SCIENTIA
Series A: *Mathematical Sciences*, Vol. 17 (2009), 19–26
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Boundedness for multilinear commutator of multiplier operator on Hardy Spaces

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ABSTRACT. In this paper, the $(H^p_{\vec{b}}, L^p)$ and (H^1, L^1) type boundedness for the multilinear commutator associated with the Multiplier operator and $BMO(R^n)$ functions are obtained.

1. Introduction.

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [1, 2]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)(p > 1)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^P_{\vec{b}}(R^n)$, then $[b, T]$ maps continuously $H^P_{\vec{b}}(R^n)$ into $L^p(R^n)$. In addition we easily known that $H^p_{\vec{b}}(R^n) \subset H^p(R^n)$. The main purpose of this paper is to consider the continuity of the multilinear commutators related to the multiplier operators and $BMO(R^n)$ functions on certain Hardy spaces. Besides this paper also proves the multilinear commutators' boundedness from $H^1(R^n)$ to $L^1(R^n)$.

2. Definitions and Lemmas.

Let us first introduce some definitions. Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

A bounded measurable function k defined on $R^n \setminus \{0\}$ is called a multiplier. The multiplier operator T_k associated with k is defined by

$$(T_k f)(x) = k(x)\hat{f}(x), \text{for } f \in S(R^n),$$

2000 *Mathematics Subject Classification.* 42B20, 42B25.
Key words and phrases. Multiplier operator; Multilinear commutator; BMO; Hardy space.

where $\hat{f}$ denotes the Fourier transform of f and $S(R^n)$ is the Schwarz test function class.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index of non-negative integers $\alpha_j (j = 1, 2, \cdots, n)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Denote by D^α the partial differential operators of order α as follows:

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad .$$

Now, we recall the definition of the class $M(s,l)$. Denote by $|x| \sim t$ the fact that the value of x lies in the annulus $\{x \in R^n : at < |x| < bt\}$, where $0 < a \leqslant 1 < b < \infty$ are values specified in each instance.

Definition 1.([17])
Let $l \geqslant 0$ be a real number and $1 \leqslant s \leqslant 2$. we say that the multiplier k satisfies the condition $M(s,l)$, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s - |\alpha|}$$

for all $R > 0$ and multi-indices α with $|\alpha| \leqslant l$, when l is a positive integer, and in addition, if

$$\left(\int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \leqslant C(\frac{|z|}{R})^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all $|z| < R/2$ and all multi-indices α with $|\alpha| = [l]$, the integer part of l, i.e., $[l]$ is the greatest integer less than or equal to l, and $l = [l] + \gamma$ when l is not an integer.

Definition 2.
Let $\overrightarrow{b} = (b_1, b_2, \cdots, b_m)(m > 1)$, $T_k f(x) = (K * f)(x)$ for $K(x) = \check{k}(x)$, we define the multilinear commutator of multiplier operator

$$T^{\overrightarrow{b}}(f)(y) = [\overrightarrow{b}, T_k] f(y) = \int_{R^n} \prod_{j=1}^m (b_j(y) - b_j(z)) K(y - z) f(z) dz,$$

and $T(f)(y) = T_k(f)(y) = \int_{R^n} f(z) K(y - z) dz$.

Definition 3.([9, 17])
Let $0 < p \leqslant 1$, a is called a $(1,q) - atom$, if a satisfies:

(1)$suppa \subset B(x_0, r)$;

(2)$\|a\|_{L^q} \leqslant |B(x_0, r)|^{1/q - 1}$;

(3)$\int a(x) x^\gamma dx = 0$, for any $0 \leqslant |\gamma| \leqslant [s] (s \geqslant 0)$.

A temperate distribution f is said to belong to $H^1(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^\infty \lambda_j a_j(x),$$

where a_j are $(1,q)$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Moreover, $||f||_{H^1(R^n)} \approx \sum_{j=1}^\infty |\lambda_j|$.

Definition 4.([9, 17])
Let b_i $(i = 1, \cdots, m)$ be a locally integrable function and $0 < p \leqslant 1$. A bounded measurable function a on R^n is said a $(p, \vec{b})$ atom, if

(1) $suppa \subset B = B(x_0, r)$

(2) $||a||_{L^\infty} \leqslant |B|^{-1/p}$

(3) $\int_B a(y)dy = \int_B a(y) \prod_{l\in\sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \leqslant j \leqslant m$.

A temperate distribution f is said to belong to $H^p_{\vec{b}}(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where a_j are $(p, \vec{b})$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H^p_{\vec{b}}(R^n)} \approx (\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$.

Lemma 1.([17])
Let $k \in M(s, l), 1 \leqslant s \leqslant 2$, and $l > \frac{n}{s}$; then the associated mapping T_k, defined a priory for $f \in \hat{D}_0(R^n)$, $T_k f(x) = (f * K)(x)$, extends to a bounded mapping from$L^p(R^n)$ into itself for $1 < p < \infty$, $K(x) = \check{k}(x)$, for $D(R^n) = \{\phi \in S(R^n) : supp(\phi)$ is compact$\}$ and $\hat{D}_0(R^n) = \{\phi \in\ S(R^n) : \hat{\phi} \in D(R^n)$ and $\hat{\phi}$ vanishes in a neighbourhood of the origin$\}$.

Lemma 2.([18],[19])
Let $1 \leqslant s \leqslant 2$, $1 < \tilde{s} < \infty$, suppose that l is a real number with $l > n/r$, $1/r = max\{1/s, 1 - 1/\tilde{s}\}$, and $k \in M(s, l)$, $K(x) = \check{k}(x)$. If one of the following three conditions is verified

1)$\{l\} < \{\frac{n}{r}\}$, $0 < m\beta < 1 + \{l\} - \{\frac{n}{r}\}$;

2)$\{l\} = \{\frac{n}{r}\}$, $0 < m\beta < 1 - \{\frac{n}{r}\}$;

3)$\{l\} > \{\frac{n}{r}\}$, $0 < m\beta < \{l\} - \{\frac{n}{r}\}$.

Then there is a positive constant t, such that

$$(\int_{2^{k+1}Q\setminus 2^kQ} |K(x - z) - K(x_Q - z)|^{\tilde{s}} dz)^{1/\tilde{s}} \leqslant C2^{-kt}(2^k h)^{-n/\tilde{s}'}.$$

Lemma 3.([4])
Let $1 < s \leqslant 2$, l is an integer, with $n/s < l \leqslant n$, and $k \in M(s, l)$, then $|\{x \in R^n : |T(f)(x)| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_{L^1}$, for any constant $C > 0$ and $\lambda > 0$.

3.Theorems and Proofs

Theorem 1.
Let $b_i \in BMO(R^n),\ 1 \leqslant i \leqslant m,\ \vec{b} = (b_1, \cdots, b_m),\ n/(n + t) < p \leqslant 1,\ t > 0$, then the multilinear commutator $T^{\vec{b}}$ is bounded from $H^p_{\vec{b}}(R^n)$ to $L^p(R^n)$.

Proof.
It suffices to show that there exists a constant $C > 0$, such that for every $(p, \vec{b})$ atom a in Definition 4,

$$||T^{\vec{b}}(a)||_{L^p} \leqslant C.$$

Let a be a $(p,\vec{b})$ atom supported on a ball $B=B(x_0,r)$.
Write

$$\int_{R^n}|T^{\vec{b}}(a)(x)|^p dx=\int_{|x-x_0|\leqslant 2r}|T^{\vec{b}}(a)(x)|^p dx+\int_{|x-x_0|>2r}|T^{\vec{b}}(a)(x)|^p dx\equiv I+II.$$

For I, taking $q>1$, by Hölder's inequality and the L^q- boundedness of $T^{\vec{b}}$, we see that

$$\begin{aligned}
I&\leqslant\left(\int_{|x-x_0|\leqslant 2r}|T^{\vec{b}}(a)(x)|^{p\cdot\frac{q}{p}}dx\right)^{p/q}\cdot|B(x_0,2r)|^{1-p/q}\\
&\leqslant C||T^{\vec{b}}(a)(x)||_{L^q}^p\cdot|B(x_0,2r)|^{1-p/q}\\
&\leqslant C||\vec{b}||_{BMO}^p||a||_{L^q}^p|B|^{1-p/q}\\
&\leqslant C||\vec{b}||_{BMO}^p.
\end{aligned}$$

For II, denoting $\lambda=(\lambda_1,\cdots,\lambda_m)$ with $\lambda_i=(b_i)_B,\ 1\leqslant i\leqslant m$, where $(b_i)_B=\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}b_i(x)dx$, by Lemma 2, Hölder's inequality and the vanishing moment of a, we get

$$\begin{aligned}
II&=\sum_{k=1}^{\infty}\int_{2^{k+1}r\geqslant|x-x_0|>2^k r}|T^{\vec{b}}(a)(x)|^p dx\\
&\leqslant C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}\left(\int_{2^{k+1}r\geqslant|x-x_0|>2^k r}|T^{\vec{b}}(a)(x)|dx\right)^p\\
&\leqslant C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}\\
&\quad\times\left[\int_{2^{k+1}r\geqslant|x-x_0|>2^k r}\left(|\int_B\prod_{j=1}^m(b_j(x)-b_j(y))K(x-y)a(y)dy|\right)dx\right]^p\\
&\leqslant C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}\\
&\quad\times\left[\int_{2^{k+1}r\geqslant|x-x_0|>2^k r}\left(|\int_B\prod_{j=1}^m(b_j(x)-b_j(y))(K(x-y)-K(x-x_0))a(y)dy|\right)dx\right]^p\\
&\leqslant C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}\\
&\quad\times\left[\int_{2^{k+1}B\setminus 2^k B}\left(\int_B\prod_{j=1}^m|(b_j(x)-\lambda_j)-(b_j(y)-\lambda_j)||K(x-y)-K(x-x_0)||a(y)|dy\right)dx\right]^p\\
&\leqslant C\sum_{k=1}^{\infty}|B(x_0,2^{k+1}r)|^{1-p}
\end{aligned}$$

$$
\begin{aligned}
& \times \left[\int_{2^{k+1}B\setminus 2^kB}\left(\int_B \sum_{i=0}^m \sum_{\sigma\in C_i^m} |(\vec{b}(x)-\lambda)_\sigma(\vec{b}(y)-\lambda)_{\sigma^c}||K(x-y)-K(x-x_0)||a(y)|dy\right)dx\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty |B(x_0,2^{k+1}r)|^{1-p} \\
& \times \left[\sum_{i=0}^m \sum_{\sigma\in C_i^m} \int_{2^{k+1}B\setminus 2^kB} |(\vec{b}(x)-\lambda)_\sigma| \left(\int_B |(\vec{b}(y)-\lambda)_{\sigma^c}||K(x-y)-K(x-x_0)||a(y)|dy\right)dx\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty |B(x_0,2^{k+1}r)|^{1-p} \\
& \times \left[\sum_{i=0}^m \sum_{\sigma\in C_i^m} \int_B |(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)| \left(\int_{2^{k+1}B\setminus 2^kB} |(\vec{b}(x)-\lambda)_\sigma||K(x-y)-K(x-x_0)|dx\right)dy\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty |B(x_0,2^{k+1}r)|^{1-p} \times [\sum_{i=0}^m \sum_{\sigma\in C_i^m} \int_B |(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)|(\left(\int_{2^{k+1}B} |(\vec{b}(x)-\lambda)_\sigma|^{\tilde{s}'}dx\right)^{1/\tilde{s}'} \\
& \times \left(\int_{2^{k+1}B\setminus 2^kB} |K(x-y)-K(x-x_0)|^{\tilde{s}}dx\right)^{1/\tilde{s}})dy]^p \\
\leqslant\ & C\sum_{k=1}^\infty |B(x_0,2^{k+1}r)|^{1-p} \\
& \times \left[\sum_{i=0}^m \sum_{\sigma\in C_i^m} \int_B |(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)|dy \left(\int_{2^{k+1}B} |(\vec{b}(x)-\lambda)_\sigma|^{\tilde{s}'}dx\right)^{1/\tilde{s}'} 2^{-kt}(2^kB)^{-1/\tilde{s}'}\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty 2^{-kpt}|B(x_0,2^{k+1}r)|^{1-p} \left[\sum_{i=0}^m \sum_{\sigma\in C_i^m} (k+1)^\sigma \|\vec{b}_\sigma\|_{BMO} \int_B |(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)|dy\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty 2^{-kpt}|B(x_0,2^{k+1}r)|^{1-p} \left[\sum_{i=0}^m \sum_{\sigma\in C_i^m} (k+1)^m|B|^{-1/p}\cdot|B|\cdot\|\vec{b}\|_{BMO}\right]^p \\
\leqslant\ & C\sum_{k=1}^\infty 2^{-kpt}(k+1)^{mp}|B(x_0,2^{k+1}r)|^{1-p}|B|^{(1-\frac{1}{p})p}\|\vec{b}\|_{BMO}^p \\
\leqslant\ & C\sum_{k=1}^\infty 2^{k(n-p(n+t))}k^{mp}\|\vec{b}\|_{BMO}^p \\
\leqslant\ & C\|\vec{b}\|_{BMO}^p.
\end{aligned}
$$

This finishes the proof of Theorem 1.

Theorem 2.Let $1 < s \leqslant 2$, $l > n/s$ is an integer, $k \in M(s,l)$. If $0 < m\beta < 1 - \{\frac{n}{s}\}$, $\vec{b} \in BMO(R^n)$, then $T^{\vec{b}}$ is weak bounded from $H^1(R^n)$ to $L^1(R^n)$, for $\{\frac{n}{s}\} = \frac{n}{s} - [\frac{n}{s}]$.

Proof. Let $f \in H^1(R^n)$, and $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j$ be the atomic decomposition for f as in Definition 3, for a_j is a $(1,q)$-atom$(q>1)$, $\lambda_j \in C$, suppose $suppa_j \subset B_j = B(x_j, r_j)$, and denote $b_{ij} = \frac{1}{|B_j|}\int_{B_j} b_i(x)dx$, then

$$\begin{aligned} T^{\vec{b}}(f)(x) &= \sum_{j=1}^{\infty} \lambda_j \prod_{i=1}^{m}(b_i(x)-b_{ij})Ta_j(x)\chi_{2B_j}(x) + \sum_{j=1}^{\infty} \lambda_j \prod_{i=1}^{m}(b_i(x)-b_{ij})Ta_j(x)\chi_{(2B_j)^c}(x) \\ &\quad -T(\sum_{j=1}^{\infty} \lambda_j \prod_{i=1}^{m}(b_i - b_{ij})a_j)(x) \\ &= J_1(x) + J_2(x) + J_3(x). \end{aligned}$$

For $J_1(x)$, noting that T is bounded from $L^q(R^n)$ to $L^q(R^n)(q>1)$(Lemma 1).

$$\begin{aligned} &\|\prod_{i=1}^{m}(b_i(x)-b_{ij})T(a_j)(x)\chi_{2B_j}(x)\|_{L^1(R^n)} \\ \leqslant\ & \int_{2B_j}|\prod_{i=1}^{m}(b_i(x)-b_{ij})T(a_j)(x)|dx \\ \leqslant\ & C\|\vec{b}\|_{BMO}|B_j|^{1-1/q}\|T(a_j)\|_{L^q} \\ \leqslant\ & C\|\vec{b}\|_{BMO}|B_j|^{(1-\frac{1}{q})+\frac{1}{q}-1} \\ \leqslant\ & C\|\vec{b}\|_{BMO}, \end{aligned}$$

so

$$\begin{aligned} &|\{x \in R^n : |J_1(x)| > \lambda/3\}| \\ \leqslant\ & 3\lambda^{-1}\sum_{j=1}^{\infty}|\lambda_j|\|\prod_{i=1}^{m}(b_i(x)-b_{ij})T(a_j)(x)\chi_{2B_j}(x)\|_{L^1(R^n)} \\ \leqslant\ & C\|\vec{b}\|_{BMO}\lambda^{-1}\sum_{j=1}^{\infty}|\lambda_j|. \end{aligned}$$

By the Hölder inequality and the size condition of a in Definition 3, we have

$$\|\prod_{i=1}^{m}(b_i(x)-b_{ij})a_j\|_{L^1(R^n)} \leqslant C\|\vec{b}\|_{BMO},$$

since T is a weak type of (1,1)(Lemma 3), we get

$$\begin{aligned} &|\{x \in R^n : |J_3(x) > \lambda/3\}| \\ \leqslant\ & C\lambda^{-1}\sum_{j=1}^{\infty}|\lambda_j|\|\prod_{i=1}^{m}(b_i(x)-b_{ij})a_j(x)\|_{L^1(R^n)} \\ \leqslant\ & C\|\vec{b}\|_{BMO}\lambda^{-1}\sum_{j=1}^{\infty}|\lambda_j|. \end{aligned}$$

Denote $\triangle K(x,y,x_j) = K(x-y) - K(x-x_j)$, $D_k(x_j) = \{x : 2^k r_j < |x-x_j| < 2^{k+1}r_j\}$, by the Hölder inequality, Lemma 2, the size condition and the vanishing moment of a in Definition 3, for $J_2(x)$, we get

$$\begin{aligned} &\|\prod_{i=1}^{m}(b_i(x)-b_{ij})T(a_j)(x)\chi_{(2B_j)^c}(x)\|_{L^1(R^n)} \\ \leqslant\ & \int_{(2B_j)^c}|\prod_{i=1}^{m}(b_i(x)-b_{ij})\left(\int_{B_j}K(x-y)a_j(y)dy\right)|dx \\ \leqslant\ & \int_{(2B_j)^c}|\prod_{i=1}^{m}(b_i(x)-b_{ij})\left(\int_{B_j}\triangle K(x,y,x_j)a_j(y)dy\right)|dx \\ \leqslant\ & \int_{B_j}|a_j(y)|\sum_{k=1}^{\infty}\left(\int_{D_k(x_j)}|\triangle K(x,y,x_j)\prod_{i=1}^{m}(b_i(x)-b_{ij})|dx\right)dy \\ \leqslant\ & C\int_{B_j}|a_j(y)|\sum_{k=1}^{\infty}\left(\int_{D_k(x_j)}|\triangle K(x,y,x_j)|^{\tilde{s}}dx\right)^{1/\tilde{s}}\left(\int_{2^{k+1}B_j}|\prod_{i=1}^{m}(b_i(x)-b_{ij})|^{\tilde{s}'}dx\right)^{1/\tilde{s}'}dy \\ \leqslant\ & C\|\vec{b}\|_{BMO}\int_{B_j}|a_j(y)|dy\sum_{k=1}^{\infty}2^{-kt}(k+1)^m \\ \leqslant\ & C\|\vec{b}\|_{BMO}|B_j|^{\frac{1}{q}-1}|B_j|^{1-\frac{1}{q}}\sum_{k=1}^{\infty}2^{-k(l-n/s)}(k+1)^m \\ \leqslant\ & C\|\vec{b}\|_{BMO}. \end{aligned}$$

Thus, we get

$$|\{x \in R^n : |J_2(x)| > \lambda/3\}| \leqslant C\|\vec{b}\|_{BMO}\lambda^{-1}\sum_{j=1}^{\infty}|\lambda_j|,$$

and

$$\begin{aligned} & |\{x \in R^n : |T^{\vec{b}}(f)(x)| > \lambda\}| \\ \leqslant & \ C\textstyle\sum_{i=1}^{3}|\{x \in R^n : |J_i(x)| > \lambda/3\}| \\ \leqslant & \ C\|\vec{b}\|_{BMO}\lambda^{-1}\textstyle\sum_{j=1}^{\infty}|\lambda_j|. \end{aligned}$$

This completes the proof of Theorem 2.

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Received 02 06 2008, revised 16 11 2008

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