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# A mapping theorem on g-metrizable spaces

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Abstract. In this paper, we give some mapping theorems on $g$-metrizable spaces in terms of some sequence-covering mappings, $\sigma$-mappings and $\pi$-mappings.

## 1. Introduction and definitions

$G$-metrizable spaces constitute an important class of generalized metric spaces in the metrization theory. In 1965, R.W.Heath[12] proved that a space is developable if and only if it is an open $\pi$-image of a metric space. In 1969, J.A.Kofner[13] proved that a space is a symmetric space satisfying weak cauchy condition if and only if it is a quotient $\pi$-image of a metric space. In 1972, D.K.Burke[14] proved that a space is semimetrizable if and only if it is a countably bi-quotient (or pseudo-open) $\pi$-image of a metric space. In 1976, K.B.Lee[15] proved that every $g$-metrizable space is a quotient $\pi$-image of a metric space. In this paper, the relationships between metric spaces and $g$-metrizable spaces are established in terms of some sequence-covering mappings, $\sigma$-mappings and $\pi$-mappings.

In this paper, all spaces are regular and $T_1$, all mappings are continuous and surjective. $N$ denotes the set of all positive integers. $\omega$ denotes the set of all natural numbers. For a family $P$ of subsets of a space $X$ and a mapping $f : X \to Y$, let $f(P) = \{f(P) : P \in P\}$. For two families $A$ and $B$ of subsets of $X$, let $A \wedge B = \{A \cap B : A \in A \text{ and } B \in B\}$. For the usual product space $\prod_{i \in N} X_i$, $p_i$ denotes the projection from $\prod_{i \in N} X_i$ onto $X_i$.

**Definition 1.1** Let $f : X \to Y$ be a mapping.

(1) $f$ is said to be a $\sigma$-mapping[1] if there exists a base $B$ for $X$ such that $f(B)$ is a $\sigma$-locally finite family of subsets of $Y$.

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(2) $f$ is said to be a strong sequence-covering mapping[6] if each convergent sequence (including its limit point) of $Y$ is the image of some convergent sequence (including its limit point) of $X$.

(3) $f$ is said to be a sequence-covering mapping[10] if each convergent sequence (including its limit point) of $Y$ is the image of some compact subset of $X$.

(4) $f$ is said to be a $\pi$-mapping[7] if $(X, d)$ is a metric space and for each $y \in Y$ and its open neighborhood $V$ in $Y$, $d(f^{-1}(y), M \setminus f^{-1}(V)) > 0$.

It is easy to check that compact mappings on metric spaces are $\pi$-mappings.

**Definition 1.2** Let $P$ be a cover of a space $X$.

(1) $P$ is said to be a $k$-network[8] for $X$ if for each compact subset $K$ of $X$ and its open neighborhood $V$, there exists a finite subfamily $P'$ of $P$ such that $K \subset \cup P' \subset V$.

(2) $P$ is said to be a cs-network for $X$ if for each $x \in X$, its open neighborhood and a sequence $\{x_n\}$ converging to $x$, there exists $P \in P$ such that $\{x_n : n \geqslant m\} \cup \{x\} \subset P \subset V$ for some $m \in N$.

(3) $P$ is said to be a cs$^*$-network for $X$ if for each $x \in X$, its open neighborhood $V$ and a sequence $\{x_n\}$ converging to $x$, there exist $P \in P$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k} : k \in N\} \cup \{x\} \subset P \subset V$.

A space $X$ is called an $\aleph$-space if $X$ has a $\sigma$-locally finite $k$-network.

**Definition 1.3** Let $P = \cup\{P_x : x \in X\}$ be a family of subsets of a space $X$ satisfying that for each $x \in X$,

(1) $P_x$ is a network of $x$ in $X$,

(2) If $U, V \in P_x$, then $W \subset U \cap V$ for some $W \in P_x$.

$P$ is called a weak-base for $X$[2] if $G \subset X$ such that for each $x \in G$, there exists $P \in P_x$ satisfying $P \subset G$, then $G$ is open in $X$.

A space $X$ is called a g-metrizable space[3] if $X$ has a $\sigma$-locally finite weak-base.

We have the following implications for a space $X$[5]:

$$\text{metrizable} \Longrightarrow g\text{-metrizable} \Longleftrightarrow \text{symmetrizable} + \aleph\text{-space}$$

## 2. The main result

**Lemma 2.1** The following are equivalent for a space $X$:

(1) $X$ is an $\aleph$-space;

(2) $X$ is a strong sequence-covering $\sigma$-image of a metric space;

(3) $X$ is a sequence-covering $\sigma$-image of a metric space.

**Proof.** (1) $\Longrightarrow$ (2). Suppose $X$ is an $\aleph$-space, then $X$ has a $\sigma$-locally finite cs-network by Theorem 4 of [4]. Let $P = \cup\{P_i : i \in N\}$ be a $\sigma$-locally finite cs-network for $X$, where each $P_i = \{P_\alpha : \alpha \in A_i\}$ is a locally finite family of subsets of $X$ which is closed under finite intersections and $X \in P_i \subset P_{i+1}$. For each $i \in N$, endow $A_i$ with discrete topology, then $A_i$ is a metric space. Put

$$M = \left\{\alpha = (\alpha_i) \in \prod_{i \in N} A_i : \{P_{\alpha_i} : i \in N\} \subset P \text{ forms a network at some point } x(\alpha) \in X\right\},$$

and endow $M$ with the subspace topology induced from the usual product topology of the family $\{A_i : i \in N\}$ of metric spaces, then $M$ is a metric space. Since $X$ is Hausdroff, $x(\alpha)$ is unique in $X$ for each $\alpha \in M$. We define $f : M \to X$ by $f(\alpha) = x(\alpha)$ for each $\alpha \in M$. Because $P$ is a $\sigma$-locally finite cs-network for $X$, then $f$ is surjective. For each $\alpha = (\alpha_i) \in M$, $f(\alpha) = x(\alpha)$. Suppose $V$ is an open neighborhood of $x(\alpha)$ in $X$, there exists $n \in N$ such that $x(\alpha) \in P_{\alpha_n} \subset V$, set $W = \{c \in M :$ the n-th coordinate of $c$ is $\alpha_n\}$, then $W$ is an open neighborhood of $\alpha$ in $M$, and $f(W) \subset P_{\alpha_n} \subset V$. Hence $f$ is continuous. We will show that $f$ is a strong sequence-covering $\sigma$-mapping.

(i) $f$ is a $\sigma$-mapping.

For each $n \in N$ and $\alpha_n \in A_n$, put

$V(\alpha_1, \cdots, \alpha_n) = \{\beta \in M$: for each $i \leqslant n$, the i-th coordinate of $\beta$ is $\alpha_i\}$.

Let $B = \{V(\alpha_1, \cdots, \alpha_n) : \alpha_i \in A_i (i \leqslant n)$ and $n \in N\}$, then $B$ is a base for $M$.

To prove $f$ is a $\sigma$-mapping, we only need to check that $f(V(\alpha_1, \cdots, \alpha_n)) = \bigcap\limits_{i \leqslant n} P_{\alpha_i}$ for each $n \in N$ and $\alpha_n \in A_n$ because $f(B)$ is $\sigma$-locally finite in $X$ by this result.

For each $n \in N$, $\alpha_n \in A_n$ and $i \leqslant n$, $f(V(\alpha_1, \cdots, \alpha_n)) \subset P_{\alpha_i}$, then $f(V(\alpha_1, \cdots, \alpha_n)) \subset \bigcap\limits_{i \leqslant n} P_{\alpha_i}$. On the other hand. For each $x \in \bigcap\limits_{i \leqslant n} P_{\alpha_i}$, there is $\beta = (\beta_j) \in M$ such that $f(\beta) = x$. For each $j \in N$, $P_{\beta_j} \in P_j \subset P_{j+n}$, then there is $\alpha_{j+n} \in A_{j+n}$ such that $P_{\alpha_{j+n}} = P_{\beta_j}$. Set $\alpha = (\alpha_j)$, then $\alpha \in V\ (\alpha_1, \cdots, \alpha_n)$ and $f(\alpha) = x$. Thus $\bigcap\limits_{i \leqslant n} P_{\alpha_i} \subset f(V(\alpha_1, \cdots, \alpha_n))$. Hence $f(V(\alpha_1,\ \cdots, \alpha_n)) = \bigcap\limits_{i \leqslant n} P_{\alpha_i}$. Therefore, $f$ is a $\sigma$-mapping.

(ii) $f$ is a strong sequence-covering mapping.

For each sequence $\{x_n\}$ converging to $x_0$, we can assume that all $x'_n s$ are distinct, and that $x_n \neq x_0$ for each $n \in N$. Set $K = \{x_m : m \in \omega\}$. Suppose $V$ is an open neighborhood of $K$ in $X$. A subfamily $A$ is said to hold the property $F(K, V)$, if

(a) $A$ is finite;

(b) for each $P \in A$, $\emptyset \neq P \cap K \subset P \subset V$;

(c) for each $z \in K$, exists unique $P_z \in A$ such that $z \in P_z$;

(d) if $x_0 \in P \in A$, then $K \setminus P$ is finite.

Since $P$ is a $\sigma$-locally finite cs-network for $X$, then the above construction can be realized, and we can assume that $\{A \subset P_i : A$ holds the property $F(K, X)\} = \{A_{ij} : j \in N\}$.

For each $n \in N$, put

$$P'_n = \bigwedge_{i,j \leqslant n} A_{ij}\ ,$$

then $P'_n \subset P_n$ and $P'_n$ also holds the property $F(K, X)$.

For each $i \in N$, $m \in \omega$ and $x_m \in K$, there is $\alpha_{im} \in A_i$ such that $x_m \in P_{\alpha_{im}} \in P'_i$. Let $\beta_m = (\alpha_{im}) \in \prod\limits_{i \in N} A_i$. By definition, $\{P_{\alpha_{im}} : i \in N\}$ is a network of $x_m$ in $X$, and $f(\beta_m) = x_m$ for each $m \in \omega$. For each $i \in N$, there is $n(i) \in N$ such that $\alpha_{in} = \alpha_{io}$ when $n \geqslant n(i)$. Hence the sequence $\{\alpha_{in}\}$ converges to $\alpha_{io}$ in $A_i$. Thus the sequence $\{\beta_n\}$ converges to $\beta_0$ in $M$. This shows that $f$ is a strong sequence-covering mapping.

(2) $\Longrightarrow$ (3) are obvious.

(3) $\Longrightarrow$ (1). Suppose $X$ is the image of a metric space $M$ under a sequence-covering $\sigma$-mapping $f$. Since $f$ is a $\sigma$-mapping, there exists a base $B$ for $M$ such that $f(B)$ is a $\sigma$-locally finite family of subsets of $X$. Because sequence-covering mappings preserve cs*-networks by Proposition 2.7.3 of [9], then $f(B)$ is a $\sigma$-locally finite cs*-network for $X$. Hence $X$ is an $\aleph$-space by [11, Lemma 1.17, Theorem 1.4].

**Lemma 2.2**[5] Suppose $(X, d)$ is a metric space and $f : X \to Y$ is a quotient mapping. Then $Y$ is a symmetric space if and only if $f$ is a $\pi$-mapping.

**Lemma 2.3** Suppose $f$ is a quotient mapping from a $k$-sapce $M$ onto a space $X$. If $P$ is a $k$-network for $M$ and $f(P)$ is point-countable in $X$, then $f(P)$ is a $k$-network for $X$.

**Proof.** Denote $F = f(P)$. Suppose $K \subset V$ with $K$ non-empty compact and $V$ open in $X$. Put

$$A = \{F \in F : F \cap K \neq \emptyset \text{ and } F \subset V\},$$

then $K \subset \cup A'$ for some finite $A' \subset A$. Otherwise, for any finite $A' \subset A$, $K \setminus \cup A' \neq \emptyset$. For each $x \in K$, put

$$A_x = \{F \in F : x \in F \subset V\},$$

then $A_x$ is countable, and $A = \cup\{A_x : x \in K\}$. Denote $A_x = \{F_i(x) : i \in N\}$ for each $x \in K$. Take $x_1 \in K$, then there exists a infinite subset $D = \{x_n : n \in N\}$ of $K$ such that each $x_{n+1} \in K \setminus \bigcup_{i,j \leqslant n} F_i(x_j)$. So $D$ has a cluster point by the compactness of $K$. Let $x$ be a cluster point of $D$, and set $B = D \setminus \{x\}$, then $B$ isn't closed in $X$. Since $f$ is a quotient mapping, $f^{-1}(B)$ isn't closed in $M$. Because $M$ is a $k$-space, then there exists a compact subset $L$ of $M$ such that $f^{-1}(B) \cap L$ isn't closed in $L$. Let $g = f|_L : L \to f(L)$, then $g$ is a closed mapping, and $g^{-1}(B \cap f(L)) = f^{-1}(B) \cap L$. So $B \cap f(L)$ isn't closed in $f(L)$. Hence $B \cap f(L)$ is a infinite subset of $X$, and $D \cap f(L)$ is so. By $K \cap f(L) \neq \emptyset$, $H = L \cap f^{-1}(K)$ is non-empty compact in $M$ and $H \subset f^{-1}(K) \subset f^{-1}(V)$, then $H \subset \cup P' \subset f^{-1}(V)$ for some finite $P' \subset P$. Thus $f(H) \subset f(\cup P') \subset V$. Denote $P' = \{P_m : m \leqslant q\}$. We can assume that $P_m \cap H \neq \emptyset$ for each $m \leqslant q$, then $f(P_m) \in A$. Because

$$D \cap f(\cup P') \supset D \cap f(H) = D \cap f(L),$$

then $D \cap f(\cup P')$ is infinite. Thus $f(P_m)$ includes infinite points of $D$ for some $m \leqslant q$. Take $x_j \in D \cap f(P_m)$, then $f(P_m) = F_i(x_j)$ for some $i \in N$. However, there exists $n > i, j$ such that $x_n \in F_i(x_j)$, a contradiction. Hence $K \subset \cup A' \subset V$ for some finite $A' \subset A$. So $F$ is a $k$-network for $X$.

**Theorem 2.4** The following are equivalent for a space $X$:

(1) $X$ is a $g$-metrizable space.

(2) $X$ is a strong sequence-covering, quotient, $\pi$, $\sigma$-image of a metric space.

(3) $X$ is a sequence-covering, quotient, $\pi$, $\sigma$-image of a metric space.

(4) $X$ is a quotient, $\pi$, $\sigma$-image of a metric space.

**Proof.** (1) $\Longrightarrow$ (2) follows from Lemma 4, Proposition 2.1.16 (2) of [9] and Lemma 5.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $X$ is the image if a metric space $(X, d)$ under a quotient, $\pi$, $\sigma$-mapping $f$. Since $f$ is a $\sigma$-mapping, then there exists a base $B$ for $M$ such that $f(B)$ is $\sigma$-locally finite in $X$. By Lemma 6, $f(B)$ is a $k$-network for $X$. Thus $X$ is an $\aleph$-space. Hence $X$ is a $g$-metrizable space by Lemma 5.

**Example 2.5** Compact-covering, quotient, compact image of locally compact metric spaces may not be $g$-metrizable.

Let
$$S = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}, \quad X = [0,1] \times S.$$
Define a typical neighborhood of $(t, 0)$ in $X$ to be of the form
$$\{(t,0)\} \cup \left(\bigcup_{k \geqslant n} V(t, 1/k)\right), \quad n \in N,$$
where $V(t, 1/k)$ is a neighborhood of $(t, 1/k)$ in $[0,1] \times \{1/k\}$. Put
$$M = (\oplus_{n \in N}[0,1] \times \{1/n\}) \oplus (\oplus_{t \in [0,1]}\{t\} \times S),$$
and define $f$ from $M$ onto $X$ such that $f$ is the natural map, that is, $f(t,s) = (t,s)$ for each $(t,s) \in M$.

Then $f$ is a compact-covering, quotient, at most two-to-one map from the locally compact metric space $M$ onto separable, regular, non-meta-Lindelöf space $X$ (see Example 2.8.16 in [9] or Example 1 in [16]). It is easy to check that $f$ is a sequence-covering map. By Lemma 2.2 in [11], $X$ has a point-regular weak-base. Because $X$ is sequential, and a regular sequential space with a $\sigma$-locally countable $k$-network is meta-Lindelöf (see [8, Proposition 1]), then $X$ has not any $\sigma$-locally countable $k$-network. So $X$ is not an $\aleph$-space. Thus $X$ is not $g$-metrizable.

This example also illustrates:

A quotient, $\pi$-image of a metric space is not necessarily a quotient, $\pi$, $\sigma$-image of a metric space.

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