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The integrals in Gradshteyn and Ryzhik. Part 7: Elementary examples

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Abstract. The table of Gradshteyn and Ryzhik contains some elementary integrals. Some of them are derived here.

1. Introduction

Elementary mathematics leave the impression that there is marked difference between the two branches of calculus. *Differentiation* is a subject that is systematic: every evaluation is a consequence of a number of rules and some basic examples. However, *integration* is a mixture of art and science. The successful evaluation of an integral depends on the right approach, the right change of variables or a patient search in a table of integrals. In fact, the theory of *indefinite* integrals of elementary functions is complete [**3, 4**]. Risch's algorithm determines whether a given function has an antiderivative within a given class of functions. However, the theory of *definite* integrals is far from complete and there is no general theory available. The level of complexity in the evaluation of a definite integral is hard to predict as can be seen in

$$\int_0^\infty e^{-x}\,dx = 1, \quad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}, \text{ and } \int_0^\infty e^{-x^3}\,dx = \Gamma\left(\tfrac{4}{3}\right). \tag{1.1}$$

The first integrand has an elementary primitive, the second one is the classical Gaussian integral and the evaluation of the third requires Euler's *gamma function* defined by

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx. \tag{1.2}$$

The table of integrals [**5**] contains a large variety of integrals. This paper continues the work initiated in [**1, 7, 8, 9, 10, 11**] with the objective of providing proofs and context of *all the formulas* in the table [**5**]. Some of them are truly elementary. In this paper we present a derivation of a small number of them.

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2. A simple example

The first evaluation considered here is that of **3.249.6**:

$$\int_0^1 (1-\sqrt{x})^{p-1}\,dx = \frac{2}{p(p+1)}. \tag{2.1}$$

The evaluation is completely elementary. The change of variables $y = 1-\sqrt{x}$ produces

$$I = -2\int_0^1 y^p\,dy + 2\int_0^1 y^{p-1}\,dy, \tag{2.2}$$

and each of these integrals can be evaluated directly to produce the result.

This example can be generalized to consider

$$I(a) = \int_0^1 (1-x^a)^{p-1}\,dx. \tag{2.3}$$

The change of variables $t = x^a$ produces

$$I(a) = a^{-1}\int_0^1 t^{1/a-1}(1-t)^{p-1}\,dt. \tag{2.4}$$

This integral appears as **3.251.1** and it can be evaluated in terms of the *beta function*

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \tag{2.5}$$

as

$$I(a) = a^{-1}B\left(p, a^{-1}\right). \tag{2.6}$$

The reader will find in [**11**] details about this evaluation.

A further generalization is provided in the next lemma.

Lemma 2.1. Let $n \in \mathbb{N}$, a, b, $c \in \mathbb{R}$ with $bc > 0$. Define $u = ac - b^2$. Then

$$\int_0^1 \frac{a+b\sqrt{x}}{b+c\sqrt{x}} x^{n/2}\,dx = \frac{2u(-b)^{n+1}}{c^{n+3}} \ln(1+c/b) + \frac{2u}{c^2}\sum_{j=0}^{n} \frac{(-1)^j}{n-j+1}\left(\frac{b}{c}\right)^j + \frac{2b}{(n+2)c}.$$

Proof. Substitute $y = b + c\sqrt{x}$ and expand the new term $(y-b)^n$. $\square$

3. A generalization of an algebraic example

The evaluation

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3} \tag{3.1}$$

appears as **3.248.4** in [**5**]. We consider here the generalization

$$q(a,b) = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sqrt{b+ax^2}}. \tag{3.2}$$

We assume that a, $b > 0$.

The change of variables $x = \sqrt{b}t/\sqrt{a}$ yields

$$q(a,b) = 2\sqrt{a}\int_0^\infty \frac{dt}{(a+bt^2)\sqrt{1+t^2}} \tag{3.3}$$

where we have used the symmetry of the integrand to write it over $[0,\infty)$. The standard trigonometric change of variables $t = \tan\varphi$ produces

$$q(a,b) = 2\sqrt{a}\int_0^{\pi/2} \frac{\cos\varphi\, d\varphi}{a\cos^2\varphi + b\sin^2\varphi}. \tag{3.4}$$

Finally, $u = \sin\varphi$, produces

$$q(a,b) = 2\sqrt{a}\int_0^1 \frac{du}{a+(b-a)u^2}. \tag{3.5}$$

The evaluation of this integral is divided into three cases:

Case 1. $a = b$. Then we simply get $q(a,b) = 2/\sqrt{a}$.

Case 2. $a < b$. The change of variables $s = u\sqrt{b-a}/\sqrt{a}$ produces $(b-a)u^2 = s^2a$, so that

$$q(a,b) = \frac{2}{\sqrt{b-a}}\int_0^c \frac{ds}{1+s^2} = \frac{2}{\sqrt{b-a}}\tan^{-1}c, \tag{3.6}$$

with $c = \sqrt{b-a}/\sqrt{a}$.

Case 3. $a > b$. Then we write

$$q(a,b) = 2\sqrt{a}\int_0^1 \frac{du}{a-(a-b)u^2}. \tag{3.7}$$

The change of variables $u = \sqrt{a}\,s/\sqrt{a-b}$ yields

$$q(a,b) = \frac{2}{\sqrt{a-b}}\int_0^c \frac{ds}{1-s^2}, \tag{3.8}$$

where $c = \sqrt{a-b}/\sqrt{a}$. The partial fraction decomposition

$$\frac{1}{1-s^2} = \frac{1}{2}\left(\frac{1}{1+s} + \frac{1}{1-s}\right) \tag{3.9}$$

now produces

$$q(a,b) = \frac{1}{\sqrt{a-b}}\ln\left(\frac{\sqrt{a}+\sqrt{a-b}}{\sqrt{a}-\sqrt{a-b}}\right). \tag{3.10}$$

The special case in **3**.**248**.**4** corresponds to $a = 3$ and $b = 4$. The value of the integral is $2\tan^{-1}(1/\sqrt{3}) = \frac{\pi}{3}$, as claimed. This generalization has been included as **3**.**248**.**6** in the latest edition of [**5**].

We now consider a generalization of this integral. The proof requires several elementary steps, given first for the convenience of the reader.

Let a, $b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$. Introduce the notation

$$I = I_n(a,b) := \int_0^\infty \frac{dt}{(a+bt^2)^n \sqrt{1+t^2}}. \tag{3.11}$$

Then we have:

Lemma 3.1. The integral $I_n(a,b)$ is given by

$$I_n(a,b) = \int_0^1 \frac{(1-v^2)^{n-1}\,dv}{(a+\alpha v^2)^n}, \tag{3.12}$$

with $\alpha = b-a$.

PROOF. The change of variables $v = t/\sqrt{1+t^2}$ gives the result. □

The identity

$$(1-v^2)^{n-1} = (1-v^2)^n + (1-v^2)^{n-1}\left\{\frac{1}{\alpha}(a+\alpha v^2) - \frac{a}{\alpha}\right\} \tag{3.13}$$

produces

$$I_n(a,b) = \frac{\alpha}{b}\int_0^1 \frac{(1-v^2)^n}{(a+\alpha v^2)^n}\,dv + \frac{1}{b}\int_0^1 \frac{(1-v^2)^{n-1}}{(a+\alpha v^2)^{n-1}}\,dv. \tag{3.14}$$

The evaluation of these integrals requieres an intermediate result, that is also of independent interest.

Lemma 3.2. Assume $z \in \mathbb{R}$ and $n \in \mathbb{N}\cup\{0\}$. Then

$$\int_0^1 \frac{dx}{(1+z^2x^2)^{n+1}} = \frac{1}{2^{2n}}\binom{2n}{n}\left(\frac{\tan^{-1} z}{z} + \sum_{k=1}^{n} \frac{2^{2k}}{2k\binom{2k}{k}}\frac{1}{(1+z^2)^k}\right). \tag{3.15}$$

PROOF. Define

$$F_n(z) := \int_0^1 \frac{dx}{(1+z^2x^2)^{n+1}} = \frac{1}{z}\int_0^z \frac{dy}{(1+y^2)^{n+1}}. \tag{3.16}$$

Take derivatives with respect to z on both sides of (3.16). The outcome is a system of differential-difference equations

$$\begin{aligned} \frac{dF_n(z)}{dz} &= \frac{2(n+1)}{z}F_{n+1}(z) - \frac{2(n+1)}{z}F_n(z) \\ \frac{dF_n}{dz} &= -\frac{1}{z}F_n(z) + \frac{1}{z(1+z^2)^{n+1}}. \end{aligned} \tag{3.17}$$

Solving for a purely recursive relation we obtain (after re-indexing $n \mapsto n-1$):

$$2nF_n(z) = (2n-1)F_{n-1}(z) + \frac{1}{(1+z^2)^n}, \tag{3.18}$$

with the initial condition $F_0(z) = \frac{1}{z}\tan^{-1} z$. This recursion is solved using the procedure described in Lemma 2.7 of [**1**]. This produces the stated expression for $F_n(z)$. □

The next required evaluation is that of the powers of a simple rational function.

Lemma 3.3. Let a, b, c, d be real numbers such that $cd > 0$. Then

$$\begin{aligned}\int_0^1 \left(\frac{ax^2+b}{cx^2+d} \right)^n dx &= \frac{a^n}{c^n} + \frac{4a^n}{c^n} \sqrt{\frac{d}{c}} \tan^{-1} \sqrt{c/d} \sum_{k=1}^{n} \left(\frac{bc-ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \\ &+ \frac{4a^n}{c^n} \sum_{k=1}^{n} \left(\frac{bc-ad}{4ad} \right)^k \binom{n}{k} \binom{2k-2}{k-1} \sum_{j=1}^{k-1} \frac{2^{2j}}{2j\binom{2j}{j}} \frac{d^j}{(c+d)^j}.\end{aligned}$$

Proof. Start with the partial fraction expansion

$$G(x) := \frac{ax^2+b}{cx^2+d} = \frac{a}{c} + \frac{bc-ad}{cd} \frac{1}{cx^2/d+1}, \tag{3.19}$$

and expand $G(x)^n$ by the binomial theorem. The result follows by using Lemma 3.2. $\square$

The next result follows by combining the statements of the previous three lemmas.

Theorem 3.4. Let a, $b \in \mathbb{R}^+$ with $a < b$. Then

$$\begin{aligned}I_{n+1}(a,b) &:= \int_0^\infty \frac{dt}{(a+bt^2)^{n+1}\sqrt{1+t^2}} \\ &= \frac{1}{a(a-b)^n} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-b}{4a} \right)^j \binom{2j}{j} \times \left(\frac{\tan^{-1}\sqrt{b/a-1}}{\sqrt{b/a-1}} + \sum_{k=1}^{j} \frac{2^{2k}}{2k\binom{2k}{k}} \left(\frac{a}{b} \right)^k \right).\end{aligned}$$

4. Some integrals involving the exponential function

In [**5**] we find **3.310**:

$$\int_0^\infty e^{-px}\, dx = \frac{1}{p}, \text{ for } p > 0, \tag{4.1}$$

that is probably the most elementary evaluation in the table. The example **3.311.1**

$$\int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}, \tag{4.2}$$

can also be evaluated in elementary terms. Observe first that the change of variables $t = px$, shows that (4.2) is equivalent to the case $p = 1$:

$$\int_0^\infty \frac{dx}{1+e^x} = \ln 2. \tag{4.3}$$

This can be evaluated by the change of variables $u = e^x$ that yields

$$I = \int_1^\infty \frac{du}{u(1+u)}, \tag{4.4}$$

and this can be integrated by partial fractions to produce the result. The parameter in (4.2) is what we call *fake*, in the sense that the corresponding integral is independent of it. The advantage of such parameter is that it provides flexibility to a formula: differentiating (4.2) with respect to p produces

$$\int_0^\infty \frac{xe^{px}\, dx}{(1+e^{px})^2} = \frac{\ln 2}{p^2}, \tag{4.5}$$

$$\int_0^\infty \frac{x^2 e^{px}(e^{px}-1)\,dx}{(1+e^{px})^3} = \frac{2\ln 2}{p^3}, \tag{4.6}$$

$$\int_0^\infty \frac{x^3 e^{px}(e^{2px}-4e^{px}+1)\,dx}{(1+e^{px})^4} = \frac{6\ln 2}{p^4}. \tag{4.7}$$

The general integral formula is obtained by differentiating (4.2) n-times with respect to p to produce

$$\int_0^\infty \left(\frac{\partial}{\partial p}\right)^n \frac{dx}{1+e^{px}} = (-1)^n \frac{n!}{p^{n+1}} \ln 2. \tag{4.8}$$

The pattern of the integrand is clear:

$$\left(\frac{\partial}{\partial p}\right)^n \frac{1}{1+e^{px}} = \frac{(-1)^n x^n e^{px}}{(1+e^{px})^{n+1}} P_n(e^{px}), \tag{4.9}$$

where P_n is a polynomial of degree $n-1$. It follows that

$$\int_0^\infty \frac{x^n e^{px} P_n(e^{px})\,dx}{(1+e^{px})^{n+1}} = \frac{n!\,\ln 2}{p^{n+1}}. \tag{4.10}$$

The change of variables $t = px$ shows that p is a fake parameter. The integral is equivalent to

$$\int_0^\infty \frac{x^n e^{x} P_n(e^{x})\,dx}{(1+e^{x})^{n+1}} = n!\,\ln 2. \tag{4.11}$$

The first few polynomials in the sequence are given by

$$\begin{aligned} P_1(u) &= 1, \\ P_2(u) &= u-1, \\ P_3(u) &= u^2-4u+1, \\ P_4(u) &= u^3-11u^2+11u-1. \end{aligned} \tag{4.12}$$

Proposition 4.1. The polynomials $P_n(u)$ satisfy the recurrence

$$P_{n+1}(u) = (nu-1)P_n(u) - u(1+u)\frac{d}{du}P_n(u). \tag{4.13}$$

PROOF. The result follows by expanding the relation

$$\frac{(-1)^{n+1}x^{n+1}e^{px}P_{n+1}(e^{px})}{(1+e^{px})^{n+2}} = \frac{\partial}{\partial p}\left(\frac{(-1)^n x^n e^{px} P_n(e^{px})}{(1+e^{px})^{n+1}}\right). \tag{4.14}$$

□

Examining the first few values, we observe that

$$Q_n(u) := (-1)^n P_n(-u) \tag{4.15}$$

is a polynomial with positive coefficients. This follows directly from the recurrence

$$Q_{n+1}(u) = (nu+1)Q_n(u) + u(1-u)\frac{d}{du}Q_n(u). \tag{4.16}$$

This comes directly from (4.13). The first few values are

$$\begin{aligned} Q_1(u) &= 1, \\ Q_2(u) &= u+1, \\ Q_3(u) &= u^2+4u+1, \\ Q_4(u) &= u^3+11u^2+11u+1. \end{aligned} \tag{4.17}$$

Writing

$$Q_n(u) = \sum_{j=0}^{n-1} E_{j,n} u^j, \tag{4.18}$$

the reader will easily verify the recurrence

$$\begin{aligned} E_{0,n+1} &= E_{0,n} \\ E_{j,n+1} &= (n-j+1)E_{j-1,n} + (j+1)E_{j,n} \\ E_{n,n+1} &= E_{n,n}. \end{aligned} \tag{4.19}$$

The numbers $E_{j,n}$ are called *Eulerian numbers.* They appear in many situations. For example, they provide the coefficients in the series

$$\sum_{k=1}^{\infty} k^j x^k = \frac{x}{(1-x)^{j+1}} \sum_{n=0}^{m-1} E_{j,n} x^n \tag{4.20}$$

and satisfy the simpler recurrence

$$E_{j,n} = nE_{j-1,n} + jE_{j,n-1}, \tag{4.21}$$

that can be derived from (4.19). These numbers have a combinatorial interpretation: they count the number of permutations of $\{1, 2, \ldots, n\}$ having j permutation ascents. The explicit formula

$$E_{j,n} = \sum_{k=0}^{j+1} (-1)^k \binom{n+1}{k} (j+1-k)^n, \tag{4.22}$$

can be checked from the recurrences. The reader will find more information about these numbers in [**6**].

5. A simple change of variables

The table [**5**] contains the example **3.195**:

$$\int_0^{\infty} \frac{(1+x)^{p-1}}{(x+a)^{p+1}}\, dx = \frac{1-a^{-p}}{p(a-1)}. \tag{5.1}$$

One must include the restrictions $a > 0$, $a \neq 1$, $p \neq 0$. The evaluation is elementary: let

$$u = \frac{1+x}{x+a}, \tag{5.2}$$

to obtain

$$I = \frac{1}{a-1} \int_{1/a}^{1} u^{p-1} \, du, \tag{5.3}$$

that gives the stated value. The formula has been supplemented with the value 1 for $a = 1$ and $\ln a/(a-1)$ when $p = 0$ in the last edition of [**5**].

Differentiating (5.1) n times with respect to the parameter p produces

$$\int_0^\infty \frac{(1+x)^{p-1}}{(x+a)^{p+1}} \ln^n \left(\frac{1+x}{x+a} \right) dx = \frac{(-1)^n a^{-p}}{(a-1)p^{n+1}} \left[n! \, (a^p - 1) - \sum_{k=1}^{n} \frac{n!(p \, \ln a)^k}{k!} \right].$$

Naturally, the integral above is just

$$\frac{1}{a-1} \int_{1/a}^{1} u^{p-1} \, \ln^n u \, du \tag{5.4}$$

and its value can also be obtained by differentiation of (5.3).

The next result presents a generalization of (5.1):

Lemma 5.1. Let a, b be free parameters and $n \in \mathbb{N}$. Then

$$\int_0^\infty \frac{(1+x)^b}{(x+a)^{b+n}} \, dx = (a-1)^{-n} \times \left\{ B(n,b) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{a^{-b-k}}{b+k} \right\},$$

where $B(n,b)$ is Euler's beta function.

Proof. Use the change of variables $u = (1+x)/(a+x)$, expand in series and then integrate term by term. $\square$

6. Another example

The integral in **3.268.1** states that

$$\int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p. \tag{6.1}$$

To compute it, and to avoid the singularity at $x = 1$, we write

$$I = \lim_{\epsilon \to 0} \int_0^{1-\epsilon} \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx. \tag{6.2}$$

This evaluates as

$$I = \lim_{\epsilon \to 0} - \ln \epsilon + \ln(1 - (1-\epsilon)^p) = \lim_{\epsilon \to 0} \ln \left(\frac{1 - (1-\epsilon)^p}{\epsilon} \right) = \ln p. \tag{6.3}$$

7. Examples of recurrences

Several definite integrals in [**5**] can be evaluated by producing a recurrence for them. For example, in **3.622.3** we find

$$\int_0^{\pi/4} \tan^{2n} x\,dx = (-1)^n \left(\frac{\pi}{4} - \sum_{j=0}^{n-1} \frac{(-1)^{j-1}}{2j-1} \right). \tag{7.1}$$

To check this identity, define

$$I_n = \int_0^{\pi/4} \tan^{2n} x\,dx \tag{7.2}$$

and integrate by parts to produce

$$I_n = -I_{n-1} + \frac{1}{2n-1}. \tag{7.3}$$

From here we generate the first few values

$$I_0 = \frac{\pi}{4},\ I_1 = -\frac{\pi}{4} + 1,\ I_2 = \frac{\pi}{4} - 1 + \frac{1}{3},\ \text{and } I_3 = -\frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5},$$

and from here one can *guess* the formula (7.1). A proof by induction is easy using (7.3).

A similar argument produces **3.622.4**:

$$\int_0^{\pi/4} \tan^{2n+1} x\,dx = \frac{(-1)^{n+1}}{2} \left(\ln 2 - \sum_{k=1}^{n} \frac{(-1)^k}{k} \right). \tag{7.4}$$

To establish this, define

$$J_n = \int_0^{\pi/4} \tan^{2n+1} x\,dx \tag{7.5}$$

and integrate by parts to produce

$$J_n = -J_{n-1} + \frac{1}{2n}. \tag{7.6}$$

The value

$$J_0 = \int_0^{\pi/4} \tan x\,dx = \frac{\ln 2}{2}, \tag{7.7}$$

and the recurrence (7.6) yield the formula.

8. A truly elementary example

The evaluation of **3.471.1**

$$\int_0^u \exp\left(-\frac{b}{x}\right) \frac{dx}{x^2} = \frac{1}{b} \exp\left(-\frac{b}{u}\right), \tag{8.1}$$

is truly elementary: the change of variables $t = -b/x$ gives the result.

9. Combination of polynomials and exponentials

Integration by parts produces

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}. \tag{9.1}$$

This appears as **2.321.1** in [**5**]. Introduce the notation

$$I_n(a) := \int x^n e^{ax}\,dx \tag{9.2}$$

so that (9.1) states that

$$I_n(a) = \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}(a). \tag{9.3}$$

This recurrence is now used to prove

$$I_n(a) = n!e^{ax}\sum_{k=0}^{n} \frac{(-1)^k x^{n-k}}{(n-k)!\,a^{k+1}} \tag{9.4}$$

by an easy induction argument. This appears as **2.321.2**. The case $1 \leqslant n \leqslant 4$ appear as **2.322.1**, **2.322.2**, **2.322.3**, **2.322.4**, respectively.

Integrating (9.4) between 0 and u produces **3.351.1**:

$$\int_0^u x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} - e^{-au}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{a^{n-k+1}}. \tag{9.5}$$

This sum can be written in terms of the incomplete gamma function. Details will appear in a future publication. Integrating (9.4) from u to ∞ produces

$$\int_u^\infty x^n e^{-ax} dx = e^{-au}\sum_{k=0}^{n}\frac{n!}{k!}\frac{u^k}{a^{n-k+1}}. \tag{9.6}$$

This appears as **3.351.2**.

The special case $n = 1$ of 3.351.1 appears as **3.351.7**. The cases $n = 2$ and $n = 3$ appear as **3.351.8** and **3.351.9**, respectively.

10. A perfect derivative

In section **4.212** we find a list of examples that can be evaluated in terms of the exponential integral function, defined by

$$\text{Ei}(x) := \int_{-\infty}^{x} \frac{e^t\,dt}{t} \tag{10.1}$$

for $x < 0$ and by the Cauchy principal value of (10.1) for $x > 0$. An exception is **4.212.7**:

$$\int_1^e \frac{\ln x\,dx}{(1+\ln x)^2} = \frac{e}{2} - 1. \tag{10.2}$$

This is an elementary integral: the change of variables $t = 1 + \ln x$ yields

$$I = \frac{1}{e} \int_1^2 \frac{(t-1)}{t^2} e^t \, dt \tag{10.3}$$

and to evaluate it observe that

$$\frac{(t-1)}{t^2} e^t = \frac{d}{dt} \frac{e^t}{t}. \tag{10.4}$$

The change of variables $t = \ln x$ in (10.2) yields

$$\int_0^1 \frac{t\, e^t \, dt}{(1+t)^2} = \frac{e}{2} - 1. \tag{10.5}$$

This is **3.353.4** in [**5**].

The previous evaluation can be generalized by introducing a parameter.

Lemma 10.1. Let $\alpha \in \mathbb{R}$. Then

$$\int_1^e \frac{\ln x \, dx}{(\alpha + \ln x)^{\alpha+1}} = \frac{e}{(\alpha+1)^\alpha} - \frac{1}{\alpha^\alpha}. \tag{10.6}$$

Proof. Substitute $t = \alpha + \ln x$ and use

$$\frac{d}{dt} \frac{e^t}{t^\alpha} = \frac{t-\alpha}{t^{\alpha+1}} e^t. \tag{10.7}$$

The case $\alpha = 1$ corresponds to (10.2). $\square$

11. Integrals involving quadratic polynomials

There are several evaluation in [**5**] that involve quadratic polynomials. We assume, for reasons of convergence, that the discriminant $d = b^2 - ac$ is strictly negative.

We start with

$$\int_0^\infty \frac{dx}{ax^2 + 2bx + c} = \frac{1}{\sqrt{ac - b^2}} \cot^{-1} \left(\frac{b}{\sqrt{ac-b^2}} \right). \tag{11.1}$$

This is evaluated by completing the square and a simple trigonometric substitution:

$$\begin{aligned} \int_0^\infty \frac{dx}{ax^2 + 2bx + c} &= \frac{1}{a} \int_{b/a}^\infty \frac{du}{u^2 - d/a^2} \\ &= \frac{1}{\sqrt{-d}} \int_{b/\sqrt{-d}}^\infty \frac{dv}{v^2+1}. \end{aligned}$$

Differentiating (11.1) with respect to c produces **3.252.1**:

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[\frac{\cot^{-1}(b/\sqrt{ac-b^2})}{\sqrt{ac-b^2}} \right].$$

We now produce a closed-from expression for this integral.

Lemma 11.1. Let $n \in \mathbb{N}$ and $u := 4(ac-b^2)/ac$. Assume $cu > 0$. Then we have the explicit evaluation
(11.2)
$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{2b}{a(cu)^n}\binom{2n-2}{n-1} \times \left\{ \frac{\sqrt{acu}}{b} \cot^{-1}\left(\frac{2b}{\sqrt{acu}}\right) - \sum_{j=1}^{n-1} \frac{u^j}{j\binom{2j}{j}} \right\}.$$

Proof. The case $n=1$ was described above:
$$h(a,b,c) := \int_0^\infty \frac{dx}{ax^2+2bx+c} = \frac{1}{\sqrt{ac-b^2}} \cot^{-1}\left(\frac{1}{\sqrt{ac-b^2}}\right). \tag{11.3}$$
Now observe that $h(a^2, abc, b^2) = h(1,b,1)/ac$. To establish (11.2), change the parameters sequentially as $a \mapsto a^2$; $c \mapsto c^2$; $b \mapsto abc$. In the new format, both sides satisfy the differential-difference equation
$$-2nc(1-b^2)f_{n+1} = \frac{df_n}{dc} + \frac{b}{ac^{2n}}. \tag{11.4}$$
The identity (11.2) is obtained by reversing the transformations of paramaters indicated above. □

Corollary 11.2. Using the notations of Lemma 11.1 we have
$$\sum_{j=1}^\infty \frac{u^j}{j\binom{2j}{j}} = \frac{\sqrt{acu}}{b} \cot^{-1}\left(\frac{2b}{\sqrt{acu}}\right). \tag{11.5}$$

The integral **3.252.2**
$$\int_{-\infty}^\infty \frac{dx}{(ax^2+2bx+c)^n} = \frac{(2n-3)!!\pi\, a^{n-1}}{(2n-2)!!\,(ac-b^2)^{n-1/2}} \tag{11.6}$$
reduces via $u = a(x+b/a)/\sqrt{ac-b^2}$ to Wallis' integral
$$\int_0^\infty \frac{du}{(u^2+1)^n} = \frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2}, \tag{11.7}$$
that appears as **3.249.1**. The reader will find in [**2**] proofs of Wallis' integral. Observe that the evaluation of **3.252.2** is much simpler than the corresponding half-line example presented in Lemma 11.1.

The last example of this type is **3.252.3**:
$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+3/2}} = \frac{(-2)^n}{(2n+1)!!}\frac{\partial^n}{\partial c^n}\left(\frac{1}{\sqrt{c}\,(\sqrt{ac}+b)}\right).$$
A simple trigonometric substitution gives the case $n=0$:
$$\begin{aligned} \int_0^\infty \frac{dx}{(ax^2+2bx+c)^{3/2}} &= \frac{\sqrt{a}}{ac-b^2}\int_{b/\sqrt{-d}}^\infty \frac{du}{(u^2+1)^{3/2}} \\ &= \frac{1}{\sqrt{c}\,(\sqrt{ac}+b)}. \end{aligned}$$

The general case follows by differentiating with respect to c and observing that

$$\left(\frac{\partial}{\partial c}\right)^j = (-1)^j \frac{(2j+1)!!}{2^j}(ax^2+bx+c)^{-3/2-j}.$$

We now provide a closed-form expression for (11.8).

Theorem 11.3. Let a, b, $c \in \mathbb{R}$ and $n \in \mathbb{N}$. Define $u = (ac-b^2)/4ac$ and assume $cu > 0$. Then

$$\int_0^\infty \frac{dx}{(ax^2+2bx+c)^{n+3/2}} = \frac{(cu)^{-n}}{\sqrt{c}\binom{2n}{n}(2n+1)} \left(\frac{1}{\sqrt{ac}+b} - \frac{b}{ac-b^2} \sum_{j=1}^{n} \binom{2j}{j} u^j \right).$$

Proof. Change parameters sequentially as $a \mapsto a^2$; $c \mapsto c^2$; $b \mapsto abc$. Then, in the new format both sides satisfy the differential-difference equation

$$-(2N(1-b^2)c)f_{N+1} = \frac{df_N}{dc} - \frac{b}{ac^{2N}}, \tag{11.8}$$

where $N = n + \frac{3}{2}$. □

12. An elementary combination of exponentials and rational functions

The table [**5**] contains two integrals belonging to the family

$$T_j := \int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^j}. \tag{12.1}$$

Indeed **3.411.19** gives T_1:

$$\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x} = -\sum_{k=0}^{n}(-1)^k \binom{n}{k} \ln(p+n-k), \tag{12.2}$$

and **3.411.20** gives T_2:

$$\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^2} = \sum_{k=0}^{n}(-1)^k \binom{n}{k} (p+n-k)\ln(p+n-k), \tag{12.3}$$

The next result presents an explicit evaluation of T_j.

Proposition 12.1. Let p be a free parameter, and let n, $j \in \mathbb{N}$ with $n+p>0$. Then

$$\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^j} = \frac{(-1)^j}{(j-1)!}\sum_{k=0}^{n}(-1)^k (p+n-k)^{j-1}\ln(p+n-k). \tag{12.4}$$

Proof. Start with the observation that

$$T_j = -\int T_{j-1}(p)dp + C. \tag{12.5}$$

Therefore we need to describe the iterative integrals $f_j(p) = \int f_{j-1}(p)\,dp$,with $f_0(p) = \ln(p+\alpha)$. This can be found in page 82 of [**2**] as

$$f_j(p) = \frac{1}{j!}(p+\alpha)^j \ln(p+\alpha) - \frac{H_j}{j!}(p+\alpha)^j + C, \tag{12.6}$$

with $\alpha = p+n-k$ and $H_j = 1+\frac{1}{2}+\cdots+\frac{1}{j}$ is the harmonic number. To build back the functions T_j employ the fact that, for any polynomial $Q(n,k)$,

$$\sum_{k=0}^{n}(-1)^k(-1)^k\binom{n}{k}Q(n,k) \equiv 0. \tag{12.7}$$

Consequently,

$$T_j = C + \frac{(-1)^{j+1}}{j!}\sum_{k=0}^{n}(-1)^k(p+n-k)^j\ln(p+n-k). \tag{12.8}$$

The last step is to check that $C=0$. This follows directly from $T_j \to 0$ as $p \to \infty$. The assertion is now validated. □

13. An elementary logarithmic integral

Entry **4.222.1** states that

$$\int_0^\infty \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)dx = (a-b)\pi. \tag{13.1}$$

In order to establish this, we consider the finite integral

$$I(m) := \int_0^m \ln\left(\frac{a^2+x^2}{b^2+x^2}\right)dx \tag{13.2}$$

and then let $m \to \infty$.

Integration by parts gives

$$\begin{aligned}\int_0^m \ln(a^2+x^2)\,dx &= m\ln(m^2+a^2) - 2\int_0^m \frac{x^2\,dx}{a^2+x^2} \\ &= m\ln(m^2+a^2) - 2m + 2a^2\int_0^m \frac{dx}{a^2+x^2} \\ &= m\ln(m^2+a^2) - 2m + 2a\tan^{-1}\left(\frac{m}{a}\right).\end{aligned}$$

Therefore

$$I(m) = m\ln\left(\frac{m^2+a^2}{m^2+b^2}\right) + 2a\tan^{-1}\left(\frac{m}{a}\right) - 2b\tan^{-1}\left(\frac{m}{b}\right).$$

The limit of the logarithmic part is zero and the arctangent part gives $(a-b)\pi$ as required.

The generalization

$$\int_0^\infty \ln\left(\frac{a^s+x^s}{b^s+x^s}\right)dx = (a-b)\frac{\pi}{\sin(\pi/s)} \tag{13.3}$$

can be established by elementary methods provided we assume the value

$$\int_0^\infty \frac{dx}{1+x^s} = \frac{\pi}{s\,\sin(\pi/s)} \tag{13.4}$$

as given. This integral is evaluated in terms of Euler's beta function in [**11**]. Indeed, integration by parts gives

$$\int_0^y \ln(a^s + x^s)\, dx = y \ln(a^s + y^s) - sy + sa^s \int_0^y \frac{dx}{a^s + y^s}, \tag{13.5}$$

and similarly for the b-parameter. Combining these evaluations gives

$$\int_0^y \ln\left(\frac{a^s + x^s}{b^s + x^s}\right) dx = y \ln\left(\frac{a^s + y^s}{b^s + y^s}\right) + sa^s \int_0^y \frac{dx}{a^s + x^s} - sb^s \int_0^y \frac{dx}{b^s + x^s}.$$

Upon letting $y \to \infty$, we observe that the logarithmic term vanishes and a scaling reduces the remaining integrals to (13.4).

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