$ and $n_1 + n_2 + ... + n_r + k = p - 1$. Then $C(G) = C_1(G)$ if and only if $r \geqslant 2$.*

PROOF. Let $r \geqslant 2$. Then $e(v) = e_1(v) = 1$ and $e(u) = e_1(u) = 2$ for each vertex $u \neq v$. Thus $C(G) = C_1(G) = \{v\}$. Conversely, let $C(G) = C_1(G)$. If $r = 1$, this means that G contains exactly one $K_t (t \geqslant 3)$ as an induced subgraph with v as one of its vertices. Also, since $p \geqslant 4$, it is clear that G contains at least one K_2 as an induced subgraph incident at v and distinct from every edge of K_t. Then it is clear that $e(v) = 1$ and $e(u) = 2$ for every vertex $u \neq v$. Also $e_1(x) = 1$ for every vertex x of K_t and $e_1(y) = 2$ for every other vertex y of G. Thus $C(G) \neq C_1(G)$, which is a contradiction. Hence $r \geqslant 2$. □

REMARK 2.4. If a graph G has a vertex of maximum degree $p - 1$, then it is not true that $C(G) = C_1(G)$. For the graph G given in Figure 2.6, $C(G) = \{v\}$ and $C_1(G) = \{v, w, x\}$ so that $C(G) \neq C_1(G)$.

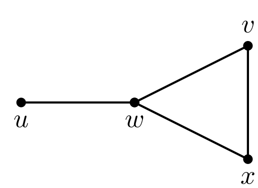

FIGURE 2.6. G

Thus we have determined certain classes of graphs G for which $C(G) = C_1(G)$. In view of this, we leave the following problem as an open question.

Problem 2.6. Characterize the class of graphs G for which $C(G) = C_1(G)$.

3. Edge-to- vertex center and edge-to- edge center of a graph

Definition 3.1. For any edge e in a connected graph G, the edge -to- vertex eccentricity $e_2(e)$ of e is $e_2(e) = max\{d(e, v) : v \in V\}$. Any edge e for which $e_2(e)$ is minimum is called an edge -to- vertex central edge of G and the set of all edge -to- vertex central edges of G is the edge-to- vertex center $C_2(G)$ of G. Any vertex v for which $e_1(e) = d(e, v)$ is called an eccentric vertex of e.

Definition 3.2. For any edge e in a connected graph G, the edge -to- edge eccentricity $e_3(e)$ of e is $e_3(e) = max\{d(e, f) : f \in E\}$. Any edge e for which $e_3(e)$ is minimum is called an edge -to-edge central edge of G and the set of all edge -to- edge central edges of G is the edge-to-edge center $C_3(G)$ of G. Any edge f for which $e_1(e) = d(e, f)$ is called an eccentric edge of e.

Example 3.1. For the graph G given in Figure 2.1, $C_2(G) = \{v_3v_5\}$ and $C_3(G) = \{v_2v_3, v_2v_4, v_3v_5, v_4v_5\}$. Both these types of eccentricities of edges of G in Figure 2.1 are given in Table 3.1. For the graph H given in Figure 3.1, $C_2(H) = \{v_2v_3, v_3v_4, v_2v_7, v_6v_7, v_3v_6\}$ and $C_3(H) = \{v_2v_3, v_3v_4, v_6v_7, v_3v_6\}$.

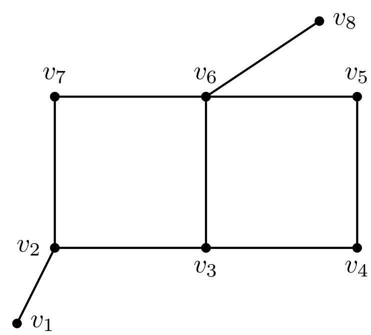

Figure 3.1: H

e	v_1v_2	v_1v_3	v_2v_3	v_2v_4	v_3v_5	v_4v_5	v_4v_7	v_5v_6	v_5v_7
$e_2(e)$	3	2	2	2	1	2	2	2	2
$e_3(e)$	2	2	1	1	1	1	2	2	2

Table 3.1.

Remark 3.1. The subgraph induced by the edge -to- vertex center $C_2(G)$ or the subgraph induced by the edge -to- edge center $C_3(G)$ of a connected graph G need not be connected. For the graph G given in Figure 3.2, $C_2(G) = C_3(G) = \{v_1v_2, v_2v_3, v_3v_4, v_5v_6, v_6v_7\}$.

Example 3.2. For an even cycle $G = C_{2n} (n \geqslant 2), e_2(e) = e_3(e) = n - 1$ for any edge e in G and for an odd cycle $G = C_{2n+1} (n \geqslant 2), e_2(e) = n$ and $e_3(e) = n - 1$

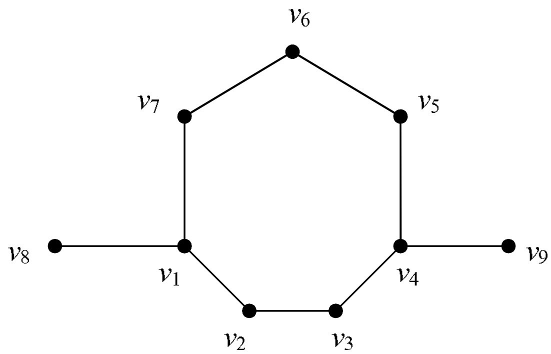

Figure 3.2: G

for any edge e in G. Also for $G = C_3, e_2(e) = e_3(e) = 1$ for any edge e. Thus for any cycle $G, C_2(G) = C_3(G) = E$. If a graph G is complete or complete bipartite, then it is easily seen that $C_2(G) = C_3(G)$.

THEOREM 3.1. *For any edge e in a graph $G, e_2(e) - 1 \leqslant e_3(e) \leqslant e_2(e)$. Further, $e_3(e) = e_2(e)$ if and only if both the vertices of any eccentric edge of e are eccentric vertices of e.*

PROOF. The proof is similar to that of Theorem 2.1. □

COROLLARY 3.1. *For any graph G, either $C_2(G) \subseteq C_3(G)$ or $C_3(G) \subseteq C_2(G)$.*

PROOF. The proof is similar to that of Corollary 2.4. □

REMARK 3.2. For any bipartite graph G, we have proved(Corollary 2.2) that $C(G) = C_1(G)$. However, it is not true that $C_2(G) = C_3(G)$ for a bipartite graph. For the graph H given in Figure 3.1, $C_2(H) \neq C_3(H)$.

THEOREM 3.2. *If G is a connected block graph with all its end-blocks K_2 or all its end-blocks $K_m (m \geqslant 3)$, then $C_2(G) = C_3(G)$.*

PROOF. The proof is similar to that of Theorem 2.4. □

COROLLARY 3.2. *For any nontrivial tree $T, C_2(T) = C_3(T)$.*

REMARK 3.3. The converse of Theorem 3.2 is false. For the graph G given in Figure 2.4, $C_2(G) = C_3(G) = \{v_2v_4, v_3v_4, v_4v_5, v_4v_6, v_4v_7\}$. However, all the end-blocks of G are neither K_2 nor all the end-blocks are $K_m (m \geqslant 3)$.

REMARK 3.4. If G is a connected block graph having end-blocks K_2 and complete graphs $K_m (m \geqslant 3)$, then $C_2(G)$ need not be equal to $C_3(G)$. For the graph G given in Figure 2.5, $C_2(G) = \{v_2v_4, v_3v_4, v_4v_5\}$ and $C_3(G) = \{v_4v_5\}$.

THEOREM 3.3. *Let $G = (K_{n_1} \cup K_{n_2} \cup ... \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geqslant 4, n_i \geqslant 2 (1 \leqslant i \leqslant r)$ and $n_1 + n_2 + ..., +n_r + k = p - 1$. Then $C_2(G) = C_3(G)$ if and only if $r \geqslant 2$.*

PROOF. Let $r \geqslant 2$. Then $e_2(f) = e_3(f) = 1$ for any edge incident at v and $e_2(g) = e_3(g) = 2$ for any other edge. Thus $C_2(G) = C_3(G) =$ the set of all edges incident at v. Conversely, let $C_2(G) = C_3(G)$. Then as in the proof of Theorem 2.5, it is easily seen that $e_2(f) = 1$ for any edge f incident at v and $e_2(g) = 2$ for any other edge g in K_t. Now, if $K_t = K_3$, then $e_3(f) = 0$ for the two edges of K_3 which are incident at v and $e_3(g) = 1$ for any other edge. If $K_t \neq K_3$, then $e_3(f) = 1$ for every edge f of G. Thus $C_2(G) \neq C_3(G)$, which is a contradiction. Hence $r \geqslant 2$. □

REMARK 3.5. If a graph G has a vertex of maximum degree $p-1$, then it is not true that $C_2(G) = C_3(G)$. For the graph G given in Figure 2.6, $C_2(G) = \{uw, wv, wx\}$ and $C_3(G) = \{wv, wx\}$ so that $C_2(G) \neq C_3(G)$.

For any subset X of edges, the subgraph induced by X has edge set X and consists of all vertices that are incident with at least one edge in X.

THEOREM 3.4. *The edge -to- vertex center $C_2(T)$ of a nontrivial tree T induces a star.*

PROOF. If T is a star and $T \neq K_2$, then $e_2(e) = 1$ for any edge e. If $T = K_2$, then $e_2(e) = 0$. Thus the subgraph induced by $C_2(T)$ is T itself. If T is not a star, we first prove that the subgraph induced by $C_2(T)$ is connected. Otherwise, there exist two edges $e = v_1v_2$ and $f = v_3v_4$ in $C_2(T)$ such that the path connecting e and f contains an edge, say $g = xy$ with g not in $C_2(T)$. Let $e_2(e) = e_2(f) = m$ and $e_2(g) = n$. Then $n > m$. Let z be an eccentric vertex of g so that $e_2(g) = d(g, z) = n$. If this distance n is attained at x, then $e_2(f) > n > m$, which is a contradiction and if the distance n is attained at y, then $e_2(e) > n > m$, which is again a contradiction . Thus $C_2(T)$ induces a connected graph. Now, we prove that $C_2(T)$ induces a star. Otherwise, the subgraph induced by $C_2(T)$ contains a path $P : v_1, v_2, v_3, v_4$ of length 3. Let $e = v_1v_2, f = v_2v_3$ and $g = v_3v_4$. Since $e, f, g \in C_2(T)$, we have $e_2(e) = e_2(f) = e_2(g) = m(say)$. Let x be an eccentric vertex of f so that $e_2(f) = d(f, x) = m$. If the distance $d(f, x)$ is attained at v_2, it is easy to see that $d(g, x) = m + 1$ so that $e_2(g) \geqslant m+1$, which is a contradiction and if the distance $d(g, x)$ is attained at v_3, it is easy to see that $d(e, x) = m + 1$ so that $e_2(e) \geqslant m + 1$, which is again a contradiction. Hence $C_2(T)$ induces a star.

□

COROLLARY 3.3. *The edge-to- edge center $C_3(T)$ of a nontrivial tree T induces a star.*

PROOF. This follows from Theorem 3.4 and Corollary 3.2. □

As in the previous section, we leave the following problem as an open question.

PROBLEM 3.5. Characterize the class of graphs G for which $C_2(G) = C_3(G)$.

The *line graph* of a given graph G is the graph $L(G)$, whose vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent in G.

The following theorem shows that the edge-to-edge center of any graph is the same as the center of its line graph.

THEOREM 3.6. *Let H denote the line graph of a graph G. Then $C_3(G) = C(H)$.*

PROOF. Let f be an edge in G and $e_3(f) = n$. Let g be an eccentric edge of f so that $d(f, g) = n$. Let $P : u_0, u_1, u_2, \ldots, u_n$ be a $f - g$ geodesic in G. Let $f_i(1 \leqslant i \leqslant n)$ be the edge $u_{i-1}u_i$. Since P is a shortest path in G, the edges $f, e_1, e_2, \ldots, e_n, g$ are all distinct and $Q : f, e_1, e_2, \ldots, e_n, g$ is a $f - g$ geodesic in H. Let d_H denote the distance in H and $e_H(f)$ denote the eccentricity of f in H. Then $d_H(f, g) = n + 1$ and g is an eccentric vertex of f in H. Hence $e_H(f) = n + 1 = e_3(f) + 1$ and the theorem follows. □

REMARK 3.6. From the proof of Theorem 3.6, we see that $e_3(f) = e_H(f) - 1$ for any edge f in G. This gives a useful criterion to settle many problems regarding the edge-to-edge center of a graph.

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Received 05 05 2009, revised 09 09 2009

RESEARCH DEPARTMENT OF MATHEMATICS
ST.XAVIER'S COLLEGE (AUTONOMOUS)
PALAYAMKOTTAI - 627 002, INDIA
E-mail address: apskumar1953@yahoo.co.in