Maximum Principles for problems of general Parabolic Partial Differential Equations

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ABSTRACT. In this paper we show that a maximum principle cannot hold for the standard spatially semidiscrete problem for general parabolic equations. We also study conditions for the solution operator acting on the discrete initial data, with homogeneous lateral boundary conditions, to be a contraction or a positive operator.

1. Introduction

Let Ω be a bounded polyhedral domain in $\mathbb{R}^d$ and consider the problem

$$(1.1) \qquad u_t + Au = 0 \quad \text{in } \Omega, \quad \text{for } t > 0,$$

$$\text{with } u = g \quad \text{on } \partial\Omega, \text{ for } t \geqslant 0, \quad u(0) = v \quad \text{in } \Omega.$$

Here

$$Au = -\sum_{k,l=1}^{d} \frac{\partial}{\partial x_k}\left(a_{kl}\frac{\partial u}{\partial x_l}\right) + \sum_{k=1}^{d} b_k \frac{\partial u}{\partial x_k},$$

where the coefficients $a_{kl}, b_k \in C^d(\overline{\Omega})$, and $(a_{kl}(x))$ is a symmetric and uniformly positive definite matrix on $\overline{\Omega}$.

The maximum principle for (1.1) asserts that, if $Q_T = \Omega \times (0,T)$, with $T > 0$, the maximum and the minumum of a solution $u \in C^2(Q_T) \cap C(\overline{Q}_T)$ over $\overline{Q}_T$ occur on the parabolic boundary, $(\partial\Omega \times [0,T]) \cup (\Omega \times \{t = 0\})$.

As a consequence of this we find at once a bound for $|u|$ in Q_T, namely

$$(1.2) \qquad \|u\|_{Q_T} = \max\left(\|g\|_{\partial\Omega \times [0,T]}, \|v\|_{\Omega}\right), \quad \text{where } \|u\|_V = \max_{\overline{V}} |u|.$$

Here V denotes a set in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$; when $V = \Omega$ we normally omit this subscript.

Since constant functions satisfy the differential equation in (1.1), it is well

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known and easy to prove that (1.2) is equivalent to the positivity condition

(1.3) $u \geqslant 0$ in Q_T,   if $v \geqslant 0$ in Ω and $g \geqslant 0$ on $\partial\Omega \times [0, T]$.

In the case of homogeneous Dirichlet boundary conditions, i.e., when $g = 0$, it follows from (1.2) that for the solution operator $E(t)$, the semigroup defined by $u(t) = E(t)v$ on the continuous functions which vanish on $\partial\Omega$, is a contraction, or

(1.4) $\|E(t)v\| \leqslant \|v\|$, for $t \geqslant 0$.

We shall be concerned with spatially semidiscrete approximations of (1.1) based on continuous, piecewise linear finite elements, defined on a family of triangulations $\Upsilon_h = \{\tau\}$ of $\overline{\Omega}$ into closed simplices τ, such that any face of any τ is either a subset of the boundary $\partial\Omega$ or a face of another $\tau \in \Upsilon_h$. We set $h = \max_{\tau \in \Upsilon_h} \operatorname{diam}(\tau)$. We associate with Υ_h the finite dimensional spaces

$$S_h = \{\chi \in C(\overline{\Omega}) : \chi|_\tau \text{ linear for } \tau \in \Upsilon_h\} \text{ and } S_h^0 = \{\chi \in S_h : \chi = 0 \text{ on } \partial\Omega\}.$$

For each t, let $g_h(t)$ be the restriction to $\partial\Omega$ of a function in S_h and let $v_h \in S_h$ with $v_h = g_h(0)$ on $\partial\Omega$. The semidiscrete standard Galerkin finite element problem associated with (1.1) is then to find $u_h(t) \in S_h$ for $t \geqslant 0$ such that

(1.5) $(u_{h,t}, \chi) + A(u_h, \chi) = 0,   \forall \chi \in S_h^0,   t > 0,$

with $u_h(t) = g_h(t)$ on $\partial\Omega$,   for $t \geqslant 0$,   and $u_h(0) = v_h$,

where $(f, g) = \displaystyle\int_\Omega f(x)\, g(x)\, dx$, $\underline{b} = (b_1, ..., b_d)^T$, and

$$A(f, g) = \int_\Omega \Big(\sum_{k,l=1}^d a_{kl} \frac{\partial f}{\partial x_l}\frac{\partial g}{\partial x_k} + \sum_{k=1}^d b_k \frac{\partial f}{\partial x_k}g\Big)dx = A_0(f, g) + (\underline{b}.\nabla f, g).$$

It is natural to ask whether an analogue of the maximum principle (1.2) holds for the discrete problem (1.5), or equivalently

(1.6) $\|u_h\|_{Q_T} = \max(\|g_h\|_{\partial\Omega\times[0,T]}, \|v_h\|_\Omega)$.

We shall demonstrate below that, in general, this is not the case. In [2] it was shown, for special families of triangulations Υ_h with all angles acute, that the backward Euler method may satisfy a maximum principle. His result requires a certain lower bound for the time step and therefore does not imply the same for the semidiscrete method by letting the time step tend to 0.

As a preparation for our analysis, we express the semidiscrete problem (1.5) in matrix form: Let $\{P_i\}_{i=1}^n$ denote the nodes of Υ_h in the interior of Ω, and $\{P_{n+i}\}_{i=1}^m$ those on $\partial\Omega$, and let $\{\Phi_i\}_{i=1}^{n+m} \subset S_h$ be the standard basis of pyramid functions defined by $\Phi_i(P_j) = \delta_{ij}$. The mass and stiffness matrices are then

$\mathcal{M} = (m_{ij})$ and $\mathcal{S} = (s_{ij})$, where $m_{ij} = (\Phi_i, \Phi_j)$ and $s_{ij} = A(\Phi_i, \Phi_j)$, $i, j = 1 : n$. To include the boundary terms, we also set $\mathcal{B} = (b_{ij})$ and $\mathcal{Z} = (z_{ij})$ with $b_{ij} = A(\Phi_i, \Phi_{n+j})$, $z_{ij} = (\Phi_i, \Phi_{n+j})$, $i = 1 : n, j = 1 : m$. We now also introduce the vector $\alpha(t) = (\alpha_1(t), ..., \alpha_n(t))^T$ of nodal values of $u_h(t)$ and correspondingly $\widetilde{g}(t) = (\widetilde{g}_{n+1}(t), ..., \widetilde{g}_{n+m}(t))^T$, where $\widetilde{g}_j(t) = g_h(P_j, t)$ and $\widetilde{v} = (v_h(P_1), ..., v_h(P_n))^T$.
Thus $u_h(t) = \sum\limits_{i=1}^{n} \alpha_i(t) \Phi_i + \sum\limits_{j=1}^{m} \widetilde{g}_{n+j}(t) \Phi_{n+j}$, and we may

hence write (1.5) as

$$(1.7) \qquad \mathcal{M} \frac{d\alpha}{dt} + \mathcal{S}\alpha = -\mathcal{B}\widetilde{g} - \mathcal{Z} \frac{d\widetilde{g}}{dt}, \qquad \text{for } t \geqslant 0 \qquad \text{with } \alpha(0) = \widetilde{v}.$$

Since the last term in (1.7) cannot be bounded by $\|g_h\|_{\partial\Omega \times [0,T]}$, it is already now clear that the full discrete maximum principle (1.6) cannot hold.

We now introduce the discrete semigroup $E_h(t)$ on S_h^0 by setting $E_h(t)v_h = u_h(t)$, where $u_h(t)$ is the solution of (1.5) with boundary data $g_h(t) = 0$. It has been shown by specific counterexamples in [6] that $E_h(t)$ does not generally satisfy the analogue of (1.4),

$$(1.8) \qquad\qquad \|E_h(t)v_h\| \leqslant \|v_h\|, \qquad \text{for } t \geqslant 0.$$

Our first goal in this paper is to show that, in fact, (1.8) cannot hold for any triangulation Υ_h which is "fine" enough. We continue to show that $E_h(t)$ cannot be positive in the sense that $E_h(t)v_h \geqslant 0$ if $v_h \geqslant 0$. Thus in neither case (1.6) can be valid. These results will be shown in Section 2 below. For weaker maximum-norm stability estimates than (1.8), see [6].

We now turn to the lumped mass method, which results from replacing the mass matrix $\mathcal{M}$ in (1.7) by a diagonal matrix $\mathcal{D}$ with diagonal elements

$$d_{ii} = \sum\limits_{i=1}^{n} m_{ij} \quad \text{and also setting } \mathcal{Z} = 0, \text{ or}$$

$$(1.9) \qquad\qquad \mathcal{D} \frac{d\alpha}{dt} + \mathcal{S}\alpha = -\mathcal{B}\widetilde{g}, \qquad \text{for } t \geqslant 0, \qquad \text{with } \alpha(0) = \widetilde{v}.$$

This may also be written in variational form, replacing the inner product in (1.5) by a quadrature approximation, or

$$(1.10) \qquad\qquad (u_{h,t}, \chi)_h + A(u_h, \chi) = 0, \qquad \forall \chi \in S_h^0, \ t > 0,$$

$$\text{with } u_h(t) = g_h(t) \quad \text{on } \partial\Omega, \quad \text{for } t \geqslant 0, \quad \text{and } u_h(0) = v_h,$$

where $(\Psi, \chi)_h = \sum\limits_{\tau \in \Upsilon_h} Q_{\tau,h}(\Psi\chi), \quad Q_{\tau,h}(f) = \frac{\text{meas}(\tau)}{d+1} \sum\limits_{P_j \in \widetilde{\tau}} f(P_j) \approx \int_\tau f dx.$

In this lumped mass case, to be discussed in Section 2 below, we shall show that the discrete maximum principle (1.6) holds if and only if the off-diagonal elements of the stiffness matrix $\mathcal{S}$ are nonpositive and if $\mathcal{B} \leqslant 0$, elementwise.

In one space dimension, this is always the case. In two space dimensions, with $A =$
$-\Delta$, this is equivalent to the condition that each edge of Υ_h , not entirely on $\partial\Omega$, is
of Delauney type, in the sense that the sums of the opposing angles in the two tri-
angles containing it are at most π. In fact, if P_i and P_j are neighbors, i.e., if P_iP_j
is an edge of Υ_h, and if α and β are the angles opposite P_iP_j, then $(\nabla\Phi_i, \nabla\Phi_j) =$
$-\sin(\alpha + \beta)/(4 \sin\alpha\sin\beta) < 0, = 0$, or > 0 when $\alpha + \beta < \pi, = \pi$,or $> \pi$, respec-
tively, see [1], [5], [6], and [7]. In [2] the condition used was that each of these an-
gles is $\leqslant \pi/2$, and, for higher space dimensions, similar conditions of "acute" type
are used. In [7] sharp conditions of Delauney type were given for $\mathcal{S}^{-1}$ to be nonposi-
tive in any number of space dimensions.

The stationary discrete elliptic problem corresponding to (1.5) and (1.10) is

$$(1.11) \qquad A(w_h, \chi) = 0, \ \ \forall\chi \in S_h^0, \ \ \text{with } w_h = g_h \ \ \text{on } \partial\Omega,$$

If $A(\chi, \chi)$ is positive definite on S_h^0, (1.11) has a unique solution and may be
written in matrix form, with α and $\widetilde{g}$ the vectors of nodal values of w_h and g_h,

$$(1.12) \qquad \mathcal{S}\alpha = -\mathcal{B}\widetilde{g}, \ \ \text{or } \alpha = -\mathcal{S}^{-1}\mathcal{B}\widetilde{g}.$$

The maximum principle for (1.11), or

$$(1.13) \qquad \|w_h\|_\Omega \leqslant \|g_h\|_{\partial\Omega} ,$$

is now equivalent to the positivity condition $\alpha \geqslant 0$ for $\widetilde{g} \geqslant 0$, and hence, by
(1.12), $\mathcal{S}^{-1}\mathcal{B} \leqslant 0$ is necessary and sufficient for (1.13). Consider now the discrete par-
abolic problem(1.10) with $v_h = 0$ at interior nodes
and $g_h(t) = g_h$ = constant in time, and assume that $A(\chi, \chi)$ is positive definite on S_h^0
and the maximum principle holds for (1.10). The solution of (1.10) then converges to the
solution of (1.11) as $t \to \infty$, and (1.13) thus also holds for the discrete stationary
problem. Hence $\mathcal{S}^{-1}\mathcal{B} \leqslant 0$ is a necessary condition for (1.6) in this case. In two di-
mensions and with $A = -\Delta$, by the results already mentioned, under Delauney con-
ditions on Υ_h, we have $\mathcal{S}^{-1} \geqslant 0$ and $\mathcal{B} \leqslant 0$ and hence (1.13) holds. In [4] exam-
ples were given of triangulations of non-Delauney type, for which the elliptic maxi-
mum principle still holds. In $\mathbb{R}^3$, [3] showed an elliptic maximum principle for tetra-
hedral decompositions of nonacute type. (For certain convection dominated ellip-
tic cases, [7]). Thus the parabolic maximum principle demands more stringent con-
ditions than (1.13). In Section 2 we also discuss conditions for contractivity and posi-
tivity of the solution operator $\widetilde{E}_h(t)$ on S_h^0 .

2. The Results and Their Proofs

We shall first show that, in general, a maximum principle cannot hold for the

semidiscrete standard Galerkin method (1.5). We shall show that, in fact, the discrete solution operator $E_h(t)$ on S_h^0, with $g_h(t) = 0$, is, in general, neither contractive nor positive. For our analysis, we define a node P_i of Υ_h to be strictly interior if all its neighbors are interior, and a near-boundary node if it is interior but not strictly interior. We set $\sigma_i = \text{supp} (\Phi_i)$.

Theorem 2.1. *Assume that div $\underline{b} = 0$ and that Υ_h is such that each near-boundary node has a strictly interior neighbor. Then $E_h(t)$ cannot be a contraction.*

The condition div $\underline{b} = 0$ is superfluous if for each near-boundary node P_j and an associated strictly interior neighbor P_i, we have $meas(\sigma_j \cap \sigma_i) \geqslant c \ meas(\sigma_i)$ with $c > 0$ and if h is sufficiently small.

Proof. Setting $t = 0$ in (1.7), with $\widetilde{g} = 0$ and $\alpha(0) = \underline{1} = (1,1,...,1)^T$, we have

$$(2.1) \qquad \mathcal{M}\beta = -\gamma = -\mathcal{S}\underline{1}, \quad \text{where} \ \beta = \alpha'(0).$$

If $E_h(t)$ were a contraction, then we would have $\beta \leqslant 0$ elementwise, and hence $\gamma \geqslant 0$. and we shall show that this is not possible under the assumptions of the theorem. Let $w_h = \sum_{j=1}^{n} \Phi_j \in S_h$ correspond to the vector $\underline{1}$ above.

We assume first that $div \ \underline{b} = 0$. In this case we have for the element γ_i of γ, corresponding to a strictly interior node P_i, using integration by parts,

$$(2.2) \qquad \gamma_i = \sum_{j=1}^{n} s_{ij} = A(\Phi_i, w_h) = (\underline{b}.\nabla\Phi_i, w_h) = -(div \ \underline{b}\Phi_i, w_h) = 0,$$

since $w_h = 1$ on σ_i. Hence $\sum_{j=1}^{n} m_{ij}\beta_j = 0$ for β as in (2.1) .

Here $m_{ij} > 0$ if and only if $j = i$ or if P_j is a neighbor of P_i, and if $\beta \leqslant 0$, then $\beta_j = 0$ for the corresponding j. Thus by the assumption on Υ_h we have $\beta = 0$ and hence $\gamma = 0$. But γ cannot be zero since

$$(2.3) \qquad \gamma.\underline{1} = \mathcal{S}\underline{1}.\underline{1} = A(w_h, w_h) = A_0(w_h, w_h) - \frac{1}{2}(\ div \ \underline{b}w_h, w_h) > 0.$$

We now turn to the case of a general $\underline{b}$ and assume again that $\beta \leqslant 0$. For P_i a strictly interior node we now have, by (2.1) and (2.2), that $m_{ii}|\beta_i| \leqslant \gamma_i \leqslant C$ meas(σ_i). Since $m_{ii} \leqslant c$ meas(σ_i), with $c > 0$, it follows that $|\beta_i| \leqslant C$ for a positive constant C. If P_j is a near-boundary node, let P_i be a strictly interior neighboring node. Then $m_{ij}|\beta_j| \leqslant \sum_{i=1}^{n} m_{il} |\beta_l| = \gamma_i \leqslant C$meas$(\sigma_i)$ and since, by assumption,

$$m_{ij} = (\Phi_i, \Phi_j) \geqslant c \ meas(\sigma_j \cap \sigma_i) \geqslant c \ meas(\sigma_i),$$

we conclude also now that $|\beta_i| \leqslant C$. Thus, for all interior nodes P_i,

$$\gamma_i \leqslant C\sum_{j=1}^{n} m_{ij} \leqslant Cmeas(\sigma_i), and \ hence \ \gamma.\underline{1} \leqslant C.$$

But, by the first part of (2.3),

$$\gamma.\underline{1} = \mathcal{S}\underline{1}.\underline{1} \geqslant c\,\|\nabla w_h\|_{L_2}^2 - C.$$

Here $\nabla w_h = 0$ on all interior simplices $\tau \in \Upsilon_h$, so that only boundary simplices contribute to the first term on the right. For a boundary simplex τ, which has a full $(d-1)$-dimensional face F_τ on $\partial\Omega$, we have, with d_τ the distance from the interior vertex of τ to the hyperplane containing F_τ, and with $|F_\tau|$ the $(d-1)$-dimensional measure of F_τ,

$$\|\nabla w_h\|_{L_2(\tau)}^2 \geqslant d_\tau^{-2}\mathrm{meas}(\tau) \geqslant c d_\tau^{-1}\,|F_\tau| \geqslant c h^{-1}\,|F_\tau|, \quad \text{with } c > 0.$$

Hence after summation over these τ, since $\cup_\tau F_\tau = \partial\Omega$, we conclude that $\gamma.\underline{1} \geqslant c h^{-1} - C$. For small h this contradicts the boundedness of $\gamma.\underline{1}$. $\square$

We note that, in one space dimension, the first assumption about Υ_h holds if there are at least three interior nodes, and when $d = 2$ the second assumption is satisfied when the triangulation is fine enough, and the angles in the triangles of $\cup_i \sigma_i$ are bounded below where the union is taken over all i such that P_i is a strictly interior neighbor of a near-boundary node. To see that some condition on Υ_h is needed, we consider the case when there is only one interior node P_1 and $div\,\underline{b} = 0$. The system (1.7) then reduces to the scalar equation

$$\|\Phi_1\|_{L_2}^2\,\alpha_1' + A_0(\Phi_1, \Phi_1)\alpha_1 = 0, \quad \text{for } t \geqslant 0, \quad \alpha_1(0) = \widetilde{v} = (v_h, \Phi_1).$$

The solution is then the exponentially decreasing function $u_h(t) = \exp(-t\lambda)\widetilde{v}$, with $\lambda = A_0(\Phi_1, \Phi_1)/\|\Phi_1\|_{L_2}^2 > 0$, and the solution operator is a contraction.

Theorem 2.2. *Assume that $div\,\underline{b} \leqslant 0$ and that Υ_h is such that there exists a strictly interior node, P_1 say, such that any neighbor of P_1 has an interior neighbor which is not a neighbor of P_1. Then $E_h(t)$ cannot be a positive operator. The condition $div\,\underline{b} \leqslant 0$ is not needed if h is sufficiently small.*

Proof. If $E_h(t)$ is a positive operator, then, by (1.7) with $\widetilde{g} = 0$, we have that $\varepsilon(t) = e^{-\mathcal{K}t} \geqslant 0$, elementwise, where $\mathcal{K} = (k_{ij}) = \mathcal{M}^{-1}\mathcal{S}$.
Since $\varepsilon(t) = I - \mathcal{K}t + O(t^2)$ as $t \to 0$, we see that then all off-diagonal elements of $\mathcal{K}$ are nonpositive. We shall show that this is impossible.

Let P_i be any interior node $\neq P_1$ which is not a neighbor of P_1. Since $\mathcal{M}\mathcal{K} = \mathcal{S}$ and since $m_{i1} = s_{i1} = 0$, we have $\displaystyle\sum_{j\neq 1}^n m_{ij}k_{j1} = 0$. Hence $k_{j1} = 0$ when $m_{ij} > 0$, i.e., when $j = i$ and when j is such that P_j is a neighbor of P_i. By our assumption about Υ_h this shows that actually $k_{j1} = 0$ for all $j \neq 1$; this is also true when P_j and P_1 are neighbors. Thus the first column of $\mathcal{K}$ only contains one possible nonzero element, namely k_{11}.

For the stiffness matrix $\mathcal{S}$ we have $\displaystyle\sum_{j=1}^n s_{j1} = A(1, \Phi_1) = 0$ while, if $div\,\underline{b} \leqslant 0$,

$$s_{11} = A_0(\Phi_1, \Phi_1) - \frac{1}{2}(div \ \underline{b}\Phi_1, \Phi_1) > 0.$$

In the case of a general $\underline{b}$, we have $s_{11} \geqslant (ch^{-2} - C)\text{meas}(\sigma_1)$, with $c > 0$, and hence $s_{11} > 0$ for h small. Thus, in either case, the first column of $\mathcal{S}$ has elements of different signs whereas this is not the case for $\mathcal{M}$, in contradiction to $\mathcal{MK} = \mathcal{S}$. $\square$

At this point we remark that our assumption about Υ_h is satisfied in one dimension if there are five or more interior nodes. For the example following Theorem 2.1 above, with only one interior node, $E_h(t)$ is also a positive operator, which shows that some condition on Υ_h is needed in Theorem 2.2.

Now we consider the lumped mass method and give necessary and sufficient conditions for the maximum principle to hold and also for the contractivity and positivity of the operator $\widetilde{E}_h(t)$ on S_h^0 .

Theorem 2.3. *The maximum principle (1.6) holds for the semidiscrete parabolic lumped mass problem (1.10) if and only if the off-diagonal elements of $\mathcal{S}$ and all elements of $\mathcal{B}$ are nonpositive.*

Proof. With the notation in the introduction we have, from (1.9),

$$(2.4) \quad \alpha(t) = \widetilde{\varepsilon}(t)\alpha(0) - \int_0^t \widetilde{\varepsilon}(t-s)\mathcal{D}^{-1}\mathcal{B}\widetilde{g}(t) \ ds, \quad \text{with} \ \widetilde{\varepsilon}(t) = e^{-\mathcal{H}t}, \ \mathcal{H} = \mathcal{D}^{-1}\mathcal{S}.$$

As in (1.3) for the continuous case, the maximum principle (1.6) is equivalent to

$$\alpha(t) \geqslant 0 \quad \text{for} \ t \geqslant 0 \quad \text{if} \ \alpha(0) \geqslant 0 \quad \text{and} \ \widetilde{g}(t) \geqslant 0 \quad \text{for} \ t \geqslant 0.$$

Thus, if (1.6) holds, it follows from (2.4), with $\widetilde{g}(t) = 0$, that $\widetilde{\varepsilon}(t) \geqslant 0$. Since

$$(2.5) \qquad\qquad \widetilde{\varepsilon}(t) = I - t\mathcal{H} + O(t^2), \qquad \text{as} \ t \to 0,$$

all off-diagonal elements of $\mathcal{H}$ must then be nonpositive, and since $\mathcal{D}$ is diagonal with positive elements, the off-diagonal elements of $\mathcal{S} = \mathcal{DH}$ are also nonpositive. Setting $\alpha(0) = 0$ and $\widetilde{g}(t) \geqslant 0$ in (2.4), we now find that it is also necessary that $\mathcal{B} \leqslant 0$.

Conversely, if the off-diagonal elements of $\mathcal{S}$ are nonpositive, this holds also for $\mathcal{H}$.Writing $\mathcal{H} = \mathcal{P} - \mathcal{Q}$ where $\mathcal{P}$ is diagonal and $\mathcal{Q} \geqslant 0$, and setting $\mathcal{J} = I + k\mathcal{P}$, we have, for k small,

$$(I + k\mathcal{H})^{-1} = (\mathcal{J} - k\mathcal{Q})^{-1} = (I - k\mathcal{J}^{-1}\mathcal{Q})^{-1}\mathcal{J}^{-1} = \sum_{l=0}^{\infty} k^l (\mathcal{J}^{-1}\mathcal{Q})^l \mathcal{J}^{-1} \geqslant 0$$

Since

$$(2.6) \qquad\qquad \widetilde{\varepsilon}(t) = \lim_{n\to\infty} (I + \frac{t}{n}\mathcal{H})^{-n},$$

this shows $\widetilde{\varepsilon}(t) \geqslant 0$. If also $\mathcal{B} \leqslant 0$, it follows that $\widetilde{\varepsilon}(t-s)\mathcal{D}^{-1}\mathcal{B} \leqslant 0$. Hence, if $\alpha(0) \geqslant 0$ and $\widetilde{g}(t) \geqslant 0$ for $t \geqslant 0$, we obtain by (2.4) that $\alpha(0) \geqslant 0$ for $t \geqslant 0$, which shows our claim. $\square$

We note that the condition of the theorem may also be expressed as

$$(2.7) \qquad A(\Phi_i, \Phi_j) \leqslant 0 \quad \text{for} \quad i = 1:n, j = 1:n+m, i \neq j.$$

We shall now give necessary and sufficient conditions for the solution operator $\widetilde{E}_h(t)$ with homogeneous boundary conditions to be contractive. In view of the above discussion, in the case $div\,\underline{b} = 0$, these conditions are essentially concerned with the properties of the rows corresponding to near-boundary nodes.

Theorem 2.4. *The semigroup $\widetilde{E}_h(t)$ on S_h^0 is a contraction if and only if $\mathcal{S}$ is diagonally dominant.*

Proof. If $\widetilde{E}_h(t)$ is a contraction, so is the matrix $\widetilde{\varepsilon}(t)$ in (2.4) with respect to the vector maximum-norm $|\cdot|_\infty$. Since $\widetilde{\varepsilon}(t) = I - \mathcal{H}t + O(t^2)$ as $t \to 0$, we have

$$|\widetilde{\varepsilon}(t)|_\infty = \max(1 - th_{ii} + t\sum_{j \neq i}^{\infty} |h_{ij}|) + O(t^2).$$

For this norm to be bounded by 1, we find at once by taking t small that it is necessary that $\sum_{j \neq i} |h_{ij}| \leqslant h_{ii}$ for $i = 1:n$, so that $\mathcal{H}$ is diagonally dominant. Since $\mathcal{D}$ is a positive diagonal matrix, $\mathcal{S} = \mathcal{D}\mathcal{H}$ is then also diagonally dominant.

Conversely, if we know that $\mathcal{S}$, and hence also $\mathcal{H} = \mathcal{D}^{-1}\mathcal{S}$, is diagonally dominant, it is easy to see that

$$(2.8) \qquad |(I + k\mathcal{H})^{-1}|_\infty \leqslant 1, \qquad \text{for } k > 0.$$

In fact, set $w = (I + k\mathcal{H})^{-1}v$ and let $|w_j| = |w|_\infty$.
Then $(1 + kh_{jj})|w_j| = |v_j - k\sum_{l \neq j} h_{jl}w_l| \leqslant |v|_\infty + kh_{jj}|w|_\infty$, from which $|w|_\infty \leqslant |v|_\infty$, which shows (2.8).
By (2.6) this implies $|\widetilde{\varepsilon}(t)|_\infty \leqslant 1$, for $t \geqslant 0$, so that $\widetilde{E}_h(t)$ is a contraction. $\square$

Theorem 2.5. *The semigroup $\widetilde{E}_h(t)$ is positive if and only if $s_{ij} \leqslant 0$ for $j \neq i$.*

Proof. In this case the positivity of $\widetilde{\varepsilon}(t)$, together with (2.5), shows at once that the off-diagonal elements of $\mathcal{H}$ are nonpositive. Since $\mathcal{S} = \mathcal{D}\mathcal{H}$, this shows the only if part of the theorem.

Conversely, if the off-diagonal elements of $\mathcal{H}$ are nonpositive, one finds as in the last part of the proof of Theorem 2.3 that $\widetilde{\varepsilon}(t) \geqslant 0$. $\square$

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