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Classical Schottky Uniformizations of Genus 2
A Package for MATHEMATICA

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Dedicated to Sevín Recillas

ABSTRACT. The general theory of Riemann surfaces asserts that a closed Riemann surface S of genus $g \geqslant 2$ may be seen as (i) the quotient by a Kleinian group G or (ii) a plane algebraic curve C (possible with singularities) or (iii) a symmetric complex $g \times g$ matrix Z with positive imaginary part (a Riemann period matrix). Numerical uniformization problem ask for numerical relations between these objects for suitable choices of G, C and Z. In this note we discuss the case of genus two for G a classical Schottky group. The algorithm has been implemented into a MATHEMATICA package for the case of M-real curves of genus 2, but it can easily be rewritten for the general case.

CONTENTS

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1. Introduction

A closed Riemann surface S of genus $g \geqslant 2$ may be seen as the quotient by a Schottky group G of rank g or as a plane algebraic curve C (possible with singularities) or as a symmetric complex $g \times g$ matrix Z with positive imaginary part (a Riemann period matrix). Numerical uniformization problem ask for numerical relations between these objects for suitable choices of G, C and Z. In this note we describe an algorithm (due in its basis to W. Burnside), for the case $g = 2$, which permits to obtain numerically both an algebraic curve and a Riemann period matrix of the genus two Riemann surface uniformized by a given classical Schottky group. The algorithm has been implemented into a MATHEMATICA package for the case of M-real curves of genus 2, but it can easily be rewritten for the general case. This algorithm also permits to find numerical real non-singular finite-gap solutions of Kadomtsev-Petviashvili partial differential equation [**19**] and, in particular, of Korteweg-de Vries partial differential equation in the case of genus two M-real curves of genus two (see Section 7). Next, we proceed to recall some definitions in order to clarify the above.

A *Schottky group of rank* 2 G is a group generated by two Möbius transformations, say A and B, with the property that there are 4 pairwise disjoint circles, say C_1, C_1', C_2 and C_2', all of them bounding a common domain of connectivity 4, say $\mathcal{D}$, so that $A(C_1) = C_1'$, $B(C_2) = C_2'$, $A(\mathcal{D}) \cap \mathcal{D} = \emptyset$ and $B(\mathcal{D}) \cap \mathcal{D} = \emptyset$. In case we may choose the above loops as circles, we say that G is a *classical Schottky group of genus* 2. It is well known (see for instance [**20**]) that if G is a Schottky group of rank 2, then the following properties hold: (i) G is a free group of rank 2, (ii) G is a Kleinian group with connected region of discontinuity $\Omega(G)$ (its complement is a Cantor set), (iii) $S = \Omega(G)/G$ is a closed Riemann surface of genus 2, (iv) $E = [A, B] = AB - BA$, EA and EB are elliptic transformation of order two in the normalizer of G, each one inducing the hyperelliptic involution on S [**14**]. The reciprocal to (iii) holds and it is given by Koebe's uniformization theorem [**16**].

Let D_0, D_1, D_2 be three pairwise disjoint simple loops, all of them bounding a domain of connectivity 3, and assume we have three elliptic transformations of order 2, say E_0, E_1, E_2, so that E_0 interchanges both topological discs bounded by D_0, E_1 interchanges both topological discs bounded by D_1 and E_2 interchanges both topological discs bounded by D_2. The group $K = \langle E_0, E_1, E_2\rangle$ is called a *Whittaker group* of rank 2. If we may choose D_0, D_1 and D_2 as circles, then we say that K is a *classical Whittaker group*. If we set $A = E_0E_1$ and $B = E_0E_2$, then we have that $G = \langle A, B\rangle$ is a Schottky group of rank 2; take $C_1 = D_1$, $C_2 = D_2$, $C_1' = E_0(D_1)$ and $C_2' = E_0(D_2)$. Clearly, if K is classical, then G is to. The Schottky group G can be characterized as the unique torsion free subgroup of minimal index (such an index equal to 2). Reciprocally, if we start with a Schottky group $G = \langle A, B\rangle$ of rank 2, then $E_0 = [A, B] = AB - BA$, $E_1 = E_0A$ and $E_2 = E_0B$ generate a Whittaker group of rank 2 [**14**]. If $S = \Omega(G)/G$, then $\Omega(G) = \Omega(K)$ and $\Omega(K)/K$ uniformizes the quotient Riemann orbifold $S/\langle j\rangle$, where $j : S \to S$ denotes the hyperelliptic involution.

A *real curve of genus* 2 is a pair (S, τ), where S is a closed Riemann surface S of genus 2 and $\tau : S \to S$ is a reflection, that is, an anticonformal involution with fixed points. The reflection τ is also called a real structure on S. The connected components of fixed points of the reflection τ are called *ovals* (or mirrors) and their are given by simple closed geodesics on the surface. As a consequence of Harnak's theorem [**2**], [**15**], the number n_τ of ovals of a reflection τ satisfies $1 \leqslant n_\tau \leqslant 3$. In case that $n_\tau = 3$, we say that τ is called a *M-reflection* and that (S, τ) called a *M-real curve of genus* 2.

A *real Schottky group of rank* 2 is by definition a Schottky group of rank 2 that keeps invariant the unit circle. In particular, a real Schottky group of rank 2 has its limit set a Cantor subset of the unit circle. Let us denote by σ the reflection on the unit circle. Let G be a real Schottky group, with region of discontinuity Ω, and $S = \Omega/G$ the closed Riemann surface uniformized by G. We have, in this case, that σ induces a reflection on S, say τ, and we have that the pair (S, τ) is a real curve. Reciprocally, Koebe's uniformization theorem [**17**] asserts that every real curve can be obtained in this way.

If we have a Whittaker group K defined by three circles D_0, D_1 and D_2, and generators E_0, E_1 and E_2 (as above), then there is a circle D_3 which is orthogonal to each D_j, for $j = 0, 1, 2$. Let us denote by τ_0 the reflection on the boundary circle of D_3. In order to ensure that τ_0 normalizes K we only need to have that both fixed points of E_j are preserved by τ_0: they may be fixed or permuted. In this way, we have exactly 4 possible configurations. In the case τ_0 normalizes K, we say that K is a *real Whittaker group* of rank 2. The Schottky group G defined by a real Whittaker group is a real Schottky group. The genus two Riemann surface $S = \Omega(G)/G$ has in this case a reflection τ which is induced by τ_0, that is, (S, τ) is a real curve of genus 2.

Given any collection of 3 different points on the complex plane, say $a_1, a_2, a_3 \in \mathbb{C} - \{0, 1\}$, we may consider the algebraic projective plane curve $C \subset \mathbb{P}^2$ (the complex projective plane) defined by the affine plane curve

$$y^2 = x(x-1)(x-a_1)(x-a_2)(x-a_3).$$

After desingularization at ∞, the above curve defines a closed Riemann surface of genus 2 and reciprocally, every closed Riemann surface of genus 2 can be so obtained. In case that the set of points $\{0, 1, \infty, a_1, a_2, a_3\}$ is invariant under a reflection of the Riemann sphere, we have that it defines naturally a real curve of genus 2.

Numerical Schottky uniformization problem ask for concrete relations (or numerical ones) between the coefficients a_1, a_2, a_3 and a Schottky group uniformizing the same (class of) Riemann surface.

In the literature there are works in the direction of numerical Fuchsian uniformizations, see for instance [**1, 5, 8, 21, 24**], which describe algebraic curves and Riemann period matrices from co-compact Fuchsian groups. In general, these Fuchsian groups are normal subgroups of NEC group groups generated by the reflections on certain compact hyperbolic polygons. In [**22**] there is an algorithm which permits to compute numerically a Schottky group uniformizing a given plane hyperelliptic algebraic curve that admits the reflection $\eta(z) = \overline{z}$ as a symmetry (see [**12**] for the general case). In [**10**] is explained the theoretical part of an algorithm (based on original ideas of W. Burnside) which permits to compute both an algebraic curve and a Riemann period matrix of the uniformized real Riemann surface in terms of a given real Schottky group. This has been used for some particular families with many automorphisms; for instance [**11**] is considered a class of real Riemann surfaces of genus two with many automorphisms commuting with a real structure. In this note we describe the values of a_1, a_2, a_3 in terms of a classical Schottky group of rank two in order to use of the above algorithm. The implemented program in MATHEMATICA [**25**] for the M-real curves can be obtained from http://docencia.mat.utfsm.cl/∼rhidalgo/files/.

The organization of this paper is as follows. In section 2 we recall some basic facts on collection of three pairwise disjoint circles. In Section 3 we construct a family $\mathfrak{F}$ of classical Whittaker groups of rank 2

$$\{K(\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2})\},$$

whose index two Schottky subgroups

$$\{G(\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2})\}$$

provides uniformization of Riemann surfaces of genus 2 which can be uniformized by a classical Schottky group. In Section 4 we make particular choices of the angles $\eta_{i,j}$ in order to obtain real Whittaker groups. For instance, for $\eta_{1,1} = \pi/2 + \theta_1$, $\eta_{1,2} = 3\pi/2 - \theta_1$, $\eta_{2,1} = 7\pi/6 + \theta_2$, $\eta_{2,2} = 13\pi/6 - \theta_2$, $\eta_{3,1} = 5\pi/6 - \theta_3$ and $\eta_{3,2} = \theta_3 - \pi/6$, we obtain real Whittaker groups whose Schottky subgroups uniformize all M-real curves of genus 2. In section 5 we describe explicitly an algebraic curve for the Riemann surface uniformized by the index two Schottky subgroup of a classical Whittaker group in the family $\mathfrak{F}$. In section 6 we show how to implement the results of previous sections to obtain numerical approximations of the algebraic curve, a Riemann period matrix (we only describe it for the M-real case, but it may be suitable modified in order to consider the general situation) and the accessory parameters of the uniformized Riemann surface. In section 6.4 we discuss two different families of symmetrical M-real curves. In section 6.5 we write down, for a concrete example, the values we obtain with our algorithm. In section 6.6 we discuss the situation for higher genus M-real curves. In section 7 we relate to the theory of finite-gap integration developed by Novikov, Dubrovin, Matveev, It·s and others to obtain explicit solutions to certain class of partial differential equations appearing in evolution problems.

2. Triples of Circles

A circle on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is either an Euclidian circle on the complex plane $\mathbb{C}$ or the union of ∞ with an Euclidian line on $\mathbb{C}$. A disc is an open domain bounded by a circle. A triple of circles C_1, C_2 and C_3 is said to be non-separating if there are pairwise disjoint discs D_1, D_2 and D_3, so that the boundary of D_j is C_j, for each $j = 1, 2, 3$. Let us note that in a triple of non-separating circles we may have that some of them may be tangent, but they cannot cross. We have the following well known fact.

Proposition 2.1. Given any triple of non-separating circles, say C_1, C_2 and C_3, there is a (unique) circle C_0 which is orthogonal to each of them.

Proof. As we may send the three circles to Euclidian circles by a suitable Möbius transformation, we may assume we are in that situation. Let L the Euclidian line determined by the centers of C_1 and C_3. We have that L is orthogonal to both C_1 and C_3. If T is a Möbius transformation that sends the center of C_1 to 0 and the center of C_3 to ∞, then, as T is conformal, it sends C_1 and C_2 to concentric Euclidian circles at the origin and it sends C_2 to an Euclidian circle contained in the interior of the annulus bounded by the above two concentric circles $T(C_1)$ and $T(C_3)$. Up to a rotation, we may assume $T(C_2)$ to be orthogonal to the real line. □

Let us now consider a triple non-separating circles, say C_1, C_2 and C_3. By the previous fact, we have a unique common orthogonal circle C_0. Let us denote by D any of the two discs bounded by C_0. Given any point $p \in D$, we may draw three (two of them non-necessarily different) circles, say L_1, L_2, L_3, so that:

(i) $p \in L_j$, for each $j = 1, 2, 3$; and
(ii) L_j is orthogonal to both C_0 and C_j, for each $j = 1, 2, 3$.

Let us denote by L_j^* the arc of L_j contained in D with one end point at p and the other at C_j.

The circles L_1, L_2 and L_3 divide the Riemann sphere into 6 disjoint sectors (two of the sectors can be empty; for instance, in the case that $L_1 = L_2$). These sectors determine 3 angles, say $\alpha \geqslant 0$, $\beta \geqslant 0$ and $\gamma \geqslant 0$, so that $\alpha + \beta + \gamma = \pi$. In the case that $\alpha = \beta = \gamma = \pi/3$, we say that $p \in D$ is a *geometric center of the non-separating triple of circles.*

The arcs L_1^*, L_2^* and L_3^* determine 3 sectors about p. If the the three sectors have equal angle equal to $2\pi/3$, then we say that p is a *main geometric center.* Clearly a main geometric center is a geometric center, but a geometric center may not be a main geometric center.

Theorem 2.1. Under the above notation, there is a unique main geometric center. Moreover,

(i) if there is no tangency between the circles, then we have exactly 4 geometric centers in D;
(ii) if we have exactly one tangency, then we have exactly 3 geometric center in D;
(iii) if we have two tangencies, we have exactly 2 geometric center in D;
(iv) if we have exactly three tangencies, then we have exactly 1 geometric center in D.

PROOF. Let us assume that the three circles are all tangent. In this case, we have three tangency points. These three tangency points determine a unique circle, this circle being the common orthogonal one. We may assume this common orthogonal circle to be the unit circle. By a Möbius transformation that keeps invariant the unit circle, we may assume these three tangency points to be 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. In this way, we see that 0 is a geometric center in the unit disc D. As a consequence of Gauss-Bonnet's formula it cannot be another geometric center.

We may now assume that at least two of the circles are not tangent, say C_1 and C_3. As mentioned in the previous section, we may assume that C_1 is the unit circle, that C_3 a circle centered at the origin and radius $R^2 > 1$, and that C_2 is orthogonal to the segment $[1, R^2] \subset \mathbb{R}$. Let us consider the circle N centered at the origin and radius R. If we consider a Möbius transformation in $PSL(2, \mathbb{R})$ that sends R to 0 and $-R$ to ∞, then we have that the circle N is sent to the imaginary line. As C_3 is the reflection about N of C_1, we have that the image of C_1 is a circle that is orthogonal to the negative real line at points $-B < -A < 0$, that the image of C_3 is a circle that is orthogonal to the positive real line at points $0 < A < B$, and that C_2 is sent to a circle orthogonal to the segment $[-A, A] \subset \mathbb{R}$. Now by use of a dilation, we may assume that $A = 1$, that is, we may assume that C_1 is orthogonal to the real line at the points $-B$ and -1, that C_3 is orthogonal to the real line at the points 1 and B, and that C_2 is orthogonal to the real line at points $-1 \leqslant U < V \leqslant 1$. Up to a reflection on the imaginary line and permuting C_1 with C_3, we may assume now on that $V > 0$.

Some reflections. Given any point $w = x + iy$, with $y > 0$, we draw the circle L_1 containing w and orthogonal to both C_1 and the real line. As the reflection τ_1 on L_1 fixes w and permutes -1 with $-B$, it follows that

$$\tau_1(z) = \frac{p + (1+p)B + p\overline{z}}{\overline{z} - p}$$

where

$$p = \frac{x^2 + y^2 - B}{1 + B + 2x}$$

We draw the circle L_2 containing w and orthogonal to both C_2 and the real line. As the reflection τ_2 on L_2 fixes w and permutes U with V, it follows that

$$\tau_2(z) = \frac{VU - r(U+V) + r\overline{z}}{\overline{z} - r}$$

where

$$r = \frac{x^2 + y^2 - UV}{2x - U - V}$$

We draw the circle L_3 containing w and orthogonal to both C_3 and the real line. As the reflection τ_3 on L_3 fixes w and permutes 1 with B, it follows that

$$\tau_3(z) = \frac{B(1-q) - q + q\overline{z}}{\overline{z} - q}$$

where

$$q = \frac{x^2 + y^2 - B}{2x - 1 - B}$$

A pair of polynomials. The condition for the angle between L_1 and L_2 to be the same as the angle between L_2 and L_3 is equivalent to have:

$$(*) \quad \tau_2(-B) = \tau_3(\tau_2(-1))$$

The condition for the angle between L_2 and L_3 to be the same as the angle between L_3 and L_1 is equivalent to have:

$$(**) \quad \tau_3(U) = \tau_1(\tau_3(V))$$

The equality of both $(*)$ and $(**)$ for the same $w = x + iy$, $y > 0$, is equivalent to solve our problem. Now, equality $(*)$ holds exactly for the zeroes (x, y), $y > 0$, of the polynomial of degree 6 (of degree 5 if $U = -V$) given by:

$$\mathbf{P_1(x,y) = A_0(x) + A_1(x)y^2 + A_2(x)y^4 + A_3(x)y^6} \in \mathbb{R}[\mathbf{x},\mathbf{y}]$$

where

$$\begin{aligned}
A_0(x) &= (-B^2U^2V - B^2UV^2) + (B^2U^2 + 4B^2UV + B^2V^2 + U^2V^2 + B^2U^2V^2)x + \\
&\quad (-3B^2U - 3B^2V - U^2V - B^2U^2V - UV^2 - B^2UV^2)x^2 \\
&\quad +(2B^2 - 2U^2V^2)x^3 + (U + B^2U + V + B^2V + 3U^2V + 3UV^2)x^4 + \\
&\quad (-1 - B^2 - U^2 - 4UV - V^2)x^5 + (U+V)x^6 \\
A_1(x) &= (B^2U + B^2V - U^2V - B^2U^2V - UV^2 - B^2UV^2) + (-2B^2 + 2U^2V^2)x + \\
&\quad (2U + 2B^2U + 2V + 2B^2V + 2U^2V + 2UV^2)x^2 - \\
&\quad -(2 + 2B^2 + 2U^2 + 8UV + 2V^2)x^3 + (3U + 3V)x^4 \\
A_2(x) &= (U + B^2U + V + B^2V - U^2V - UV^2) - (1 + B^2 + U^2 + 4UV + V^2)x + \\
&\quad +(3U + 3V)x^2 \\
A_3(x) &= (U + V)
\end{aligned}$$

Similarly, equality $(**)$ holds exactly for the zeroes (x, y), $y > 0$, of the polynomial of degree 6 given by:

$$\mathbf{P_2(x,y) = B_0(x) + B_1(x)y^2 + B_2(x)y^4 + B_3(x)y^6} \in \mathbb{R}[\mathbf{x},\mathbf{y}]$$

where

$$\begin{aligned}
B_0(x) &= (-B^3U - B^3V + 3B^2UV + 3B^3UV) + \\
&\quad (2B^3 - 2B^2U - 2B^3U - 2B^2V - 2B^3V - 2BUV - 6B^2UV - 2B^3UV)x + \\
&\quad (B^2 + B^3 + 3BU + 5B^2U + 3B^3U + 3BV + 5B^2V + \\
&\quad 3B^3V - UV - BUV - B^2UV - B^3UV)x^2 + \\
&\quad (-4B - 4B^2 - 4B^3 + 4UV + 4BUV + 4B^2UV)x^3 + \\
&\quad (1 + B + B^2 + B^3 - 3U - 5BU - 3B^2U - 3V - 5BV - 3B^2V - UV - BUV)x^4 + \\
&\quad (2 + 6B + 2B^2 + 2U + 2BU + 2V + 2BV - 2UV)x^5 + \\
&\quad (-3 - 3B + U + V)x^6 \\
B_1(x) &= (-3B^2 - 3B^3 + 3BU + 9B^2U + 3B^3U + 3BV + 9B^2V + 3B^3V - UV - 9BUV - \\
&\quad 9B^2UV - B^3UV) + (-4B - 12B^2 - 4B^3 + 4UV + 12BUV + 4B^2UV)x + \\
&\quad (2 + 10B + 10B^2 + 2B^3 - 6U - 14BU - 6B^2U - 6V - 14BV - 6B^2V + 2UV + \\
&\quad 2BUV)x^2 + (4 + 12B + 4B^2 + 4U + 4BU + 4V + 4BV - 4UV)x^3 + \\
&\quad (-9 - 9B + 3U + 3V)x^4 \\
B_2(x) &= (1 + 9B + 9B^2 + B^3 - 3U - 9BU - 3B^2U - 3V - 9BV - 3B^2V + 3UV + 3BUV) + \\
&\quad 2 + 6B + 2B^2 + 2U + 2BU + 2V + 2BV - 2UV)x + (-9 - 9B + 3U + 3V)x^2 \\
B_3(x) &= (-3 - 3B + U + V)
\end{aligned}$$

In order to solve our problem, we need to find the common zeroes (x, y), $y > 0$, of the polynomials $P_1(x, y)$ and $P_2(x, y)$. We divide the arguments into the following cases (1) $V < 1$, (2) $V = 1$ and $U > -1$ and (3) $V = 1$ and $U = -1$.

(1) Case $V < 1$**.** In this case we have no tangencies between the circles.

Lemma 2.1. If $-V < U$, then the zeroes of A_0 (respectively, B_0) are exactly 6 different real points. Moreover:

(i) there is one zero of A_0 inside $(-\infty, U)$;
(ii) there is one zero of A_0 inside (U, V);

(iii) there is one zero of A_0 inside $(1,B)$;
(iv) there is one zero of A_0 inside $(B,+\infty)$;
(v) there are two zeroes of B_0 inside $(-\infty,U)$, which are separated by the zero of A_0;
(vi) there is one zero of B_0 inside (U,V) which is greater than the zero of A_0 in there;
(vii) there is one zero of B_0 inside $(1,B)$ which is greater than the zero of A_0 in there;
(viii) the other two zeroes of A_0 are U and V;
(ix) the other two zeroes of B_0 are 1 and B.

In the case $U=-V$ we have the same except that $A_0(x)$ has exactly 5 zeroes; it has no zeroes in $(B,+\infty)$.

Proof. We have by direct inspection that

$$A_0(U)=0=A_0(V)$$

$$B_0(1)=0=B_0(B)$$

$$\frac{dA_0}{dx}(U)=(B^2-U^2)(1-U^2)(V-U)^2>0$$

$$\frac{dA_0}{dx}(V)=(V^2-B^2)(V^2-1)(V-U)^2>0$$

$$\frac{dB_0}{dx}(1)=4(B-1)^2(1+B)(1-U)(V-1)<0$$

$$\frac{dB_0}{dx}(B)=4(B-1)^2B(1+B)(U-B)(B-V)<0$$

It follows that in the open interval (U,V) we must have a zero of A_0 with negative derivative and in the interval $(1,B)$ we must have a zero of B_0 with positive derivative.

As

$$B_0(U)=(U-1)(U-B)(U^2-1)(U^2-B^2)(U-V)<0$$

$$B_0(V)=(U-V)(V-1)(V-B)(B^2-V^2)(V^2-1)>0$$

we obtain that inside the open interval (U,V) we also have a zero of B_0. Similarly, as we have

$$A_0(1)=(B-1)(B+1)(U-1)^2(V-1)^2>0$$

$$A_0(B)=(1-B)B(1+B)(B-U)^2(B-V)^2<0$$

we also have a zero of A_0 in the open interval $(1,B)$.

If $-V<U$, then as the coefficient of x^6 of $A_0(x)$ is $(U+V)>0$ and $A_0(B)<0$, we have a zero of A_0 inside $(B,+\infty)$. As $A_0(-B)=B(B^2-1)(B+U)^2(B+V)^2>0$ and $A_0(-1)=-(B^2-1)(U+1)^2(V+1)^2<0$, we must have a zero of A_0 in $(-B,-1)$.

The coefficient of x^6 of B_0 is $(U+V-3-3B)<0$. As we have that $B_0(-1)=4(B-1)(1+B)^2(1+U)(1+V)>0$ and $B_0(-B)=-4(B-1)B^2(1+B)^2(B+U)(B+V)<0$, then in the interval $(-B,-1)$ there is zero of B_0. As $Q_2(U)<0$, we also have that one zero of B_0 is inside $(-1,U)$.

At this point we have essentially the desired result, except that we have said nothing about the relation between the zeroes of A_0 and B_0 in the intervals $(-B,-1)$, (U,V) and $(1,B)$. We need to check that the zero of A_0 in $(-B,-1)$ (respectively, in $(1,B)$) is greater than the zero of B_0 in the same interval. Similarly, we need to check that the zero of A_0 in (U,V) is smaller than the zero of B_0 in the same interval. We first observe that the zeroes of $A_0(x)$ and $B_0(x)$ are disjoint. In fact, the condition $A_0(x)=0$ asserts that

$$U \in \left\{x, \frac{B^2V-2B^2x+x^3+B^2x^3-Vx^4}{-B^2+Vx+B^2Vx-2Vx^3+x^4}\right\}.$$

Any of the two cases obligates to have $B_0(V)=0$, a contradiction. By direct computation of the zeroes of A_0 and B_0 for particular values of U, V and B together the previous fact and continuity arguments we obtain the inequality desired part.

□

Let us denote the real zeroes of $A_0(x)$ by a_1,..., a_6 and the real zeroes of B_0 by b_1,..., b_6, so that

$$-B < a_1 < b_1 < -1 < a_2 < b_2 < b_3 < a_3 < b_4 < a_4 < a_5 < b_5 < a_6 < b_6 \leqslant +\infty$$
$$b_2 = U,\ b_4 = V,\ a_4 = 1,\ a_6 = B$$

We call each component of zeroes of P_j an oval of P_j. Observe that both polynomials have a central oval; the one that separates the other two ovals.

(1.1) Case $-V<U$. As $A_3(x)=U+V>0$ and $B_3(x)=-3-3B+U+V<0$, we may consider the polynomials in $x\in\mathbb{R}$ and $w\in[0,+\infty)$ given by

$$Q_1(x,w) = P_1(x,+\sqrt{w})/A_3(x) = w^3 + \frac{A_2(x)}{A_3(x)}w^2 + \frac{A_1(x)}{A_3(x)}w + \frac{A_0(x)}{A_3(x)}$$
$$Q_2(x,w) = P_2(x,+\sqrt{w})/B_3(x) = w^3 + \frac{B_2(x)}{B_3(x)}w^2 + \frac{B_1(x)}{B_3(x)}w + \frac{B_0(x)}{B_3(x)}$$

We have that if $P_j(x,y)=0$ for some $(x,y)\in\mathbb{R}^2$, then we have $Q_j(x,w)=0$ for $(x,w=y^2)\in\mathbb{R}\times[0,+\infty)$. A suitable study of the above polynomials of degree 3 in the positive variable w permits us to obtain the following fact.

Lemma 2.2. The locus of zeroes of both $P_1(x,y)$ and $P_2(x,y)$ are as described in figure 1.

In this way, we have exactly 4 geometric centers. The common zero of both polynomials given by the intersection of the the two central ovals corresponds to the main geometric center.

Example 2.1. We have the following examples:

(i) If we set $U=-0.4$, $V=0.5$ and $B=10$, then the 4 points in upper-half plane of theorem 2.1 are given by: $0.705127+0.307764i$, $-0.631108+0.352922i$, $0.0783451+1.30366i$ and $0.155614+7.65529i$.

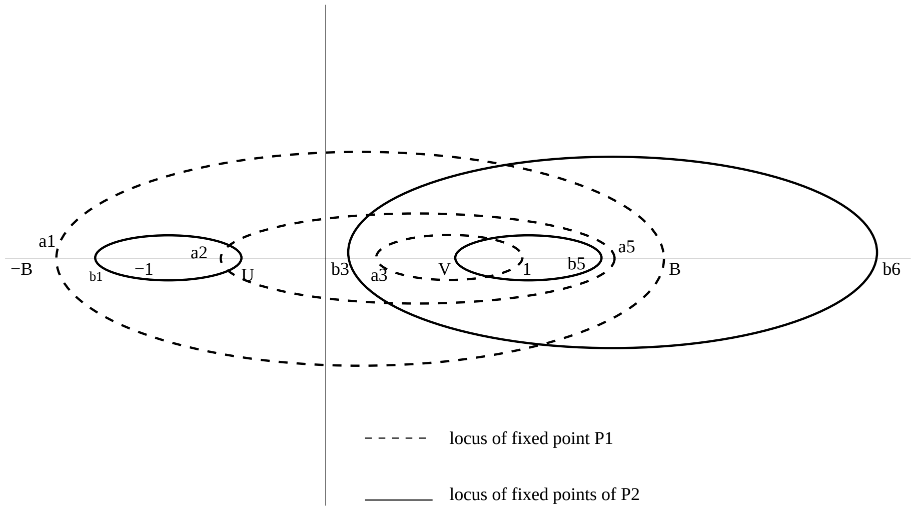

FIGURE 1. $-1 < -V < U < V < 1$: No tangencies and non-symmetric

(ii) If we set $U = -0.5$, $V = 0.5$ and $B = 10$, then the 2 points in upper-half plane of theorem 2.1 are given by: $7.65687i$ and $1.30602i$.

(1.2) Case $U = -V$. As $A_3(x) = 0$ and $B_3(x) = -3 - 3B < 0$, we may consider the polynomials in $x \in \mathbb{R}$ and $w \in [0, +\infty)$ given by

$$Q_1(x,w) = P_1(x, +\sqrt{w}) = A_2(x)w^2 + A_1(x)w + A_0(x)$$
$$Q_2(x,w) = P_2(x, +\sqrt{w})/B_3(x) = w^3 + \frac{B_2(x)}{B_3(x)}w^2 + \frac{B_1(x)}{B_3(x)}w + \frac{B_0(x)}{B_3(x)}$$

We have that if $P_j(x,y) = 0$ for some $(x,y) \in \mathbb{R}^2$, then we have $Q_j(x,w) = 0$ for $(x, w = y^2) \in \mathbb{R} \times [0, +\infty)$. A suitable study of the above polynomials of degree 3 in the positive variable w permits us to obtain the following fact.

Lemma 2.3. The locus of zeroes of both $P_1(x,y)$ and $P_2(x,y)$ are as described in figure 2.

In this way, we have exactly 4 geometric centers. The common zero of both polynomials given by the intersection of the the imaginary line (the central oval of P_2) with the central oval of P_1 corresponds to the main geometric center.

(2) Case $V = 1$ and $U > -1$. In this case we have exactly one tangency, $A_3(x) = 1 + U > 0$ and $B_3(x) = -2 - 3B + U < 0$.

Lemma 2.4. The locus of zeroes of both $P_1(x,y)$ and $P_2(x,y)$ are as described in figure 3.

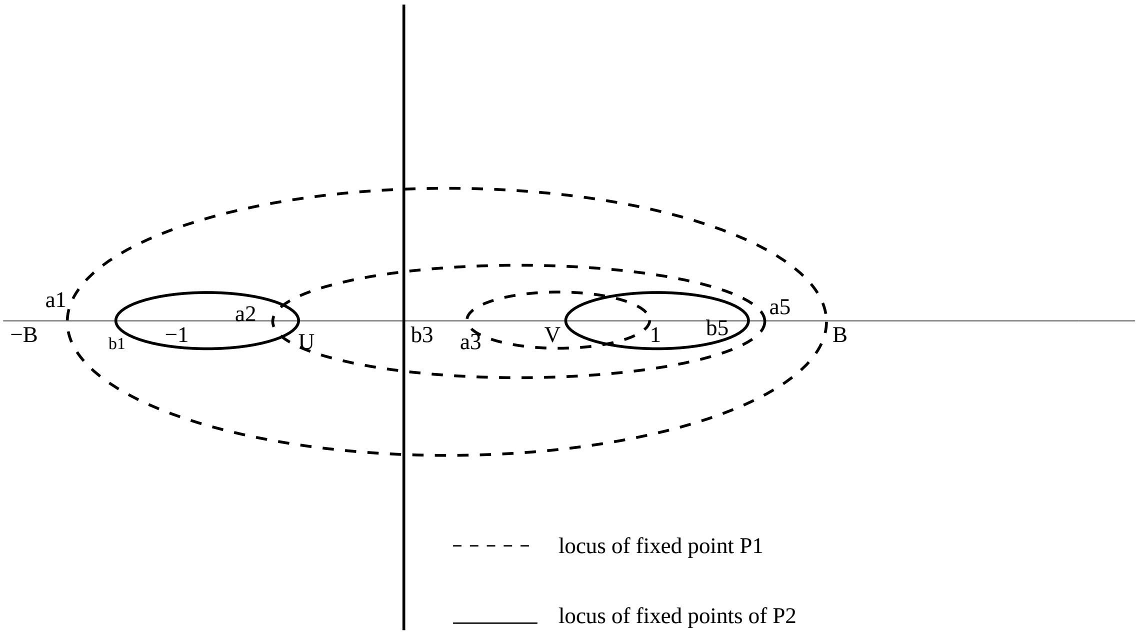

FIGURE 2. $0 < -U = V < 1$: Non tangencies but symmetric

In this way, we have exactly 3 geometric centers. The common zero of both polynomials given by the intersection of the the two central ovals corresponds to the main geometric center.

(3) Case $V = 1$ and $U = -1$. In this case we have exactly two tangencies, $A_3(x) = 0$ and $B_3(x) = -3(1 + B) < 0$.

Lemma 2.5. The locus of zeroes of both $P_1(x, y)$ and $P_2(x, y)$ are as described in figure 4.

In this way, we have exactly 2 geometric centers. The common zero of both polynomials given by the intersection of the the imaginary line (the central oval of P_2) with the central oval of P_1 corresponds to the main geometric center.

□

Remark 2.1. In the symmetrical case $U = -V$ we have that the main geometric center belongs to the same orthogonal circle (the imaginary line in our normalization) as one of the other three geometric centers. But this is not true when $U \neq -V$.

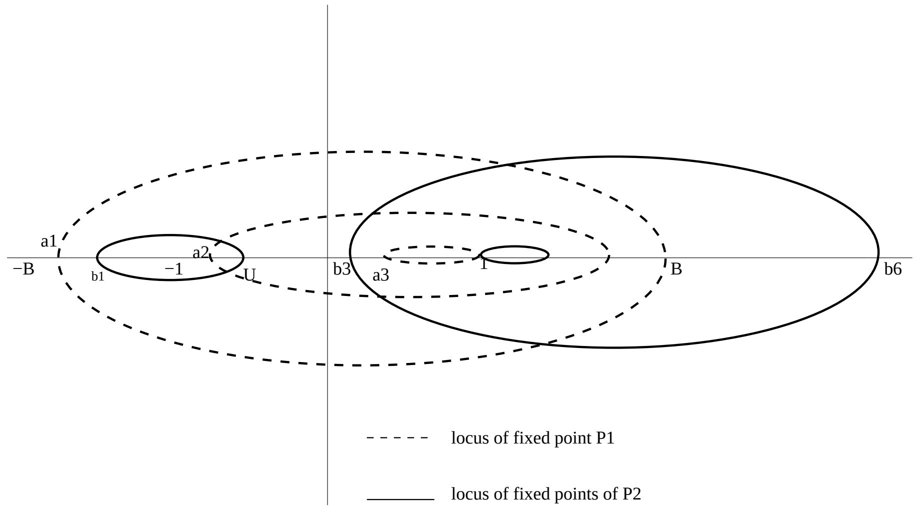

FIGURE 3. $V = 1$, $-1 < U < 1$: One tangency

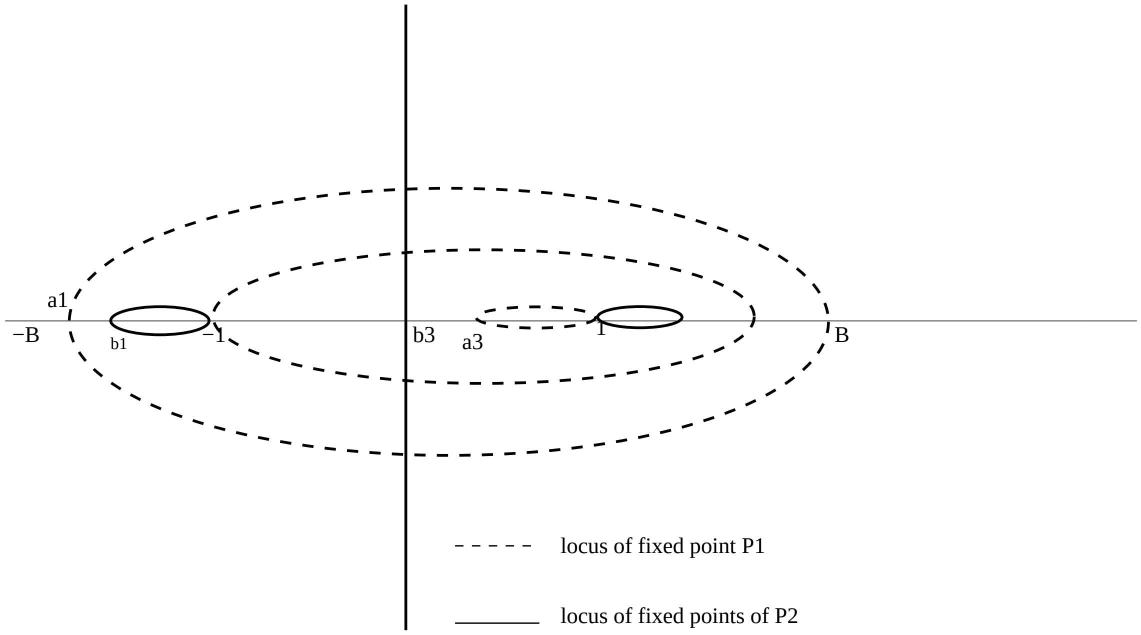

FIGURE 4. $-U = V = 1$: Two tangencies

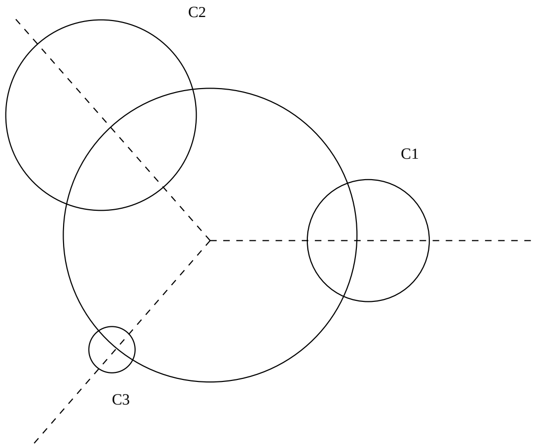

FIGURE 5

3. Classical Whittaker groups of rank 2

In this section we provide a family of classical Whittaker groups of rank 2 so that they respective classical Schottky subgroups provide uniformization of all Riemann surfaces of genus 2.

3.1. A geometrical construction. Let us consider the open set in $\mathbb{R}^3$ given by:

$$\mathcal{Q} = \{(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 : 0 < \theta_1, \theta_2, \theta_3,\ \theta_1 + \theta_2 < \frac{2\pi}{3},\ \theta_2 + \theta_3 < \frac{2\pi}{3},\ \theta_3 + \theta_1 < \frac{2\pi}{3}\}.$$

We have that $\mathcal{Q}$ is the interior of the convex hull of the points $(0,0,0)$, $(2\pi/3,0,0)$, $(0,2\pi/3,0)$, $(0,0,2\pi/3)$, $(\pi/3,\pi/3,\pi/3)$. We draw circles on the Riemann sphere (see figure 5), say C_1, C_2 and C_3, each one orthogonal to the unit circle C_0, and so that:

(i) C_1 intersects the unit circle at the points $e^{\pm i\theta_1}$;
(ii) C_2 intersects the unit circle at the points $e^{(2\pi/3 \pm \theta_2)i}$; and
(iii) C_3 intersects the unit circle at the points $e^{(4\pi/3 \pm \theta_3)i}$.

Remark 3.1. The triples $(\theta_1, \theta_2, \theta_3) \in \mathcal{Q}$ should in particular satisfy that $\theta_1 + \theta_2 + \theta_3 < \pi$. Also, one of the circles C_j may contain ∞, that is, to be the union of ∞ with an Euclidian line that containing 0. For instance, if $\theta_1 = \pi/2$, then C_1 is an Euclidian line. The restrictions given in $(*)$ asserts that the circles C_1, C_2 and C_3 form a non-separating triple as defined in section 2, in particular, they are mutually disjoint and that they bound a common domain $\mathcal{D}$ of connectivity 3. If we have that one of the C_j is an Euclidian line, then the other two are Euclidian circles. If for instance $\theta_1 \in (\pi/2, 2\pi/3)$, we have that all are Euclidian circles and C_2 and C_3 are separated from ∞ by C_1.

Now, for each $j = 1,2,3$, we consider two angles $\eta_{j,1}, \eta_{j,2} \in [0, 2\pi)$ so that $\eta_{j,1} \neq \eta_{j,2}$. These two angles, for each fixed j, determines (in a natural way) two different points on the circle C_j and, in particular, a unique elliptic transformation of order two, say E_j, that has them as fixed points. We have that E_j permuts the two discs bounded by C_j. We set

$$K(\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2})$$

the group generated by these transformations E_1, E_2 and E_3.

3.2. Parameter space. We denote by $\mathfrak{F}$ the collection of tuples

$$w = (\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2})$$

so that $(\theta_1, \theta_2, \theta_3) \in \mathcal{Q}$ and $\eta_{j,1}, \eta_{j,2} \in [0, 2\pi)$ so that $\eta_{j,1} \neq \eta_{j,2}$.

Similarly, for fixed values $\eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}$, we denote by $\mathfrak{F}_{(\eta_{1,1},\eta_{1,2},\eta_{2,1},\eta_{2,2},\eta_{3,1},\eta_{3,2})}$ the corresponding subfamily of $\mathfrak{F}$. It is not hard to see that each of these subfamilies is a copy of $\mathcal{Q}$ and that they are all disjoint.

The following fact is easy to see from the construction.

Theorem 3.1. If $w \in \mathfrak{F}$, then the group $K(w)$ is a classical Whittaker group of rank 2.

If we have $w \in \mathfrak{F}$, then we have the classical Whittaker group of rank 2 $K(w) = \langle E_1, E_2, E_3 \rangle$. The group $K(w)$ has unique index two torsion free subgroup $G(w)$. It is not hard to see that $G(w)$ is a classical Schottky group of rank 2 generated by the loxodromic transformations $A_1 = E_1E_2$ and $A_2 = E_1E_3$. Each one of the elliptics involutions E_j induces the hyperelliptic involution on the uniformized surface S by the Schottky group $G(w)$. The set of the fixed points of E_1, E_2 and E_3 project onto the 6 fixed points of the hyperelliptic involution.

Conjecture 3.1. Given any genus two Riemann surface S which can be uniformized by a classical Schottky group, there is some $w \in \mathfrak{F}$ so that, if Ω is the region of discontinuity of $K(w)$ (the same as for $G(w)$), then $S = \Omega/G(w)$ and $S/\langle j \rangle = \Omega/K(w)$, where $j : S \to S$ denotes the hyperelliptic involuition.

The following Conjecture is part of the general one, due to L. Bers, saying that every closed Riemann surface may be uniformized by a classical Schottky group.

Conjecture 3.2. Given any genus two Riemann surface S, there is some tuple $w \in \mathfrak{F}$ so that, if Ω is the region of discontinuity of $K(w)$ (the same as for $G(w)$), then $S = \Omega/G(w)$ and $S/\langle j \rangle = \Omega/K(w)$, where $j : S \to S$ denotes the hyperelliptic involuition.

Remark 3.2.

(i) The parameter space $\mathfrak{F}$ depends on 9 real parameters, but the moduli space of genus 2 Riemann surfaces depends only on 6.

(ii) The group of permutations on three letters $\mathcal{S}_3$ acts naturally on $\mathfrak{F}$ as follows. The cyclic permutation (1 2 3) acts by the rule

$$(\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}) \mapsto (\theta_2, \theta_3, \theta_1, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}, \eta_{1,1}, \eta_{1,2})$$

and the involution (1 2) acts by the rule

$$(\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}) \mapsto (\theta_2, \theta_1, \theta_3, \eta_{2,1}, \eta_{2,2}, \eta_{1,1}, \eta_{1,2}, \eta_{3,1}, \eta_{3,2})$$

(iii) The collection of subfamilies $\mathfrak{F}_{(\eta_{1,1},\eta_{1,2},\eta_{2,1},\eta_{2,2},\eta_{3,1},\eta_{3,2})}$ is kept invariant under the previous action of $\mathcal{S}_3$. Of course, some of them have non-trivial stabilizers.

(iv) It is not clear the relations must have two tuples $w, w' \in \mathfrak{F}$ in order for $G(w)$ and $G(w')$ to produce isomorphic Riemann surfaces.

4. Real curves

In order for a tuple

$$w = (\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}) \in \mathfrak{F}$$

to determine a real Whitaker group, we need to have that, for each $j = 1, 2, 3$, both fixed points of E_j be either on the unit circle or on the same ray from 0. In this way, we have the following cases:

(1) $(\eta_{1,1}, \eta_{1,2}) \in \{(0, \pi), (\pi/2 + \theta_1, 3\pi/2 - \theta_1)\}$;
(2) $(\eta_{2,1}, \eta_{2,2}) \in \{(2\pi/3, 5\pi/3), (7\pi/6 + \theta_2, 13\pi/6 - \theta_2)\}$;
(3) $(\eta_{3,1}, \eta_{3,2}) \in \{(\pi/3, 4\pi/3), (5\pi/6 - \theta_3, \theta_3 - \pi/6)\}$.

In this way, we obtain 8 different subfamilies in $\mathfrak{F}$ as above. Up to the natural action of $\mathcal{S}_3$, as described in Remark 3.2, we obtain only 4 different classes of such families; they describe exactly the 4 different possible actions of a reflection in genus 2. Two of these (classes of) subfamilies are invariant under the action of $\mathcal{S}_3$ (one corresponds to M-real actions and the other to the case when the reflection as one diving oval) and the other two are only invariant under the action of a $\mathbb{Z}_2$ (they correspond to the other two actions). In Section 5.2 we describe the action of $\mathcal{S}_3$ on the subfamily corresponding to M-real curves.

4.1. M-real curve's case. We next proceed to describe some basics facts for one of the subfamilies, M-real curves, but it may be written (with the suitable modifications) for all other cases. The subfamily providing the M-real situation is given by

$$\mathfrak{F}_{(\pi/2+\theta_1, 3\pi/2-\theta_1, 7\pi/6+\theta_2, 13\pi/6-\theta_2, 5\pi/6-\theta_3, \theta_3-\pi/6)}$$

which we denote by short as $\mathfrak{F}_{(3,0)}$. The next, which provides some of the properties of these M-type groups, is not hard to check from the construction.

Theorem 4.1. Let $w \in \mathfrak{F}_{(3,0)}$ and denote by τ_j the reflection on the circle C_j, for $j = 0, 1, 2, 3$. Then (see figure 6),

(1) $K(w) = \langle E_1 = \tau_0\tau_1, E_2 = \tau_0\tau_2, E_3 = \tau_0\tau_3 \rangle$;

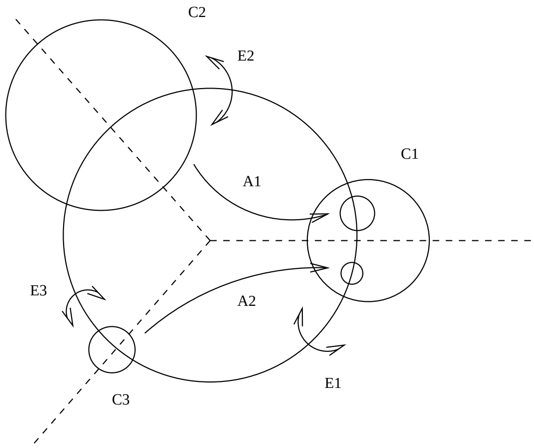

FIGURE 6

(2) $G(w) = \langle A_1 = \tau_1\tau_2, A_2 = \tau_1\tau_3 \rangle$;
(3) E_k is an elliptic transformation of order 2 keeping invariant the unit circle which permutes both discs bounded by the unit circle, for $k = 1, 2, 3$; and
(4) the reflection τ_0 induces on $S = \Omega(G(w))/G(w)$ a reflection τ with exactly 3 ovals, that is, we have that (S, τ) is a M-real curve of genus 2.

Theorem 4.2. If (S, τ) is a M-real curve of genus two, then there exists $w \in \mathfrak{F}_{(3,0)}$ so that S is uniformized by the Schottky group $G(w)$ and τ is induced by τ_0.

PROOF. As a consequence of the above construction, quasiconformal deformation theory and the fact that the set of fixed points of reflections on the Riemann sphere are circles, we have that every M-real curve can be uniformized by the index two orientation preserving half (a Schottky group) of an extended Kleinian group generated by the reflections on a triple of non-separating disjoint circles bounding a common domain. As a consequence of the results of section 2, any such a collection of three circles has a common orthogonal circle which, up to a Möbius transformation, we may assume to be the unit circle C_0. Let us consider the main geometric center (inside the unit disc) of such a triple which, up to a conformal automorphism of the unit disc, can be assume to be 0. Now, the (transformed) circles satisfy the needed conditions (i), (ii) and (iii) for a suitable triple in $\mathcal{Q}$.

□

5. Algebraic-Schottky description

5.1. Burnside's arguments. Let us consider a tuple

$$w = (\theta_1, \theta_2, \theta_3, \eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2}) \in \mathfrak{F}$$

the Whittaker group $K(w)$ and its Schottky subgroup $G(w)$. We denote by Ω the region of discontinuity of these two groups and denote by $S=\Omega/G(w)$ the uniformized Riemann surface by $G(w)$. We have the following convergence fact due to Burnside.

Theorem 5.1 (Burnside [**4**]). The serie

$$\sum_{\gamma\in G(w)} \gamma'(z)$$

converges locally uniformly on the region of discontinuity to a meromorphic function. The poles of such a function are exactly at the points in $\mathbb{C}$ which are $G(w)$-equivalent to ∞ and they are all of order 2.

As a consequence of the above, we have the holomorphic forms

$$\begin{cases} w_1(z) &= \dfrac{1}{2\pi i}\sum_{\gamma\in G(w)}\left(\dfrac{\gamma'(z)}{\gamma(z)-A_1^{-1}(\infty)}\right)dz,\\ \\ w_2(z) &= \dfrac{1}{2\pi i}\sum_{\gamma\in G(w)}\left(\dfrac{\gamma'(z)}{\gamma(z)-A_2^{-1}(\infty)}\right)dz. \end{cases}$$

and, in particular, we have the following holomorphic map $\Phi:\Omega\to\mathbb{C}$ defined by

$$\Phi(z)=\frac{\sum_{\gamma\in G(w)}\left(\dfrac{\gamma'(z)}{\gamma(z)-A_1^{-1}(\infty)}\right)}{\sum_{\gamma\in G(w)}\left(\dfrac{\gamma'(z)}{\gamma(z)-A_2^{-1}(\infty)}\right)}.$$

If we give to the circle C_j (for $j=2,3$) the counterclockwise orientation, then we have that

$$\int_{C_k} w_r = \begin{cases} 1, & \text{for } k=r+1;\\ 0, & \text{otherwise.}\end{cases}$$

If we denote by $\alpha_k\subset S$ the projection of the circle C_{k+1} (for $k=1,2$), and we denote by β_1^* (respectively, β_2^*) the projection of a simple arc inside the region bounded by the circles C_1, C_2 and C_3 that connects one of the fixed of E_2 (respectively, one of the fixed points of E_3) to one fixed point of E_1, both of them disjoint (in particular, each one ends at a different fixed point of E_1).

Remark 5.1. In the case $w\in\mathfrak{F}_{(3,0)}$, we may choose β_1^* (respectively, β_2^*) the projection of the arc of the unit circle with arguments between θ_1 and $2\pi/3-\theta_2$ (respectively, the projection of the arc of the unit circle with arguments between $4\pi/3+\theta_3$ and $2\pi-\theta_1$).

We have a canonical homology basis for S given by

$$\{\alpha_1,\alpha_2,\beta_1=\beta_1^*\cup\tau(\beta_1^*),\beta_2=\beta_2^*\cup\tau(\beta_2^*)\}.$$

The projection of w_1 and w_2 on S are respectively the holomorphic differentials ω_1 and ω_2 so that

$$\int_{\alpha_k} \omega_r = \begin{cases} 1, & \text{for } k=r; \\ 0, & \text{otherwise.} \end{cases}$$

In this way, the holomorphic map

$$\Phi : \Omega \to \widehat{\mathbb{C}} : z \mapsto \Phi(z) = x$$

is a lifting to Ω of the two-fold branched covering $\phi : S \to \widehat{\mathbb{C}}$ induced by the hyperelliptic involution. If we compose at the left of Φ by the Möbius transformation

$$U(x) = \frac{(x - \Phi(e^{-i\theta_1}))(\Phi(e^{i\theta_1}) - \Phi(e^{i(4\pi/3+\theta_3)}))}{(x - \Phi(e^{i(4\pi/3+\theta_3)}))(\Phi(e^{i\theta_1}) - \Phi(e^{-\theta_1}))},$$

we obtain the map $T : \Omega \to \widehat{\mathbb{C}}$ defined by

$$T(z) = \frac{(\Phi(z) - \Phi(e^{-i\theta_1}))(\Phi(e^{i\theta_1}) - \Phi(e^{i(4\pi/3+\theta_3)}))}{(\Phi(z) - \Phi(e^{i(4\pi/3+\theta_3)}))(\Phi(e^{i\theta_1}) - \Phi(e^{-\theta_1}))}.$$

The fixed points of E_j, for $j = 1, 2, 3$, which correspond to the six fixed points of the hyperelliptic involution, are send under T to the points $T(e^{-i\theta_1}) = 0$, $T(e^{i\theta_1}) = 1$, $T(e^{(2\pi/3-\theta_2)i}) = a$, $T(e^{(2\pi/3+\theta_2)i}) = b$, $T(e^{(4\pi/3-\theta_3)i}) = c$ and $T(e^{(4\pi/3+\theta_3)i}) = \infty$.

Remark 5.2. In the M-real type situation, that is $w \in \mathfrak{F}_{(3,0)}$, we observe that the reflection τ keeps invariant (and does not change the orientations) of the loops β_1 and β_2, and keeps invariant (but changes the orientation) of the loops α_1 and α_2. It follows that at the level of holomorphic forms the action of τ is given by

$$\begin{cases} \tau(\omega_1) &= -\overline{\omega_1}, \\ \tau(\omega_2) &= -\overline{\omega_2}. \end{cases}$$

The above asserts that under the projection $T : \Omega \to \widehat{\mathbb{C}}$, the reflection τ_0 induces the reflection on the real line, in particular, we have $a, b, c \in \mathbb{R}$ and, moreover, $1 < a < b < c$.

The above permits to obtain the following.

Theorem 5.2. The Riemann surface uniformized by the classical Schottky group $G(w)$, where $w \in \mathfrak{F}$, is represented by the algebraic curve

$$y^2 = x(x-1)(x-a)(x-b)(x-c),$$

where $a, b, c \in \mathbb{C} - \{0, 1\}$ are so that

$$a = T(e^{(2\pi/3-\theta_2)i}), b = T(e^{(2\pi/3+\theta_2)i}), c = T(e^{(4\pi/3-\theta_3)i}).$$

In the M-real case, that is $w \in \mathfrak{F}_{(3,0)}$, we have

$$a, b, c \in \mathbb{R}$$

$$1 < a = T(e^{(2\pi/3-\theta_2)i}) < b = T(e^{(2\pi/3+\theta_2)i}) < c = T(e^{(4\pi/3-\theta_3)i}) < +\infty$$

5.2. Some remarks: M-real case. Let us set

$$\mathcal{P} = \{(a, b, c) \in \mathbb{R}^3 : 1 < a < b < c\}.$$

The general theory (see for instance [**6, 23**]) asserts that for every M-real curve of genus two there is at least one triple $(a, b, c) \in \mathcal{P}$ so that is is given algebraically by

$$y^2 = x(x-1)(x-a)(x-b)(x-c).$$

It is also known that two triples $(a, b, c), (\widehat{a}, \widehat{b}, \widehat{c}) \in \mathcal{P}$ define real isomorphic M-real curves if and only if there is a Möbius transformation $A \in PGL(2, \mathbb{R})$ so that

$$A(\{0, 1, a, b, c, \infty\}) = \{0, 1, \widehat{a}, \widehat{b}, \widehat{c}, \infty\},$$

in other words, if they define the same orbit under the group $D_6 = \langle \alpha, \beta \rangle$, where

$$\alpha(a, b, c) = \left(\frac{c}{b}, \frac{c-1}{b-1}, \frac{c-a}{b-a}\right)$$

$$\beta(a, b, c) = \left(a, \frac{a(c-1)}{c-a}, \frac{a(b-1)}{b-a}\right)$$

The fixed points by α are also fixed points of β. These corresponds to the real algebraic curves of type $(2, 3, 0)$ having the dihedral group D_6 as group of real automorphisms. The map α and β are induced, respectively, by the following Möbius transformations

$$\alpha'(z) = \frac{a(z-1)}{z-a}, \quad \text{and} \quad \beta'(z) = \frac{z-c}{z-b}.$$

On the other hand, theorem 4.2 gives us a real analytic map

$$\Psi : \mathcal{Q} \to \mathcal{P} : (\theta_1, \theta_2, \theta_3) \mapsto (a, b, c),$$

so that for each triple $p = (a, b, c) \in \mathcal{P}$ there is a triple $\theta = (\theta_1, \theta_2, \theta_3)$ so that $\Psi(\theta)$ is equivalent to p under the dihedral group D_6. In this way, the M-real curve determined by the triple (a, b, c) is uniformized by $G(\theta_1, \theta_2, \theta_2)$. Theorem 5.2 gives us transcendental relations between both triples (a, b, c) and $(\theta_1, \theta_2, \theta_3)$. This gives explicit transcendental relations between the triples in $\mathcal{Q}$ defining real isomorphic M-real curves.

6. Numerical implementation

In this section we provide the numerical implementation of the previous process.

6.1. Obtaining an algebraic curve. Let us consider a tuple $w \in \mathfrak{F}$. We denote by $p_{j,i}$ the point in C_j determined by the angle $\eta_{j,i}$.

We denote by $G_m(w)$ the subset of $G(w)$ formed by all reduced words (in A_1 and A_2) of length at most m. We now consider Φ_m in similar fashion as Φ, but the summands are done over $G_m(w)$ instead of $G(w)$. Then we consider $T_m : \Omega \to \widehat{\mathbb{C}}$ defined as

$$T_m(z) = \frac{(\Phi_m(z) - \Phi_m(p_{1,2}))(\Phi_m(p_{1,1}) - \Phi_m(p_{3,2}))}{(\Phi_m(z) - \Phi_m(p_{3,2}))(\Phi_m(p_{1,1}) - \Phi_m(p_{1,2}))}.$$

Remark 6.1. In the case $w \in \mathfrak{F}_{(3,0)}$ we use $p_{1,1} = e^{i\theta_1}$ $p_{1,2} = e^{-i\theta_1}$ and $p_{3,2} = e^{i(4\pi/3+\theta_3)}$.

We consider the values:

$$\begin{cases} a_m &= T_m(e^{(2\pi/3-\theta_2)i}), \\ b_m &= T_m(e^{(2\pi/3+\theta_2)i}), \\ c_m &= T_m(e^{(4\pi/3-\theta_3)i}), \end{cases}$$

and we may then obtain as an approximation curve the following one:

$$y^2 = P(x) = x(x-1)(x-a_m)(x-b_m)(x-c_m).$$

Remark 6.2. The numerical program permits also to draw the image of the three circles C_1, C_2 and C_3 under T_m. We have observed that their images still arcs of circle, say N_1, N_2 and N_3 respectively. Moreover, if we denote by M_j the circle determined by N_j we obtain that $M_j \cap N_k = \emptyset$ for $j \neq k$. We conjecture that this holds always. Reciprocally, given any three different points $a, b, c \in \mathbb{C} - \{0, 1\}$ we are able to draw three circles M_1, M_2 and M_3, each one containing only two points of the set $\{0, 1, \infty, a, b, c\}$, so that on each of the M_j we may choose an arc N_j, determined by the two points, with the property that $M_j \cap N_k = \emptyset$ for $j \neq k$. This observation, together with quasiconformal deformation theory, permits to see that for every Riemann surface S of genus 2 there is a parameter $w \in \mathfrak{F}$ so that $S = \Omega(G(w))/G(w)$.

6.2. Computing a Riemann period matrix: M-real case. Once we have the approximated curve as in the previous section, we may easily compute numerically a Riemann period matrix for it (see figure 7). There are already implemented packages in Maple and Mathematica which permits this. We describe a process for the M-real case, which may be adapted for the general case, we have used in our program. Assume we have obtained the M-real algebraic curve

$$y^2 = x(x-1)(x-a)(x-b)(x-c),$$

where $1 < a < b < c$. Let us consider first the base of holomorphic forms

$$\Theta_1 = \frac{dx}{\sqrt{P(x)}}, \quad \Theta_2 = \frac{xdx}{\sqrt{P(x)}}.$$

We have that

$$\omega_j = s_{1j}\Theta_1 + s_{2j}\Theta_2, \quad j = 1, 2.$$

As we should have that

$$\int_{\alpha_j} \omega_k = \begin{cases} 1, & k = j \\ 0, & k \neq j, \end{cases}$$

we have

$$\begin{cases} s_{11}L_{11} + s_{21}L_{21} = \dfrac{-1}{2} \\ s_{11}L_{12} + s_{21}L_{22} = 0 \end{cases}$$

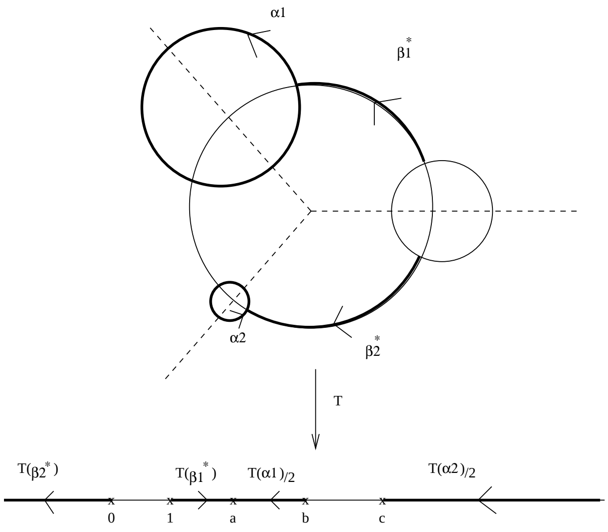

FIGURE 7

$$\begin{cases} s_{12}L_{11} + s_{22}L_{21} = 0 \\ s_{12}L_{12} + s_{22}L_{22} = \dfrac{-1}{2} \end{cases}$$

where

$$\begin{cases} L_{11} &= \int_a^b \Theta_1,\ L_{12} = \int_c^{+\infty} \Theta_1, \\ \\ L_{21} &= \int_a^b \Theta_2,\ L_{22} = \int_c^{+\infty} \Theta_2. \end{cases}$$

Once we have the values of s_{ij}, we have obtained the differentials ω_1 and ω_2. In this way we may obtain a Riemann period matrix

$$Z = \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix},$$

where

$$\begin{cases} t_{11} &= 2\left(s_{11}\int_1^a \Theta_1 + s_{21}\int_1^a \Theta_2\right), \\ \\ t_{12} &= 2\left(s_{12}\int_1^a \Theta_1 + s_{22}\int_1^a \Theta_2\right), \\ \\ t_{22} &= 2\left(s_{12}\int_{-\infty}^0 \Theta_1 + s_{22}\int_{\infty}^0 \Theta_2\right). \end{cases}$$

6.3. Computing the accesory parameters. To the holomorphic branched covering $T : \Omega \to \widehat{\mathbb{C}}$ we have associated the second order Fuchsian differential equation

$$w''(x) + \frac{1}{2}\{T^{-1}, x\}w(x) = 0,$$

where

$$\{T^{-1},x\} = \frac{(T^{-1})'''(x)}{(T^{-1})'(x)} - \frac{3}{2}\left(\frac{(T^{-1})''(x)}{(T^{-1})'(x)}\right)^2$$

is the Schwarzian derivative of T^{-1} respect to $x = T(z)$. Usual computations asserts that

$$\{T^{-1},x\} =$$
$$\frac{3}{8}\left(\frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x-a)^2} + \frac{1}{(x-b)^2} + \frac{1}{(x-c)^2} - \right.$$
$$\left. - \frac{4x^3 + c_2x^2 + c_1x + c_0}{x(x-1)(x-a)(x-b)(x-c)}\right),$$

where a, b and c are as in theorem 5.2. The values c_0, c_1 and c_2 are classically known as the *accesory parameters* associated to the branch covering $T : \Omega \to \widehat{\mathbb{C}}$. See for instance [**18**] for some details on these accesory parameters. As for any Möbius transformation A we have that $\{A, x\} = 0$, we get the equality

$$\{T^{-1},x\} = -\frac{\{T,z\}}{(T'(z))^2},$$

where $T(z) = x$. Since we have a numerical approximation T_m for T (as described above), we may compute

$$-\frac{\{T_m,z\}}{(T_m'(z))^2},$$

obtaining an approximation of $\{T^{-1},x\}$ and, in particular, approximations of the accesory parameters.

6.4. Some symmetrical cases of M-real curves. We consider a tuple $w \in \mathfrak{F}_{(3,0)}$.

6.4.1. *First symmetrical situation.* If we choose the values $\theta_2 = \theta_3$ in our construction of section 3, then we have that the reflection on the real line, say τ_4, normalizes $G(w)$. This extra reflection induces a reflection on the uniformized surface S. Under the projection $T : \Omega \to \widehat{\mathbb{C}}$ this reflection induces a reflection σ satisfying the following:

$$\sigma(0) = 1, \quad \sigma(a) = \infty, \quad \sigma(b) = c.$$

It follows that

$$\sigma(z) = \frac{a(\overline{z}-1)}{\overline{z}-a},$$

and

$$1 < a < b < c + \sqrt{a(a-1)},$$
$$c = \frac{a(b-1)}{b-a}.$$

In particular, in this case we have that the algebraic curve is given by:

$$y^2 = x(x-1)(x-a)(x-b)\left(x - \frac{a(b-1)}{b-a}\right).$$

In our homology basis we have that the action of σ is the following:

$$\begin{cases} \alpha_1 & \mapsto & -\alpha_2 \\ \beta_1 & \mapsto & \beta_2 \end{cases}$$

In particular, the extended symplectic representation of η is given by the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If

$$Z = \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}$$

denotes the Riemann period matrix of S in such a canonical homology basis, then we have that σ should fix it, that is,

$$Z = -A^{-1}\overline{Z}A,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{H}_2$$

It follows that

$$t_{22} = -\overline{t_{11}}, \quad t_{12} \in i\mathbb{R}.$$

6.4.2. *Second symmetrical situation.* If we now choose the values $\theta_1 = \theta_2 = \theta_3$ in our construction of section 3, then we have that the reflection on the real line, say τ_4, and the reflection on the line through 0 and $e^{2\pi/3}$, say τ_5 both normalize $G(w)$. This two extra reflections induce on S the reflections σ (as above) and a new one, say η, so that $\sigma\eta$ has order 3. As above, under $T : \Omega \to \widehat{\mathbb{C}}$ we have that

$$\sigma(z) = \frac{a(\overline{z}-1)}{\overline{z}-a},$$

and

$$1 < a < b < a + \sqrt{a(a-1)},$$
$$c = \frac{a(b-1)}{b-a}.$$

We also have that

$$\eta(0) = \infty, \quad \eta(a) = b, \quad \eta(1) = c.$$

It follows that:

$$\eta(z) = \frac{ab}{\overline{z}}$$
$$x_3 = ab.$$

All the above gives us:

$$b = \frac{1 + a + \sqrt{(1+a)^2 - 4}}{2},$$
$$c = a\frac{1 + a + \sqrt{(1+a)^2 - 4}}{2}.$$

In particular, the algebraic curve is given by:

$$y^2 = x(x-1)(x-a)\left(x - \frac{1+a+\sqrt{(1+a)^2-4}}{2}\right)\left(x - a\frac{1+a+\sqrt{(1+a)^2-4}}{2}\right).$$

In our homology basis we have that the action of η is the following:

$$\begin{cases} \alpha_1 & \mapsto & -\alpha_1 \\ \alpha_2 & \mapsto & \alpha_1+\alpha_2 \\ \beta_1 & \mapsto & \beta_1-\beta_2 \\ \beta_2 & \mapsto & -\beta_2 \end{cases}$$

In particular, the extended symplectic representation of σ is given by the matrix

$$\left[\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

By the previous case we have that the Riemann period matrix of S in such a canonical homology basis has the form

$$Z = \left[\begin{array}{cc} u & iv \\ iv & -\overline{u} \end{array}\right], \quad v \in \mathbb{R}.$$

We have that η should fix it, that is,

$$Z = -B\overline{Z}\,{}^tB,$$

where

$$B = \left[\begin{array}{rr} -1 & 0 \\ 1 & 1 \end{array}\right] \in \mathcal{H}_2$$

It follows that

$$Z = ix\left[\begin{array}{rr} 2 & -1 \\ -1 & 2 \end{array}\right], \quad x \in (0,+\infty).$$

Remark 6.3. The class of M-real curves described in the second case corresponds to those with maximal group of automorphisms commuting with a M-reflection. Such a group is generated by the hyperelliptic involution and the reflections τ, ω and η, and it has order 24.

6.5. An Example. In MATHEMATICA [**25**] we call the package by the command

In[1]:=<<Schottky‘RealSchottky‘

To execute the package for a concrete example of angles

$$\theta_1 = \theta_2 = \theta_3 = 0.275384001 = \frac{5^2 \cdot 1163543}{2^4 \cdot 3 \cdot 43^2 \cdot 3739}\pi,$$

and $n = 6$, we call the command

In[2]:=RealSchottky[0.275384001, 0.275384001, 0.275384001, 6]

The program will print out generators of the Schottky group:

$$A_1[z] = \frac{(-1.55873 - 0.899934i) + (0.039921 - 1.8012i)z}{(-1.57984 - 0.866025i) - (0. + 1.79987i)z}$$

$$A_2[z] = \frac{(-1.55873 + 0.899934i) + (0.039921 + 1.8012i)z}{(-1.57984 + 0.866025i) + (0. + 1.79987i)z}$$

two bi-rationally equivalent real curves

$$y^2 = 8.999961x - 22.4999x^2 + 19.9999x^3 - 7.49999x^4 + x^5 = \\ x(x-1)(x-1.49777)(x-1.99999)(x-2.99998)$$

$$w^2 = (-0.500001 - 0.86602i + u)(-0.499999 + 0.866026i + u) \\ (1. - 1.4088 \times 10^{-6}i + u)(u^3 - 1)$$

and a Riemann period matrix

$$Z = \begin{bmatrix} 1.1546995388533579i & -0.5773497684485667i \\ -0.5773497684485667i & 1.1546995388533579i \end{bmatrix} \cong \begin{bmatrix} 2i/\sqrt{3} & -i/\sqrt{3} \\ -i/\sqrt{3} & 2i/\sqrt{3} \end{bmatrix}.$$

One of the famous M-real curves of genus two is Bolza's curve

$$y^2 = x(x-1)(x-1.5)(x-2)(x-3),$$

which corresponds to the unique (conformal class) of Riemann surfaces of genus two with maximal automorphisms group. We may then see that the Schottky group

$$G\left(\frac{5^2 \cdot 1163543}{2^4 \cdot 3 \cdot 43^2 \cdot 3739}\pi, \frac{5^2 \cdot 1163543}{2^4 \cdot 3 \cdot 43^2 \cdot 3739}\pi, \frac{5^2 \cdot 1163543}{2^4 \cdot 3 \cdot 43^2 \cdot 3739}\pi\right)$$

seems to be a good approximation Schottky uniformizing group of Bolza's curve.

6.6. M-real curves of higher genera. A for real Schottky groups of genus $g \geqslant 3$ the number of reduced words of length at most m is $2g \cdot (2g-1)^{m-1}$, we see that the above procedure is computationally very expensive for $m \geqslant 2$. We may consider other approaches as follows in order to get higher genus computation. For instance, every M-real curve S_2 of genus two

$$y^2 = x(x-1)(x-a)(x-b)(x-c),\ 1 < a < b < c,$$

has as a double unbranched cover a M-real curve of genus three S_3 of the form

$$w^2 = (u^2-1)(u^2-\alpha^2)(u^2-\beta^2)(u^2-\gamma^2),\ 1 < \alpha < \beta < \gamma,$$

where the two-fold cover is given by

$$Q(u) = u^2.$$

This double cover is given by cutting S_2 along the oval corresponding to the projection of the circle C_3. If $G(w) = \langle A_1, A_2 \rangle$, where $w \in \mathfrak{F}_{(3,0)}$, is the real Schottky group that uniformizes S_2, then the real Schottky group $G_3(w) = \langle A_1, A_2^2, A_2 A_1 A_2^{-1} \rangle$ uniformizes S_3.

7. Connection to partial differential equations

One of the interests of M-curves is related to the theory of finite-gap integration developed by Novikov, Dubrovin, Matveev, It·s and others. Let us consider a M-curve (S,τ) and $p_0 \in S$ a real point (that is, a fixed point of τ). If we choose a canonical homology basis of S, say

$$\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$$

and a basis of holomorphic one-forms

$$\{\omega_1, ..., \omega_g\}$$

so that

$$\int_{\alpha_j} \omega_k = \delta_{jk},$$

then the associated Riemann period matrix is given by $Z = (z_{jk})$, where

$$z_{jk} = \int_{\beta_j} \omega_k.$$

If Z is a symmetric matrix with positive imaginary part, such a Riemann period matrix, then its theta function is defined by

$$\theta(z;Z) = \sum_{m \in \mathbb{Z}^g} exp\{\pi(i\langle Zm, m\rangle + 2\langle z, m\rangle)\}.$$

Near the point p_0 we choose a local coordinate z so that $z(p_0) = 0$. In this coordinate $\omega_j = f_j(z)dz$, for each $j = 1, ..., g$. Set

$$u_j = 2\pi i f_j(0),\ v_j = 2\pi i f_j'(0),\ w_j = \pi i f_j''(0),$$

and the vectors

$$U = (u_1, ..., u_g),\ V = (v_1, ..., v_g),\ W = (w_1, ..., w_g) \in \mathbb{C}^g.$$

Let η be the unique meromorphic one-form which is holomorphic in $S - \{p_0\}$ so that

$$\int_{\alpha_j} \eta = 0, \quad \text{for every } j = 1, .., g,$$

and so that has pole of order 2 at p_0. In the above local coordinate η looks like

$$\eta = \left(\frac{-1}{z^2} - a_0 + a_1 z + \cdots\right) dz$$

For each vector $D \in \mathbb{C}^g$, I.M. Krichever proved in [**19**] that

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln \theta\left(\frac{Ux + Vy + Wt + D}{2\pi i}; Z\right) + 2a_0,$$

gives all the real nonsingular finite-gap solutions of the Kadomtsev-Petviashvili equation [**7**]

$$(KP) \quad \frac{3}{4}u_{yy} = \frac{\partial}{\partial x}\left(u_t - \frac{1}{4}(6uu_x + u_{xxx})\right).$$

If, moreover, S is hyperelliptic and p_0 is a fixed point of the hyperelliptic involution, then $V = 0$ [**13**] and u turns out to be a real nonsingular finite-gap solution of the Korteweg-de Vries equation

$$(KdV) \quad 4u_t = 6uu_x + u_{xxx}.$$

If we consider a real Schottky group $G(w)$, where $w \in \mathfrak{F}_{(3,0)}$, uniformizing a M-real curve (S, τ), then all the above data, by the exception of a_0, needed to obtain the above finite-gap solution $u(x, y, t)$ can be obtained from the previous numerical algorithm. It is a well known fact that

$$\eta = \left(\frac{-1}{z^2} - \frac{\sum_{\gamma \in G - \{I\}} \gamma'(1/z)}{z^2} \right) dz,$$

from which we obtain

$$a_0 = \sum_{\gamma \in G(w) - \{I\}} c_\gamma^{-2},$$

where

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL(2, \mathbb{C})$$

As a consequence of the results of this note, we have then a numerical algorithm which permits to obtain finite-gap solutions of genus two of the (KP) equation (then of the (KdV) equation) (see also [**3**]).

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