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On the Surface Group Conjecture

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Abstract. We consider the following conjecture. Suppose that G is a non-free non-cyclic one- relator group such that each subgroup of finite index is again a one-relator group and each subgroup of infinite index is a free group. Must G be a surface group? We show that if G is a freely indecomposable fully residually free group and satisfies the property that every subgroup of infinite index is free then G is either a cyclically pinched one-relator group or a conjugacy pinched one-relator group. Further such a group G is either hyperbolic or free abelian of rank 2.

1. Introduction

Let G be the fundamental group of a compact surface of genus g. Then G has a one-relator presentation

$$< a_1, b_1, ..., a_g, b_g; [a_1, b_1]....[a_g, b_g] >$$

in the orientable case and

$$< a_1, ...a_g; a_1^2...a_g^2 >$$

in the non-orientable case. From covering space theory it follows that any subgroup of finite index is again a surface group of higher genus while any subgroup of infinite index must be a free group. These results, although known since the early 1900's were proved purely algebraically using Reidemeister-Schreier rewriting by Hoare,Karrass and Solitar in 1971 [HKS 1,2]. It is well known (see [FR]) that an orientable surface group can be faithfully represented as a discrete subgroup of $PSL_2(\mathbb{C})$ and hence each such group is linear. It follows that surface groups are residually finite. G.Baumslag [GB] showed that any orientable surface group of genus ≥ 2 must be residually free and 2-free from which it can be deduced using results of Remeslennikov [Re] and Gaglione and Spellman [GS] that they are fully residually free (see section 2). The article [AFR] surveys most of the properties of surface groups and shows how they are the primary

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motivating examples for much of combinatorial group theory.

In this paper we consider the **surface group conjecture**. In the Kourovka notebook Melnikov proposed the following problem.

Conjecture 1.1. ***Surface Group Conjecture A*** *Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is a surface group.*

Since subgroups of infinite index in surface groups must be free groups this conjecture was modified to:

Conjecture 1.2. ***Surface Group Conjecture B*** *Suppose that G is a non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group and every subgroup of infinite index is a free group. Then G is a surface group.*

Using the structure theorem for fully residually free groups in terms of its JSJ decomposition (see section 2) we can make some progress on these conjectures. We say that a group G satisfies **Property IF** if every subgroup of infinite index is free. Recall that the one-relator presentation for a surface group allows for a decomposition as a cyclically pinched one-relator group in both the orientable and non-orientable cases and as a conjugacy pinched one relator group in the orientable case (see section 3 and [FRS]). In particular we prove the following:

Theorem 1.1. *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one relator group.*

Corollary 1.1. *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.*

Further if the surface group conjecture is true then a group satisfying the conditions of the conjecture must be hyperbolic or free abelian of rank 2. We then prove the following:

Theorem 1.2. *Let G be a finitely generated fully residually free group with property IF. Then either G is hyperbolic or G is free abelian of rank 2.*

In light of these results we give a modified version of the surface group conjecture.

Conjecture 1.3. ***Surface Group Conjecture C*** *Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group.*

We then give some conditions under which Surface Group Conjecture C is true.

Finally we note that although here we concentrate on Property IF there has been some evidence for the Surface Group Conjecture based on the subgroups of finite index. Note that an orientable surface group of genus $g \geq 2$ with the presentation

$$G =< a_1, b_1, ..., a_g, b_g; [a_1, b_1]....[a_g, b_g] >$$

also has a presentation

$$G =< x_1, ..., x_n; x_1...x_n x_1^{-1}...x_n^{-1} = 1 >$$

with n even. P. M. Curran [C] has proved the following.

Theorem 1.3. *Let G be a one-relator group with the presentation*

$$G =< x_1, ..., x_n; x_1^{\nu_1}...x_n^{\nu_n} x_1^{-\nu_1}...x_n^{-\nu_n} = 1 > .$$

Then, if n is odd, there exist normal subgroups of finite index which do not have one-relator presentations. In particular if

$$G =< x_1, ..., x_n; x_1...x_n x_1^{-1}...x_n^{-1} = 1 >$$

then every subgroup of finite index is again a one-relator group if and only if n is even and hence a surface group.

2. Fully Residually Free Groups

Our results depend on the properties of fully residually free groups. A group G is *fully residually free* if given finitely many nontrivial elements $g_1, ..., g_n$ in G there is a homomorphism $\phi : G \to F$, where F is a free group, such that $\phi(g_i) \neq 1$ for all $i = 1, ..., n$. Fully residually free groups have played a crucial role in the study of equations and first order formulas over free groups and in particular the solution of the Tarski problem (see [KhM] and [Se 1-6]). Finitely generated fully residually free groups are also known as **limit groups**. In this guise they were studied by Sela (see [Se 1-6] and [BeF 2]) in terms of studying homomorphisms of general groups into free groups.

A *universal sentence* in the language of group theory is a first order sentence using only universal quantifiers (see [FGMRS]. The *universal theory* of a group G consists of all universal sentences true in G. All free groups share the same universal theory. A group G is called a *universally free group* if it shares the same universal theory as the class of free groups. Since any universal sentence is equivalent to an existential sentence the universally free groups have the same existential theory. Remeslennikov calls the universally free groups $\exists$-free groups. Remeslennikov [Re] and independently Gaglione and Spellman [GS] proved the following remarkable theorem.

Theorem 2.1 (Re,GS)**.** *Suppose G is residually free. Then the following are equivalent:*

(1) G is fully residually free
(2) G is commutative transitive
(3) G is universally free

From the linearity it is easy to see that orientable surface groups are commutative transitive. A result of G. Baumslag [GB] shows that they are residually free and hence we have

Theorem 2.2. *An orientable surface group is fully residually free.*

The class of fully residually free groups coincides with the class of universally free groups. From the solution of the Tarski problem all free groups share the same elementary or first-order theory. An *elementary free group* is a group G which shares the same elementary theory as the class of free groups. As an outgrowth of the solution to the Tarski problem Kharlampovich and Myasnikov and independenly Sela have completely characterized the elementary free groups. In particular, in the language of Kharlampovich and Myasnikov, they are precisely the NTQ-groups - the coordinate groups of regular NTQ-systems of equations over free groups. What is important here is that an elementary free group must be universally free and hence fully residually free. Therefore results proved about fully residually free groups apply to elementary free groups.

Kharlampovich, Myasnikov, Remesslenikov, Sela and others have done extensive work on describing both the subgroups and the subgroup structure of fully residually free groups. We mention a few results that are relevant to our main result. The following is a summary of several results:

Theorem 2.3. *(see [KhM] and the references there) Let G be a finitely generated fully residually free group. Then*

(1) G can be embedded as a subgroup in the free exponential group $F^{\mathbb{Z}[t]}$

(2) G is finitely presentable

(3) G can be constructed in a systematic way starting with free groups and abelian groups using free products with cyclic amalgamation and extensions of centralizers.

The construction mentioned in the theorem leads to the existence of nontrivial JSJ decompositions. This is crucial to our results.

JSJ-decompositions were introduced by Rips and Sela ([RiS]) and have played a fundamental role in the study of both hyperbolic groups and fully residually free groups. Roughly a JSJ-decomposition of a group G is a splitting of G as a graph of groups with abelian edges which is canonical in that it encodes all other such abelian splittings. If each edge is cyclic it is called a *cyclic JSJ-decomposition.* For a formal definition we refer to [KhM]. There are also full discussions in [BeF 2] and the work of Sela [Se 1-6]. The relevant fact for fully residually free groups is the following.

Theorem 2.4. *(see [KhM]) (a) A finitely generated fully residually free group which is indecomposable relative to JSJ-decompositions is either the fundamental group of a closed surface, a free group or a free abelian group.*

(b) A finitely generated fully residually free group admits a non-trivial cyclic JSJ-decomposition if it is not abelian or a surface group.

3. Main Results

Recall that a surface group has a one-relator presentation

$$< a_1, b_1, ..., a_g, b_g; [a_1, b_1]....[a_g, b_g] >$$

in the orientable case and

$$< a_1, ...a_g; a_1^2...a_g^2 >$$

in the non-orientable case. Orientable surface groups are fully residually free, residually finite and have the property that subgroups of finite index are again surface groups while subgroups of infinite index are free groups.

A **cyclically pinched one-relator group** is a one-relator group of the following form

$$G =< a_1, ..., a_p, a_{p+1}, ..., a_n; U = V >$$

where $1 \neq U = U(a_1, ..., a_p)$ is a cyclically reduced, non-primitive (not part of a free basis) word in the free group F_1 on $a_1, ..., a_p$ and $1 \neq V = V(a_{p+1}, ..., a_n)$ is a cyclically reduced, non-primitive word in the free group F_2 on $a_{p+1}, ..., a_n$.

Clearly such a group is the free product of the free groups on $a_1, ..., a_p$ and $a_{p+1}, ..., a_n$ respectively amalgamated over the cyclic subgroups generated by U and V.

Consider the standard one-relator presentation for an orientable surface group of genus $g \geq 2$:

$$S_g =< a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1 > .$$

If we let $U = [a_1, b_1]...[a_{g-1}, b_{g-1}], V = [a_g, b_g]^{-1}$ then S_g decomposes as the free product of the free groups F_1 on $a_1, b_1, ..., a_{g_1}, b_{g-1}$ and F_2 on a_g, b_g amalgamated over the maximal cyclic subgroups generated by U in F_1 and V in F_2. Hence these are cyclically pinched one-relator groups. There is a similar decompositon in the nonorientable case.

Cyclically pinched one-relator groups have been shown to be extremely similar to surface groups. G.Baumslag [GB] has shown that such a group is residually finite. A group G is **conjugacy separable** if given elements g, h in G either g is conjugate to h or there exists a finite quotient where they are not conjugate. J. Dyer [Dy] has proved the conjugacy separability of cyclically pinched one-relator groups. Note that conjugacy separability in turn implies residual finiteness. S. Lipschutz proved that cyclically pinched one-relator groups have solvable conjugacy problem [Li]. P.Scott [Sc] proved that surface groups are subgroup separable and then Brunner, Burns and Solitar [BBS] showed that in general cyclically pinched one-relator groups are subgroup separable. Cyclically pinched one-relator groups with neither U nor V (in the presentation (2.1)) proper powers were shown to be linear by Wehrfritz [W] while using a result of Shalen [Sh], Fine and Rosenberger (see [FR]) showed that such a cyclically pinched one-relator group has a faithful representation in $PSL_2(\mathbb{C})$. If neither U nor V is a proper power Juhasz and Rosenberger [JR], and independently Bestvinna and Feign [BeF 1] and Kharlampovich and Myasnikov [KhM 4] have proved that cyclically pinched one-relator groups are hyperbolic. Rosenberger [Ro], using Nielsen cancellation, has given a positive solution to the isomorphism problem for cyclically pinched one-relator groups, that is, he has given an algorithm to determine if an arbitrary one-relator group is isomorphic or not to a given cyclically pinched one-relator group.

The HNN analogs of cyclically pinched one-relator groups are called **conjugacy pinched one-relator groups** and are also motivated by the structure of orientable surface groups. In particular suppose

$$S_g = < a_1, b_1, ..., a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1 > .$$

Let $b_g = t$ then S_g is an HNN group of the form

$$S_g =< a_1, b_1, ..., a_g, t; tUt^{-1} = V >$$

where $U = a_g$ and $V = [a_1, b_1]...[a_{g-1}, b_{g-1}]a_g$. Generalizing this we say that a **conjugacy pinched one-relator group** is a one-relator group of the form

$$G =< a_1, ..., a_n, t; tUt^{-1} = V >$$

where $1 \neq U = U(a_1, ..., a_n)$ and $1 \neq V = V(a_1, ..., a_n)$ are cyclically reduced in the free group F on $a_1, ..., a_n$.

More information about both cyclically pinched one-relator groups and conjugacy pinched one-relator groups is in [FR] or [FRS].

We say that a group G satisfies **Property IF** if every subgroup of infinite index is free and we concentrate on this property in conjunction with the property of being fully residually free.

Proposition 3.1. *If a one-relator group has Property IF and is freely decomposable then its a free group. More generally if G is a finitely generated freely decomposable fully residually free group with property IF. Then G is a free group.*

Proof. Suppose that G is a one-relator group and is a free product. Then each factor has infinite index and so is a free group and hence the whole group is a free group. The same proof follows in general. □

Now we present the main results.

Theorem 3.1. *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one relator group.*

Proof. Suppose that G is a finitely generated fully residually free group. Then if it is indecomposable relative to JSJ-decompositions it is either the fundamental group of a closed surface, a free group or a free abelian group. The first two cases are covered by the theorem. If is free abelian of rank > 2 then it cannot satisfy property IF. Therefore if it is abelian it must be either infinite cyclic and hence free or free abelian of rank 2. In this case G has the presentation

$$G =< x, y; [x, y] = 1 > \Longrightarrow G =< x, y; xyx^{-1} = y >$$

and G can be considered as a conjugacy pinched one-relator group.

If G is not abelian or a surface group then from Theorem E, G admits a non-trivial cyclic JSJ-decomposition. Let e be a a non-trivial edge with edge stabilizer G_e. Collapse everything at one vertex into a single group G_1 and everything at the other vertex to a single group G_2. That is G_1 is the subgroup generated by all vertex groups on one side of the vertex e and G_2 is the subgroup generated by all vertex groups on the other side of e. If both collapse totally, that is both G_1 and G_2 are trivial, then G is cyclic which is a contradiction. Hence we can assume that at least one of G_1 and G_2 is nontrivial.

Suppose first that both G_1 and G_2 are non-trivial. Then G is a free product of G_1 and G_2 with cyclic amalgamation along the edge e and hence along G_e. Since this is a free product with amalgamation, both factors G_1 and G_2 have infinite index. By assumption G satisfies property IF and hence both factors are free groups. Therefore G is a free product with amalgamation of two free groups with a cyclic amalgamated subgroup; i.e a cyclically pinched one-relator group or a free group.

Now suppose that G_2 is trivial. Then G has a tree decomposition with a single edge emanating from G_1. Hence G is an HNN extension of G_1 with cyclic associated subgroups. As before from property IF, G_1 must be a free group, and hence G in this case is a conjugacy pinched one-relator group. □

Corollary 3.1. *Suppose that G is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.*

Proof. From Theorem 3.1, G is either free or a cyclically pinched or conjugacy pinched one-relator group. From Property IF it follows that G is torsion-free. Suppose that H is a subgroup of finite index. H is then also finitely generated and fully residually free. Since subgroups of infinite index in H also have infinite index in G the subgroup H also satisfies property IF. If H is freely decomposable then from Proposition 1 H is a free group. In this case G is a torsion-free finite extension of H and hence G is also free. Therefore if G is not free then every subgroup of finite index must be freely indecomposable. From Theorem 3.1 it follows then that every subgroup of finite index must be a one-relator group. □

Surface groups of genus $g \geq 2$ are hyperbolic. Hence if the surface group conjecture were to be true then the resulting group must be hyperbolic unless the group were free abelian of rank 2. We can consider a free abelian group as a surface group of genus $g = 1$; i.e. $G =< x, y; [x, y] = 1 >$. We then have.

Theorem 3.2. *Let G be a finitely generated fully residually free group with property IF. Then either G is hyperbolic or G is free abelian of rank 2.*

Proof. Suppose that G is a finitely generated fully residually free group with property IF. Since free groups are hyperbolic we may assume that G is not free. Therefore from Theorem 1 G is either a cyclically pinched one-relator group or a conjugacy pinched one-relator group.

Suppose first that G is cyclically pinched so that

$$G =< a_1, ..., a_p, a_{p+1}, ..., a_n; U = V >$$

where $1 \neq U = U(a_1, ..., a_p)$ is a cyclically reduced, non-primitive (not part of a free basis) word in the free group F_1 on $a_1, ..., a_p$ and $1 \neq V = V(a_{p+1}, ..., a_n)$ is a cyclically reduced, non-primitive word in the free group F_2 on $a_{p+1}, ..., a_n$. Suppose that U, V

are both proper powers so that $U = U_1^m$ and $V = V_1^k$ with both $m, k > 1$ for some non-trivial words U_1, V_1 in the free groups on the generators that U and V involve. Therefore in G we have the relation $U_1^m = V_1^n$ and further in G U_1 and V_1 do not commute so that the commutator $[U_1, V_1] \neq 1$. Since G is fully residually free then there is a homomorphism $\phi : G \to F$ with F a free group with

$$\phi(U_1) \neq 1, \phi(V_1) \neq 1, \phi([U_1, V_1]) = [\phi(U_1), \phi(V_1)] \neq 1.$$

But then we have a relation $(\phi(U_1))^m = (\phi(V_1)^k$ between noncommuting elements $\phi(U_1), \phi(V_1)$ in a free group which is impossible. It follows than that under these conditions not both U and V can be proper powers. From a result proved independently by Kharlampovich and Myasnikov [KM], Juhasz and Rosenberger [JR] and Bestvinna and Feighn [BeF] a cyclically pinched one-relator group with not both U, V proper powers is hyperbolic completing the proof in the cyclically pinched case.

Now suppose that G is conjugacy pinched so that

$$G = < a_1, ..., a_n, t; tUt^{-1} = V >$$

where $1 \neq U = U(a_1, ..., a_n)$ and $1 \neq V = V(a_1, ..., a_n)$ are cyclically reduced in the free group F on $a_1, ..., a_n$. Throughout the rest of the proof F will denote the free group on $a_1, ..., a_n$.

Suppose first that U and V are proper powers in the free group F so that $U = U_1^m, V = V_1^k$ with $m, k \geq 2$ and both U_1 and V_1 are both not proper powers in F. Then the commutator $[tU_1t^{-1}, v_1] \neq 1$ in G. Since G is fully residually free there is then an homomorphism ϕ into a free group F_1 with $\phi([tU_1t^{-1}, v_1] \neq 1$ in F_1. However then there is a relation $(\phi(tu_1t^{-1}))^m = (\phi(V_1)^k$ between noncommuting elements $\phi(tU_1t^{-1})$ and $\phi(V_1)$ in the free group F_1 which is impossible. Hence either $m = 1$ or $k = 1$. Without loss of generality we may assume then that $k = 1$ so that $V = V_1$ is not a proper power in F.

We claim that if G is fully residually free then either U and V are conjugately separated in F or $U = gVg^{-1}$ for some $g \in F$. In the latter case G must be free abelian of rank 2. Recall that U, V are **conjugately separated** if no conjugate of U intersects the cyclic subgroup $< V >$. Suppose that U and V are not conjugately separated. Then there is an $q \neq 1$ with

$$g^{-1}Ug = V^q$$

for some element $g \in F$. Since $t^{-1}Ut = V$ in G this would imply that in G we have

$$t^{-1}U^qt = V^q \text{ and hence } g^{-1}Ug = t^{-1}U^mt.$$

Since G is fully residually free there is then a homomorphism ϕ of G into a free group F_1 with

$$\phi(U) \neq 1, \phi(g) \neq 1, \phi(t) \neq 1, \phi(gt^{-1}) \neq 1$$

Then $\phi(U)$ would be conjugate to $(\phi(U))^q$ in the free group F which is impossible if $|q| > 1$.

Therefore U, V are conjugately separated unless $q = \pm 1$. Hence if U and V are not conjugately separated then $U = gV^{\epsilon}g^{-1}$ for some $g \in F$, $\epsilon = \pm 1$ and U and V are both not proper powers in F. Hence G has a presentation

$$G =< a_1, ..., a_n, t; tUt^{-1} = V >$$

with $U = gV^{\epsilon}g^{-1}$ in F . Now we replace t by $t_1 = gt$ and hence G has a presentation

$$G =< a_1, ..., a_n, t; tUt^{-1} = U^{\epsilon} >$$

where $\epsilon = \pm 1$ and U is not a proper power in F.

Suppose first that $\epsilon = 1$. Then G is the rank one extension of centralizers

$$G =< a_1, ..., a_n, t; tUt^{-1} = U >$$

G then contains the free abelian group or rank 2, $< t, U >$. If U were not just a single generator a_i then $< t, U >$ would have infinite index in G. But this is not free contradicting property IF and therefore $U = a_i$ for some generator a_i. Then G has the form

$$G =< a_1, ..., a_n, t; ta_it^{-1} = a_i >$$

This is just the free product of the free abelian group $< a_i, t >$ and the free group on the remaining generators. The free abelian group would have infinite index if there were any remaining generators contradicting property IF. Therefore if U and V are not conjugately separated G is a free abelian group of rank 2.

Now suppose that $\epsilon = -1$. Then G contains the subgroup

$$H =< t, U; tUt^{-1} = U^{-1} > .$$

In this case G cannot be fully residually free.

Therefore either U and V are conjugately separated in F or G is free abelian of rank 2. From a result of Gildenhuys, Kharlampovich and Myasnikov [GKM] a nonabelian conjugacy pinched one-relator group where U, V are conjugately separated is hyperbolic. Since a free abelian group of rank 2 is a fully residually free group satisfying property IF this completes the proof. □

In the preceding results we assumed that G was fully residually free and used the JSJ decomposition. However property IF will imply the finite index property if we assume not the fully residually free property but that we start with a graph of groups decomposition. In particular which get the following which gives further evidence towards the full surface group conjecture;

Theorem 3.3. *Let G be a nonfree cyclically pinched or conjugacy pinched one-relator group with property IF. Then each subgroup of finite index is again a cyclically pinched or conjugacy pinched one-relator group.*

Proof. We use the subgroup theorems for free products with amalgamation and HNN groups as described by Karrass and Solitar. We need these in the following form:

Theorem 3.4. *(see [KS 1]) Let $G = G_1 *_A G_2$ be a non-trivial free product with amalgamation. If H is a subgroup of G then H is an HNN group*

$$H =< t_1, ..., t_n, .., S; rels\ S, t_1^{-1}S_1t_1 = f_1(S_1), ... >$$

whose base S is a tree product . Each vertex group in the base is a conjugate of $G_1 \cap H$ or $G_2 \cap H$ and each amalgamated subgroup is a conjugate of A intersected with H. Further the associated subgroups $\{S_i, f_i(S_i)\}$are also conjugates of A intersected with H and each associated subgroup is contained in a vertex group.

An HNN group whose base is a tree product and where each associated subgroup is a subgroup of a vertex group is called a **treed HNN group**. The above theorem says that subgroups of free products with amalgamation are treed HNN groups.

Karrass and Solitar also explicitly describe the vertex groups, amalgamated subgroups and associated subgroups in terms of double coset representatives for H (see [KS 1]).

The corresponding subgroup theorem for HNN groups states:

Theorem 3.5. *Suppose G is an HNN group with a presentation*

$$G =< t_1, ..., t_n, .., K;\ rels\ K, t_i^{-1}L_it_i = f_i(L_i), i = 1, ... > .$$

Then any subgroup H of G is a treed HNN group. Further the vertex groups of the base of H are conjugates of the base K intersected with H while the amalgamated subgroups in the base of H are conjugates of the L_i intersected with H and the non-trivial associated subgroups in H are conjugates of K or of the L_i intersected with H.

Now suppose that G is a cyclically pinched one relator group with property IF. Then G can be expressed as the amalgamted free product

$$G = F_1 \underset{U=V}{\rightarrow} \star F_2$$

where F_1 and F_2 are free groups.

Suppose that H is a subgroup of finite index in G. Then H is a treed HNN group as described by Theorem F. The base group K is a tree which has vertices of the form

$$... \star \{F_1^\star \underset{U=V}{\rightarrow} \star F_2^\star\} \star ....$$

where as described, $F_1^\star$ and $F_2^\star$ are conjugates of subgroups of F_1 and F_2 respectively. Since F_1 and F_2 are free each of these are free groups. If H has no nontrivial vertex of this form then the base K is a free group and the conjugations by the free part are trivial. Hence H is a free group. But since it has finite index in G this would imply that G were free contrary to assumption. Therefore the base K in H must have at least one nontrivial vertex of the form above. If the free part of H (free part as an HNN extension) were nontrivial then the vertex subgroup

$$\{F_1^\star \underset{U=V}{\rightarrow} \star F_2^\star\}$$

would have infinite index in G. But this is not free contradicting property IF. Therefore the free part is empty. Similarly if there were more than one vertex of this type then each vertex of this type would have infinite index again contradicting property IF. It follows that H must consist of just one vertex of this type and therefore be a cyclically pinched one-relator group or a conjugacy pinched one relator group.

The same type of analysis using Theorem G gives the result for conjugacy pinched one-relator groups with property IF. This gives further credence to the surface group conjecture since a surface group can be expressed as either a cyclically pinched one-relator group or a conjugacy-pinched one-relator group. $\square$

We mention further that the following is known. It appears in [KRW] but not stated exactly in the same way.

Theorem 3.6. *(see [KRW]) Suppose that* $G =< a_1, ..., a_n; a_1^{\alpha_1} \cdots a_n^{\alpha_n} = 1 >$ *with* $n \geq 2$ *and all* $\alpha_i \geq 2$*. Then* G *has Property IF if and only if* $\alpha_1 = ... = \alpha_n = 2$*.*

In light of these results we give a modified version of the surface group conjecture.

Conjecture 3.1. ***Surface group Conjecture C*** *Suppose that* G *is a finitely generated freely indecomposable fully residually free group with property IF. Then* G *is a surface group.*

We note that Surface group Conjecture C is true under either of the following two conditions

(1) The original relator is strictly quadratic

(2) There is only one (QH) vertex in the JSJ decomposition for G

That is:

Theorem 3.7. *Suppose that* G *is a nonfree finitely generated freely indecomposable fully residually free group with property IF. If either*

(1) G *is a one-relator group with a strictly quadratic relator*

(2) there is only one (QH) vertex in the JSJ decomposition for G

then G *is a surface group.*

Proof. It is well known (see [LS]) that if a one-relator group has a strictly quadratic relator then it is a free product of a free group and a surface group. If G is freely indecomposable the result follows immediately.

A QH-vertex in the cyclic JSJ decomposition is described in the following way.

Let P be a group which admits one of the following presentations

(1) $< p_1, .., p_m, a_1, b_1, ..., a_g, b_g; \prod_{k=1}^m p_k \prod_{i=1}^g [a_i, b_i] >$

(2) $< p_1, .., p_m, v_1, ..., v_g; \prod_{k=1}^m p_k \prod_{i=1}^g v_i^2 >$.

Suppose $\Gamma(V, E)$ is a splitting of a group G as a graph of groups and suppose $P = G_v$ where $v \in V$. Then G is a *QH-vertex* if $e_1, ...e_n$ are the all edges with initial vertex v then $\alpha(e_i) = p_i$ where α is the boundary monomorphism taking G_{e_i} into G_v.

QH stands for *quadratically hanging.* Notice that a QH-vertex is a free group and geometrically is the fundamental group of a punctured surface.

If there is only one QH-vertex then after the collapse as described in the proof of Theorem 1 the resulting relator must be strictly quadratic and the result follows; $\square$

We close the paper by mentioning a somewhat related conjecture due to Bogopolski. Recall the following theorem due to Magnus (see [LS]) related to automorphisms of one-relator groups.

Theorem 3.8. *Suppose that R and S are elements of a free group which have the same normal closure. Then R is conjugate to $S^{\pm 1}$.*

Recently Bogopolski [B] (and independently Jim Howie [H]) proved the same result for one-relator quotients of surface groups. Bogopolski calls this the **Magnus Property**. His conjecture is the following.

Conjecture 3.2. *Let G be a torsion-free hyperbolic one-relator group with Property IF. Then G satisfies the Magnus Property.*

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