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ON QUOTIENT π-IMAGES OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. We prove that a space is quotient π-image of a locally separable metric space if and only if it has a π- and double cs^*-cover. We also investigate quotient π-s-images of locally separable metric spaces.

1. Introduction

Characterizations of images of metric spaces under certain covering-mappings have attracted many authors. In the past, various results have been obtained by means of certain networks [13]. Recently, π-images of metric spaces have caught the attention once again [5, 6, 8, 14]. It is known that quotient π-images of metric spaces (resp. separable metric spaces) have been obtained, see [9, Theorem 3.1.6] (resp. [5, Theorem 3.4]), for example. In a private communication the first author of [14] informed that, in general, it is difficult to get "nice" characterizations of π-images of locally separable metric spaces (instead of metric domains). These lead us to investigate quotient π-images of locally separable metric spaces. That is, we are interested in the following question.

Question 1.1. *How are quotient π-images of locally separable metric spaces characterized?*

Taking this question into account, we give an internal characterization on subsequence-covering (sequentially-quotient) π-images of locally separable metric spaces. As an application of this result, we

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get a characterization on quotient π-images of locally separable metric spaces. We also investigate subsequence-covering (sequentially-quotient) π-s-images of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be Hausdorff, all mappings are continuous and onto, a convergent sequence includes its limit point, and $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a collection of subsets of X, we denote by $st(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *frequently* in A if $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$. For terms which are not defined here, please refer to [2, 13].

2. Main results

Definition 2.1. Let $\mathcal{P}$ be a collection of subsets of a space X, and K be a subset of X.

(1) $\mathcal{P}$ is a *cover for K in X*, if $K \subset \bigcup \mathcal{P}$.
(2) For each $x \in X$, $\mathcal{P}$ is a *network at x* in X, if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.
(3) $\mathcal{P}$ is a *cs^*-cover for K in X*, if for each convergent sequence S in K, S is frequently in some $P \in \mathcal{P}$.
(4) $\mathcal{P}$ is a *cs^*-network for X* [13], if for each convergent sequence S converging to $x \in U$ with U open in X, S is frequently in $P \subset U$ with some $P \in \mathcal{P}$.

Remark 2.2. Let X be a space.

(1) When $K = X$, a cover (resp. cs^*-cover) for K in X is a *cover of X* (resp. *cs^*-cover for X*) in the sense of [2] (resp. [14]).
(2) If $\mathcal{P}$ is a cover (resp. cs^*-cover) for X, then $\mathcal{P}$ is a cover (resp. cs^*-cover) for K in X, for every subset K of X.
(3) A cover (resp. cs^*-cover) for X is abbreviated to a *cover* (resp. *cs^*-cover*).

Definition 2.3. Let $\mathcal{P}$ be a collection of subsets of a space X. We say that $\mathcal{P}$ is *point-countable* [2], if every point of X is in at most countably many members of $\mathcal{P}$.

Definition 2.4. Let $\mathcal{P}_n$ be a cover for X for each $n \in \mathbb{N}$.

(1) $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *refinement* [4] for X, if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for each $n \in \mathbb{N}$.
(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* [14] for X, if $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a refinement for X, and for each $x \in X$, $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x.

Definition 2.5. Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-strong network for a space X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and A_n is endowed with discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} A_n$, is a metric space with metric d described as follows. For $a = (\alpha_n), b = (\beta_n) \in M$, if $a = b$, then $d(a, b) = 0$, and if $a \neq b$, then $d(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Define $f : M \longrightarrow X$ by choosing $f(a) = x_a$, then f is a mapping, and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev's system* [14].

Definition 2.6. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is a *subsequence-covering mapping* [4], if for every convergent sequence S of Y, there is a compact subset K of X such that $f(K)$ is a subsequence of S.
(2) f is a *sequentially-quotient mapping* [4], if for every convergent sequence S of Y, there is a convergent sequence L of X such that $f(L)$ is a subsequence of S.
(3) f is a *quotient mapping* [12], if U is open in Y whenever $f^{-1}(U)$ is open in X.
(4) f is a *pseudo-open mapping* [7], if $y \in \mathrm{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X.
(5) f is a *π-mapping* [14], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

(6) f is an *s-mapping* [8], if for each $y \in Y$, $f^{-1}(y)$ is a separable subset of X.
(7) f is a *π-s-mapping* [8], if f is both π-mapping and s-mapping.

Definition 2.7 ([2])**.** Let X be a space, then

(1) X is a *sequential space*, if a subset of X is closed if and only if together with any sequence it contains all its limits.
(2) X is a *Fréchet space*, if for each $A \subset X$ and each $x \in \overline{A}$ there exists a sequence in A converging to x.

Definition 2.8. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X, where each X_λ has a refinement $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ consisting of countable covers for X_λ.

(1) $\{X_\lambda : \lambda \in \Lambda\}$ is a *π-cover* for X if for each $x \in U$, with U open in X, there exists $n \in \mathbb{N}$ such that
$$\bigcup\{st(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U.$$
(2) $\{X_\lambda : \lambda \in \Lambda\}$ is a *double cs^*-cover* for X if for each convergent sequence S of X, there exists a $\lambda \in \Lambda$ such that S is frequently in X_λ and $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence S_λ of S in X_λ for each $n \in \mathbb{N}$.

Remark 2.9. (1) If $\{X_\lambda : \lambda \in \Lambda\}$ is a π-cover for X, then $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a σ-strong network for X_λ for each $\lambda \in \Lambda$.
(2) If $\{X_\lambda : \lambda \in \Lambda\}$ is a double cs^*-cover for X, then it is a cs^*-cover for X and $\bigcup\{\mathcal{P}_{\lambda,n} : \lambda \in \Lambda, n \in \mathbb{N}\}$ is a cs^*-network for X.

Lemma 2.10. *Let $f : X \longrightarrow Y$ be a mapping and S be a convergent sequence in X. If $\mathcal{P}$ is a cs^*-cover for S in X, then $f(\mathcal{P})$ is a cs^*-cover for $f(S)$ in Y.*

Proof. Let H be a convergent sequence in $f(S)$. Then $G = f^{-1}(H) \cap S$ is a convergent sequence in S and $f(G) = H$. Since $\mathcal{P}$ is a cs^*-cover for S in X, G is frequently in some $P \in \mathcal{P}$. Then H is frequently in some $f(P) \in f(\mathcal{P})$. It implies that $f(\mathcal{P})$ is a cs^*-cover for $f(S)$ in Y. $\square$

Lemma 2.11. *Let $(f, M, X, \mathcal{P}_n)$ be a Ponomarev's system and S be a convergent sequence in X. If $\mathcal{P}_n$ is point-countable for each $n \in \mathbb{N}$, then the following are equivalent.*

(1) *For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cs^*-network for S in X,*
(2) *There exists a compact subset K of M such that $S = f(K)$.*

Proof. (1) $\Longrightarrow$ (2). As in the proof of [14, Lemma 2.2 (iv)].

(2) $\Longrightarrow$ (1). Recall that, as in the proof of [15, General Theorem], the following hold,

(a) for each $a = (\alpha_n) \in M$, $\mathcal{B}(a) = \{U(\alpha_1, \ldots, \alpha_k) : k \in \mathbb{N}\}$ is a basis at a in M, where

$$U(\alpha_1, \ldots, \alpha_k) = \{b = (\beta_n) \in M : \beta_i = \alpha_i \text{ for all } i \leq k\},$$

for each $k \in \mathbb{N}$.

(b) $f(U(\alpha_1, \ldots, \alpha_k)) = \bigcap_{i=1}^{k} P_{\alpha_i}$.

For each $n \in \mathbb{N}$, we get $\{U(\alpha_1, \ldots, \alpha_n) : a \in M\}$ is an open cover for M. Since K is compact, $K \subset \bigcup \mathcal{F}$ with some finite subfamily $\mathcal{F}$ of $\{U(\alpha_1, \ldots, \alpha_n) : a \in M\}$. Note that M is metric, $K = \bigcup\{K_F : F \in \mathcal{F}\}$ where $K_F \subset F$ and K_F is compact for each $F \in \mathcal{F}$. It implies that $S = \bigcup\{f(K_F) : F \in \mathcal{F}\}$. Since each $f(K_F)$ is closed (in fact, each $f(K_F)$ is compact), S is frequently in some $f(K_F) \subset f(F)$. From (b) in the above, $f(F) \subset P_{\alpha_n}$ for some $P_{\alpha_n} \in \mathcal{P}_n$. It implies that $\mathcal{P}_n$ is a cs^*-cover for S in X. □

Theorem 2.12. *The following are equivalent for a space X.*

1. *X is a subsequence-covering π-image of a locally separable metric space,*
2. *X is a sequentially-quotient π-image of a locally separable metric space,*
3. *X has a double cs^*- and π-cover.*

Proof. (1) $\Longrightarrow$ (2). By [4, Proposition 2.1].

(2) $\Longrightarrow$ (3). Let $f : M \longrightarrow X$ be a sequentially-quotient π-mapping from a locally separable metric space M with metric d onto X. Since M is a locally separable metric space, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is a separable metric space by [2, 4.4.F]. For each $\lambda \in \Lambda$, denote D_λ is a countable dense subset of M_λ, and put $f_\lambda = f|_{M_\lambda}$ and $X_\lambda = f_\lambda(M_\lambda)$. For each $a \in M_\lambda$ and $n \in \mathbb{N}$, put $B(a, 1/n) = \{b \in M_\lambda : d(a, b) < 1/n\}$, $\mathcal{B}_{\lambda,n} = \{B(a, 1/n) : a \in D_\lambda\}$, and $\mathcal{P}_{\lambda,n} = f_\lambda(\mathcal{B}_{\lambda,n})$. It is clear that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a refinement consisting of countable covers for X_λ.

(a) $\{X_\lambda : \lambda \in \Lambda\}$ *is a π-cover.*

Consider $x \in U$ with U open in X. Since f is a π-mapping, $d(f^{-1}(x), M - f^{-1}(U)) > 2/n$ for some $n \in \mathbb{N}$. Then, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/n$ where $U_\lambda = U \cap X_\lambda$. Let $a \in D_\lambda$ and

$x \in f_\lambda(B(a,1/n)) \in \mathcal{P}_{\lambda,n}$. We shall prove that $B(a,1/n) \subset f_\lambda^{-1}(U_\lambda)$. In fact, if $B(a,1/n) \not\subset f_\lambda^{-1}(U_\lambda)$, then pick $b \in B(a,1/n) - f_\lambda^{-1}(U_\lambda)$. Note that $f_\lambda^{-1}(x) \cap B(a,1/n) \neq \emptyset$, pick $c \in f_\lambda^{-1}(x) \cap B(a,1/n)$, then $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \leq d(c,b) \leq d(c,a) + d(a,b) < 2/n$. It is a contradiction. So $B(a,1/n) \subset f_\lambda^{-1}(U_\lambda)$, thus $f_\lambda(B(a,1/n)) \subset U_\lambda$. Then $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. It implies that $\bigcup\{st(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$.

(b) $\{X_\lambda : \lambda \in \Lambda\}$ *is a double* cs^**-cover.*

For each convergent sequence S of X, since f is sequentially-quotient, there exists a convergent sequence L in M such that $f(L)$ is a subsequence of S. Note that L is eventually in some M_λ. Then S is frequently in X_λ. Put $S_\lambda = f(L \cap M_\lambda)$, then S_λ is a subsequence of S. For each $n \in \mathbb{N}$, since $\mathcal{B}_{\lambda,n}$ is an open cover for M_λ, $\mathcal{B}_{\lambda,n}$ is a cs^*-cover for the convergent sequence $L \cap M_\lambda$ in M_λ. It follows from Lemma 2.10 that $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for S_λ in X_λ.

(3) $\Longrightarrow$ (1). it follows from Remark 2.9.(1) that the Ponomarev's system $(f_\lambda, M_\lambda, X_\lambda, \mathcal{P}_{\lambda,n})$ exists for each $\lambda \in \Lambda$. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space with metric d_λ described as follows. For $a = (\alpha_n), b = (\beta_n) \in M_\lambda$, if $a = b$, then $d_\lambda(a,b) = 0$, and if $a \neq b$, then $d_\lambda(a,b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Put $M = \oplus_{\lambda \in \Lambda} M_\lambda$ and define $f : M \longrightarrow X$ by choosing $f(a) = f_\lambda(a)$ for every $a \in M_\lambda$ with some $\lambda \in \Lambda$. Then f is a mapping and M is a locally separable metric space with metric d defined as follows. For $a, b \in M$, if $a, b \in M_\lambda$ for some $\lambda \in \Lambda$, then $d(a,b) = d_\lambda(a,b)$, and otherwise, $d(a,b) = 1$. We shall prove that f is a subsequence-covering π-mapping.

(a) f *is a* π*-mapping.*

Let $x \in U$ with U open in X, then $\bigcup\{st(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$ for some $n \in \mathbb{N}$. So, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$ where $U_\lambda = U \cap X_\lambda$. It implies that $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n$. In fact, if $a = (\alpha_k) \in M_\lambda$ such that $d_\lambda(f_\lambda^{-1}(x), a) < 1/n$, then there is $b = (\beta_k) \in f_\lambda^{-1}(x)$ such that $d_\lambda(a,b) < 1/n$. So $\alpha_k = \beta_k$ if $k \leq n$. Note that $x \in P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Then $f_\lambda(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Hence $a \in f_\lambda^{-1}(U_\lambda)$. It implies that $d_\lambda(f_\lambda^{-1}(x), a) \geq 1/n$ if $a \in M_\lambda - f_\lambda^{-1}(U_\lambda)$. So $d_\lambda(f_\lambda^{-1}(x), M_\lambda -$

$f_\lambda^{-1}(U_\lambda)) \geq 1/n$. Therefore

$$\begin{aligned} d(f^{-1}(x), M - f^{-1}(U)) &= \inf\{d(a,b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\ &= \min\left\{1, \inf\{d_\lambda(a,b) : a \in f_\lambda^{-1}(x), b \in M_\lambda - f_\lambda^{-1}(U_\lambda), \lambda \in \Lambda\}\right\} \\ &\geq 1/n > 0. \end{aligned}$$

It implies that f is a π-mapping.

(b) *f is subsequence-covering.*

For each convergent sequence S of X, there exists $\lambda \in \Lambda$ such that S is frequently in X_λ and $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence S_λ of S in X_λ for each $n \in \mathbb{N}$. It follows from Lemma 2.11 that $S_\lambda = f_\lambda(K_\lambda)$ for some compact subset K_λ of M_λ. Note that K_λ is also a compact subset of M. It implies that f is subsequence-covering. □

Corollary 2.13. *The following are equivalent for a space X.*

(1) *X is a quotient π-image of a locally separable metric space,*
(2) *X is a sequential space having a double cs^*- and π-cover.*

Proof. (1) $\Longrightarrow$ (2). Let $f : M \longrightarrow X$ be a quotient π-mapping from a locally separable metric M onto X. It follows from [11, Lemma 3.5] that X is a sequential space and f is sequentially-quotient. Then X is a sequential space with a double cs^*- and π-cover by Theorem 2.12.

(2) $\Longrightarrow$ (1). It follows from Theorem 2.12 that X is a sequential space and a sequentially-quotient π-image of a locally separable metric space. By [11, Lemma 3.5], X is a quotient π-image of a locally separable metric space. □

Theorem 2.14. *The following are equivalent for a space X.*

(1) *X is a subsequence-covering π-s-image of a locally separable metric space,*
(2) *X is a sequentially-quotient π-s-image of a locally separable metric space,*
(3) *X has a double cs^*- and point-countable π-cover.*

Proof. (1) $\Longrightarrow$ (2). By [4, Proposition 2.1].

(2) $\Longrightarrow$ (3). By using notations and arguments in the proof (1) $\Longrightarrow$ (2) of Theorem 2.12, we only need to prove that $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable. For each $x \in X$, since the mapping f is an s-mapping, $f^{-1}(x)$ is separable. Then $f^{-1}(x)$ meets only countably many M_λ's,

i.e., x meets only countably many X_λ's. It implies that $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable.

(3) $\Longrightarrow$ (1). By using notations and arguments in the proof (3) $\Longrightarrow$ (1) of Theorem 2.12, we only need to prove that the mapping f is an s-mapping. For each $x \in X$, since $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable, $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is countable. For each $\lambda \in \Lambda_x$, since M_λ is separable metric, $f_\lambda^{-1}(x)$ is separable. Then $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$ is separable. It implies that f is an s-mapping. $\square$

Corollary 2.15. *The following are equivalent for a space X.*

1. *X is a quotient π-s-image of a locally separable metric space,*
2. *X is a sequential space with a double cs^*- and point-countable π-cover.*

Remark 2.16. It follows from [3, Proposition 2.3] that "*quotient*" and "*sequential*" in Corollary 2.13 and Corollary 2.15 can be replaced by "*pseudo-open*" and "*Fréchet*", respectively.

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References

[1] A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys, **21** (1966), 115 – 162.
[2] R. Engelking, *General topology*, PWN-Polish Scientific Publishers, Warszawa 1977.
[3] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math., **57** (1965), 107 – 115.
[4] Y. Ge, *On compact images of locally separable metric spaces*, Topology Proc., **27** (**1**) (2003), 351 – 360.
[5] Y. Ge, *On π-images of metric spaces*, Acta Math. APN, **22** (2006), 209 – 215.
[6] Y. Ge and J. S. Gu, *On π-images of separable metricspaces*, Math. Sci., **10** (2004), 65 – 71.
[7] G. Gruenhage, E. Michael and Y. Tanaka, *Spaces determined by point-countable covers*, Pacific J. Math., **113** (**2**) (1984), 303 – 332.
[8] Z. Li, *On π-s-images of metric spaces*, Int. J. Math. & Math. Sci., **7** (2005), 1101 – 1107.
[9] S. Lin, *Point-countable covers and sequence-covering mappings*, Chinese Science Press., Beijing 2002.
[10] S. Lin and P. Yan, *Notes on cfp-covers*, Comment. Math. Univ. Carolinae, **44** (**2**) (2003), 295 – 306.

[11] S. Lin and P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl., **109** (2001), 301 – 314.
[12] E. Michael, $\aleph_0$-*spaces*, J. Math. Mech., **15** (1966), 983 – 1002.
[13] Y. Tanaka, *Theory of k-networks II*, Q & A in General Top., **19** (2001), 27 – 46.
[14] Y. Tanaka and Y. Ge, *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math., **32** (**1**) (2006), 99 – 117.
[15] Y. Tanaka and Z. Li, *Certain covering-maps and k-networks, and related matters*, Topology Proc., **27** (2003), 317 – 334.

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