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Series associated with Polygamma functions

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Abstract. We use integral identities to establish a relationship with sums that include polygamma functions, moreover we obtain some closed forms of binomial sums. In particular cases, we establish some identities for Polygamma functions

1. Introduction

The aim of this paper is to give a proof and some examples of the following theorem:

Theorem 1. Let a be a positive real number, $m > 0$, $|t| \leqslant 1$, $j \geqslant 0$, $q \in \mathbb{N}$, $p \in \mathbb{N}$ and $j \geqslant 0$, then

$$
\begin{aligned}
S(a,j,m,p,q,t) &= \sum_{n=0}^{\infty} t^n n^p \binom{n+m-1}{n} Q^{(q)}(a,j) \\
&= q \int_0^1 (1-x)^{j-1} \left[\lambda_m(f)\right]^{(p)} \left[\log(1-x)\right]^{q-1} dx \\
&\qquad + j \int_0^1 (1-x)^{j-1} \left[\lambda_m(f)\right]^{(p)} \left[\log(1-x)\right]^{q} dx,
\end{aligned}
\tag{1.1}
$$

where $Q^{(q)}(a,j) = \frac{d^q Q(a,j)}{dj^q}$ is the q^{th} derivative operator of the binomial coefficient $Q(a,j) = \binom{an+j}{j}^{-1}$ and $\left[\lambda_m(f)\right]^{(p)}$ is the p^{th} consecutive derivative operator of $\lambda_m(f) = \sum_{n=0}^{\infty} \binom{n+m-1}{n} f^{\ n} = (1-f)^{-m}$ where $f = f(x) = tx^a$ for $x \in (0,1)$.

First we state a number of lemmas that will be useful in the proof of Theorem 1. For specific values of the parameters (a,j,m,p,q,t) we then highlight a number of examples, some of which include the summation of harmonic numbers.

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2. Technical Lemmas

LEMMA 1. *For a and m positive real numbers, and $t \in \mathbb{R}$ let*

$$f = f(x) = tx^a \tag{2.1}$$

and

$$\lambda_m(f) = \sum_{n=0}^{\infty} \binom{n+m-1}{n} f^{\ n} = \frac{1}{(1-f)^m}.$$

The consecutive derivative operator of the continuous function $(1-f)^{-m}$ for $x \in (0,1)$ is defined as

$$\begin{aligned} [\lambda_m(f)]^{(0)} &= \frac{1}{(1-tx^a)^m} \\ &\vdots \\ [\lambda_m(f)]^{(p)} &= \underbrace{x\frac{d}{dx}\left(x\frac{d}{dx}\left(\cdots x\frac{d}{dx}\left(\frac{1}{(1-f)^m}\right)\right)\right)}_{p-times} \end{aligned}$$

so that

$$[\lambda_m(f)]^{(p)} = a^p \sum_{n=0}^{\infty} n^p\ f^{\ n} = \frac{a^p}{(1-f)^{m+p}} \sum_{r=1}^{p} (-1)^{p+r+1}\ C_{p,m}(r)\ f^{\ r}, \tag{2.2}$$

where the convolution coefficient

$$C_{p,m}(r) = \sum_{\nu=1}^{r} (-1)^{\nu} (m)_{\nu} \binom{p-\nu}{r-\nu} S(p,\nu) \tag{2.3}$$

and

$$S(p,\nu) = \left\{ {p \atop \nu} \right\} = \frac{1}{\nu!} \sum_{\mu=0}^{\nu} (-1)^{\mu} \binom{\nu}{\mu} (\nu-\mu)^p$$

are Stirling numbers of the second kind.

PROOF. We note from (2.1), that $x\frac{df}{dx} = af$ and

$$[\lambda_m(f)]^{(1)} = a\sum_{n=0}^{\infty} n\binom{n+m-1}{n} f^{\,n} = \frac{maf}{(1-f)^{m+1}}$$

$$[\lambda_m(f)]^{(2)} = a^2\sum_{n=0}^{\infty} n^2\binom{n+m-1}{n} f^{\,n} = \frac{a^2}{(1-f)^{m+2}}\left\{mf(1-f)+m(m+1)f^2\right\}$$

$$\begin{aligned}[\lambda_m(f)]^{(3)} &= a^3\sum_{n=0}^{\infty} n^3\binom{n+m-1}{n} f^{\,n} \\ &= \frac{a^3}{(1-f)^{m+3}}\{mf(1-f)^2+3\cdot m(m+1)f^{\,2}(1-f)+m(m+1)(m+2)f^{\,3}\} \\ &\ \ \vdots \\ [\lambda_m(f)]^{(p)} &= a^p\sum_{n=0}^{\infty} n^p\binom{n+m-1}{n} f^{\,n} \\ &= \frac{a^p}{(1-f)^{m+p}}\sum_{r=1}^{p} S(p,r)\,(m)_r\ \ f^{\,r}(1-f)^{p-r} \\ &= \frac{a^p}{(1-f)^{m+p}}\sum_{r=1}^{p} S(p,r)\,(m)_r\ \ f^{\,r}\sum_{j=0}^{p-r}(-1)^j\binom{p-r}{j} f^{\,j}.\end{aligned}$$

Collecting powers of f we have that

$$[\lambda_m(f)]^{(p)} = \frac{a^p}{(1-f)^{m+p}}\sum_{r=1}^{p}(-1)^{p+r+1}\,C_{p,m}(r)\ f^{\,r},$$

where $C_{p,m}(r)$ is given by (2.3).

By induction we see that

$$\begin{aligned}[\lambda_m(f)]^{(p+1)} &= a^p\frac{d}{dx}[\lambda_m(f)]^{(p)} = a^p\ x\frac{d}{dx}\left[\sum_{r=1}^{p} S(p,r)\,(m)_r\ \ f^{\,r}(1-f)^{-m-r}\right] \\ &= a^{p+1}\Big[S(p,1)\,mf\ \ (1-f)^{-m-1}+\cdots \\ &\qquad + f^{\,p}(1-f)^{-m-p}\left\{(m+p-1)(m)_{p-1}S(p,p-1)\right. \\ &\qquad \left. +p(m)_pS(p,p)\right\}+(m+p)\,(m)_pS(p,p)f^{\,p+1}(1-f)^{-m-p-1}\Big].\end{aligned}$$

From properties of Stirling numbers of the second kind,

$$S(p,1) = S(p+1,1) = 1, \qquad S(p,p) = S(p+1,p+1) = 1.$$

Furthermore, $S(p,p-1)+pS(p,p) = S(p+1,p)$ is the recurrence relation of Stirling numbers of the second kind and from the fact that

$$(m+p-1)(m)_{p-1} = (m)_p, \qquad (m+p)\,(m)_p = (m)_{p+1},$$

we may write

$$a^{p+1}\Big[S(p,1)\,mf\ \ (1-f)^{-m-1}+\cdots$$
$$+f^{\ p}(1-f)^{-m-p}\{(m+p-1)(m)_{p-1}S(p,p-1)$$
$$+p(m)_pS(p,p)\}+(m+p)(m)_pS(p,p)f^{\ p+1}(1-f)^{-m-p-1}\Big]$$
$$=\frac{a^{p+1}}{(1-f)^{m+p+1}}\left[S(p+1,1)(m)_1\ f\ \ (1-f)^p+\cdots\right.$$
$$\left.+S(p+1,p)(m)_p\ f^{\ p}\ \ (1-f)+(m)_{p+1}S(p+1,p+1)f^{\ p+1}\right]$$
$$=\frac{a^{p+1}}{(1-f)^{m+p+1}}\sum_{r=1}^{p+1}S(p+1,r)(m)_r\ f^{\ r}(1-f)^{p+1-r}$$

so that (2.2) follows. ∎

The next lemma deals with the derivatives of binomial coefficients.

Lemma 2. *Let a be a positive real number with $j\geqslant 0$, $n>0$ and let $Q(a,j)=\binom{an+j}{j}^{-1}$ be an analytic function in j then,*

(2.4) $Q^{(1)}(a,j)$

$$=\frac{dQ}{dj}=\begin{cases}-Q(a,j)P(a,j),\ \ where\ P(a,j)=\sum\limits_{r=1}^{an}\frac{1}{r+j}\quad for\ j>0\\ \\ -Q(a,j)\left[\psi(j+1+an)-\psi(j+1)\right]\end{cases},$$

and for $\lambda\geqslant 2$

$$Q^{(\lambda)}(a,j)=\frac{d^\lambda Q}{dj^\lambda}=-\sum_{\rho=0}^{\lambda-1}\binom{\lambda-1}{\rho}Q^{(\rho)}(a,j)P^{(\lambda-1-\rho)}(a,j), \tag{2.5}$$

where $P^{(0)}(a,j)=\psi(j+1+an)-\psi(j+1)$, for $n=1,2,3,...$, and $Q^{(0)}(a,j)=Q(a,j)$. For $i=1,2,3,...$

$$P^{(i)}(a,j)=\frac{d^iP}{dj^i}=\frac{d^i}{dj^i}\left(\psi(j+1+an)-\psi(j+1)\right) \tag{2.6}$$
$$=(-1)^i i!\sum_{r=1}^{an}\frac{1}{(r+j)^{i+1}}$$
$$=(-1)^i i!\left[\zeta(i+1,j+1)-\zeta(i+1,j+1+an)\right].$$

Proof. Let

$$Q(a,j)=\binom{an+j}{j}^{-1}=\frac{\Gamma(an+1)\Gamma(j+1)}{\Gamma(an+j+1)}=\frac{\Gamma(an+1)}{\prod_{r=1}^{an}(r+j)}.$$

Taking logs of both sides and differentiating with respect to j we obtain the result (2.4).

Now from (2.4) and for $\lambda \geqslant 2$

$$Q^{(\lambda)}(a,j) = \frac{d^{\lambda}Q}{dj^{\lambda}} = Q^{(\lambda)}(a,j) = \frac{d^{\lambda-1}}{dj^{\lambda-1}}(-QP) = -\sum_{\rho=0}^{\lambda-1}\binom{\lambda-1}{\rho} Q^{(\rho)} P^{(\lambda-1-\rho)}$$

and $P^{(\lambda-1-\rho)}(a,j)$ is given by (2.6). ∎

Remark 1. *We list the following*

$$Q^{(1)}(a,j) = -\binom{an+j}{j}^{-1}\left[H_{an+j}^{(1)} - H_j^{(1)}\right]$$

$$Q^{(2)}(a,j) = \binom{an+j}{j}^{-1}\left[\left(H_{an+j}^{(1)} - H_j^{(1)}\right)^2 + H_{an+j}^{(2)} - H_j^{(2)}\right]$$
$$= \binom{an+j}{j}^{-1}\left[\sum_{r=1}^{an}\sum_{s=1}^{r} \frac{2}{(r+j)(s+j)}\right],$$

$$Q^{(3)}(a,j) = -\binom{an+j}{j}^{-1}\left[\left(H_{an+j}^{(1)} - H_j^{(1)}\right)^3 + 2\left[H_{an+j}^{(3)} - H_j^{(3)}\right] + 3\left[H_{an+j}^{(2)} - H_j^{(2)}\right]\left[H_{an+j}^{(1)} - H_j^{(1)}\right]\right].$$

In the special case when $a = 1$ and $j = 0$ we may write

$$\begin{cases} Q^{(1)}(1,0) = -H_n^{(1)}, \\ Q^{(2)}(1,0) = \left(H_n^{(1)}\right)^2 + H_n^{(2)}, \\ Q^{(3)}(1,0) = \left(H_n^{(1)}\right)^3 + 3H_n^{(1)}H_n^{(2)} + 2H_n^{(3)}. \end{cases}$$

The generalised harmonic numbers are given by

$$H_n^{(r)} = \sum_{k=1}^{n}\frac{1}{k^r}; \quad H_n = H_n^{(1)}.$$

The Digamma function $\psi(z)$ is defined as

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,dt,$$

and has [**5**] *the series representation*

$$\psi(z) = \sum_{r=0}^{\infty}\left(\frac{1}{r+1} - \frac{1}{r+z}\right) - \gamma,$$

where γ is the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \log(n)\right) = -\psi(1) \approx 0.577215664901532860606512\ldots,$$

and $\Gamma(z)$ *is the Gamma function. Similarly the polygamma function* $\psi^{(k)}(z)$, $k \in \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$. *is defined by*

$$\psi^{(k)}(z) = \frac{d^{k+1}}{dz^{k+1}} \log \Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)$$
$$= -\int_{t=0}^{1} \frac{[\log(t)]^k \; t^{z-1}}{1-t} dt, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

$\psi^{(0)}(z) = \psi(z)$. *The polygamma function is connected to the Hurwitz zeta function by* $\psi^{(k)}(z) = (-1)^{k+1}\zeta(k+1, z)$. *The Digamma function is connected to the classical result*

$$\sum_{n=1}^{\infty} \frac{(m)_n}{n\,(p)_n} = \psi(p) - \psi(p-m) \tag{2.7}$$

for $\mathbb{R}(p-m) > 0$; $p \notin \mathbb{Z}_0^- := \{-1, -2, -3, \dots\}$, *where*

$$(m)_n = \frac{\Gamma(m+n)}{\Gamma(m)} = \begin{cases} 1; & n = 0 \\ m(m+1)(m+2)\cdots(m+n-1); & n \in \mathbb{N} \end{cases}$$

denotes the Pochhammer symbol, or the shifted factorial symbol. The well documented Gauss summation formula

$$\sum_{n=0}^{\infty} \frac{(m)_n (q)_n}{n!\,(p)_n} = \frac{\Gamma(p)\Gamma(p-m-q)}{\Gamma(p-m)\Gamma(p-q)}, \quad \mathbb{R}(p-m-q) > 0;\ p \notin \mathbb{Z}_0^-$$

is also closely related to the summation (2.7). One dimensional Euler sums may be written in the form (other forms are possible)

$$E_n = \sum_{n=1}^{\infty} \frac{t^n \left[H_n^{(r)}\right]^p}{n^q}, \quad t = \{1, -1\}.$$

In the study of Euler sums E_n, *there inevitably appears a rich zoo of special functions including gamma, digamma, polygamma, polylogarithms, zeta and many other functions, see for example* [**3**], [**2**], [**4**], [**1**] *and* [**7**] . *Some of these functions are related in a special way, such as*

$$H_n^{(1)} = \psi(n+1) - \psi(1) = \psi(n+1) + \gamma$$

and

$$H_n^{(r+1)} = \frac{(-1)^r}{r!}\left(\psi^{(r)}(n+1) - \psi^{(r)}(1)\right).$$

We state the following theorem which was given in [**6**].

THEOREM 2. Let a be a positive real number, $m > 0$, $|t| \leqslant 1$ and $j \geqslant 0$, then

$$\sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{\binom{an+j}{j}} = j \int_0^1 \frac{(1-x)^{j-1}}{(1-tx^a)^m} dx \tag{2.8}$$

$$= {}_{a+1} F_a \left[\begin{array}{c} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{array} \middle| t \right].$$

REMARK 2. *Many specific examples of (2.8) were given in* [**6**], *such as:*

$$\sum_{n=0}^{\infty} \frac{\binom{n+m-1}{n} \Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+j+1\right)} = j \int_0^1 \frac{(1-x)^{j-1}}{(1-\sqrt{x})^m} \, dx = \frac{2j \; {}_2F_1 \left[\begin{array}{c} 2, 1-j \\ j+2-m \end{array} \middle| -1 \right]}{(j+1-m)\,(j-m)}$$

$$= \frac{1}{m! \binom{j-1}{j-1-m}} \sum_{\mu=0}^{m} \frac{\binom{m}{\mu}}{\prod_{\nu=1}^{j-m} \left(\nu + \frac{\mu}{2}\right)}$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{\alpha}\right)^n \binom{n+3}{n}}{\binom{2n+4}{4}} = 4 \int_0^1 \frac{(1-x)^3}{\left(1-\frac{x^2}{\alpha}\right)^4} dx = {}_3 F_2 \left[\begin{array}{c} 4, \frac{1}{2}, 1 \\ \frac{5}{2}, 3 \end{array} \middle| \frac{1}{\alpha} \right], \alpha \neq 1$$

$$= \frac{1}{4} \left[3\alpha + \frac{5\sqrt{\alpha}}{2} \ln\left(\frac{\alpha+1}{\alpha-1}\right) - \frac{3\,(\alpha)^{3/2}}{2} \ln\left(\frac{\alpha+1}{\alpha-1}\right) \right].$$

Now we give a proof of Theorem 1 of this paper.

PROOF. From (2.8) we can write

$$\sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{\binom{an+j}{j}} = j \int_0^1 \frac{(1-x)^{j-1}}{(1-tx^a)^m} dx,$$

if we now apply the operator $[\lambda_m(f)]^{(p)}$, from (2.2), we see that

$$\sum_{n=0}^{\infty} \frac{t^n \; n^p \binom{n+m-1}{n}}{\binom{an+j}{j}} = j \int_0^1 (1-x)^{j-1} \left[\lambda_m(f)\right]^{(p)} dx.$$

Now we utilise the operator $Q^{(q)}(a,j)$, from (2.5), to obtain

$$\sum_{n=0}^{\infty} t^n \; n^p \binom{n+m-1}{n} Q^{(q)}(a,j)$$

$$= q \int_0^1 (1-x)^{j-1} \left[\lambda_m(f)\right]^{(p)} \left[\log(1-x)\right]^{q-1} dx$$

$$+ j \int_0^1 (1-x)^{j-1} \left[\lambda_m(f)\right]^{(p)} \left[\log(1-x)\right]^{q} dx.$$

As a matter of interest it is worthwhile to note that for the special case of $j = 0$ we may express

$$\sum_{n=0}^{\infty} t^n n^p \binom{n+m-1}{n} Q^{(q)}(a,0) = q \int_0^1 (1-x)^{-1} \left[\lambda_m(f)\right]^{(p)} \left[\log(1-x)\right]^{q-1} dx.$$

∎

The following examples can now be given.

3. Examples

Corollary 1. *We consider the case $p = 2$ so that from (1.1) we have:*

$$\sum_{n=1}^{\infty} t^n \; n^2 \binom{n+m-1}{n} \frac{d^q}{dj^q} \left[Q(a,j)\right]$$
$$= qmt \int_0^1 \frac{(1-x)^{j-1} x^a (1+mtx^a)}{(1-tx^a)^{m+2}} \left[\log(1-x)\right]^{q-1} dx$$
$$+ jmt \int_0^1 \frac{(1-x)^{j-1} x^a (1+mtx^a)}{(1-tx^a)^{m+2}} \left[\log(1-x)\right]^{q} dx$$

From this corollary we can make a number of observations.

Remark 3. *For $q = 1, a = 2, j = 4, m = 2$ and $t = -1$*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \; n^2 \binom{n+1}{n}}{\binom{2n+4}{4}} \sum_{r=1}^{2n} \frac{1}{r+4}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \; n^2 \binom{n+1}{n}}{\binom{2n+4}{4}} \left[H_{2n+4}^{(1)} - \frac{25}{12}\right]$$
$$= 2\int_0^1 \frac{(1-x)^3 x^2 (1-2x^2)}{(1+x^2)^4} dx + 8\int_0^1 \frac{(1-x)^3 x^2 (1-2x^2)}{(1+x^2)^4} \log(1-x) dx$$
$$= \frac{23}{8} + 14G - 5\zeta(2) - \frac{\pi}{8}\left(29 + 14\ln(2)\right) + \frac{39}{4}\ln(2) + 2\left(\log(2)\right)^2,$$

from which we extrapolate the result

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \; n^2 (n+1)}{\binom{2n+4}{4}} H_{2n+4}^{(1)} = \frac{103}{12} + 5\zeta(2) + \frac{\pi}{8}\left(29 + 14\ln(2) - \frac{25 \cdot 7}{3}\right)$$
$$+ \frac{83}{12}\ln(2) - 14G - 2\left(\log(2)\right)^2,$$

where G is Catalan's constant, defined by

$$G = \frac{1}{2}\int_0^1 K(s) ds = \sum_{r=1}^{\infty} \frac{(-1)^r}{(2r+1)^2} \approx 0.915965...,$$

and $K(s)$ is the complete elliptic integral of the first kind, given by

$$K(s)=\int_0^{\pi/2}\frac{dt}{\sqrt{1-s^2\sin^2 t}}.$$

REMARK 4. *For non-integer, $a=\frac{1}{2}, j=3, m=2, q=3$ and $t=-1$*

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}\;\; n^2\,(n+1)\,Q^{(3)}\left(\frac{1}{2},3\right)\\
&=6\int_0^1\frac{(1-x)^2\,x^{\frac{1}{2}}\left(1-2x^{\frac{1}{2}}\right)}{\left(1+x^{\frac{1}{2}}\right)^4}\left[\log(1-x)\right]^2 dx\\
&\qquad+6\int_0^1\frac{(1-x)^2\,x^{\frac{1}{2}}\left(1-2x^{\frac{1}{2}}\right)}{\left(1+x^{\frac{1}{2}}\right)^4}\left[\log(1-x)\right]^3 dx\\
&=\frac{32803}{2}+2\log(4)\left[33\left\{\log(4)\right\}^3-130\left\{\log(4)\right\}^2+1230\log(4)-4080\right]\\
&\qquad+24\left[65\log(4)-205-33\left\{\log(4)\right\}^2\right]\zeta(2)\\
&\qquad+\left[3168\log(4)-3120\right]\zeta(3)-3564\zeta(4),
\end{aligned}$$

and $Q^{(3)}\left(\frac{1}{2},3\right)$ can be evaluated from (2.5).

REMARK 5. *The very special case of $a=1, j>m+2, q\in\mathbb{N}$ and $t=1$, gives*

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^2\binom{n+m-1}{n}Q^{(q)}(1,j)\\
&=qm\int_0^1(1-x)^{j-m-3}\,x\,(1+mx)\left[\log(1-x)\right]^{q-1}dx\\
&\qquad+jm\int_0^1(1-x)^{j-m-3}\,x\,(1+mx)\left[\log(1-x)\right]^{q}dx\\
&=(-1)^q\;mq!\left[\frac{(m+1)(m+2)}{(j-m-2)^{q+1}}-\frac{(m+1)(2m+1)}{(j-m-1)^{q+1}}+\frac{m^2}{(j-m)^{q+1}}\right].
\end{aligned}$$

We can note that

$$\sum_{j=m+3}^{\infty}\sum_{n=1}^{\infty} n^2\binom{n+m-1}{n}Q^{(q)}(1,j)\approx O\left(\zeta(q+1)\right).$$

For the case $q=4$,

$$\sum_{j=m+3}^{\infty}\sum_{n=1}^{\infty} n^2\binom{n+m-1}{n}Q^{(4)}(1,j)=24m+72m^2+\frac{93}{4}m^3+24m\zeta(5).$$

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