SCIENTIA
Series A: *Mathematical Sciences*, Vol. 10 (2004), 47–52
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Characterization of Matrix Operators on l^p-spaces

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Abstract. A necessary and sufficient condition for a class of matrix operators acting from l^{p_1} into l^{p_2} to be continuous and compact is given in this paper, which answers I. J. Maddox's open problem.

Suppose $A = (a_{ij})$ be an infinite real matrix. It determines a linear operator **A** from l^{p_1} into l^{p_2} $(1 \leqslant p_1, p_2 \leqslant \infty)$ according to the following raw:

$$Ax = \left(\sum_{j=1}^{\infty} a_{1j} t_j, \sum_{j=1}^{\infty} a_{2j} t_j, \cdots \right)^T \in l^{p_2}, \text{ for any } x = (t_1, t_2, \cdots)^T \in l^{p_1}.$$

The set of all such bounded matrix operators is denoted by $\mathcal{B}(l^{p_1}, l^{p_2})$, in which the set of all compact matrix operators is denoted by $\mathcal{B}_c(l^{p_1}, l^{p_2})$. The characterization of $\mathcal{B}(l^{p_1}, l^{p_2})$ (or $\mathcal{B}_c(l^{p_1}, l^{p_2})$) has remained a problem (see I. J. Maddox [2], Ch.7, §5, Problem 12) and attracted a lot of researchers (see[1], [3], [4]). Maddox[2] characterized $\mathcal{B}(l^{p_1}, l^{p_2})$ in terms of their elements when p_1 or p_2 is 1 or ∞. Crone[1] answered the question when $p_1 = p_2 = 2$. This paper is devoted to the case $1 < p_1, p_2 < \infty$.

Let $\mathcal{N}$ be the set of natural numbers and $1 < p < \infty$. The Lebesgue matrix space $l^p(\mathcal{N} \times \mathcal{N})$ is expressed as the set

$$\left\{ A = (a_{ij}) : \|A\|_p = \left(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

If $B = (b_{ij}) \in l^q(\mathcal{N} \times \mathcal{N})$ with $1/p + 1/q = 1$, we denote

$$\langle A, B \rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} b_{ij}.$$

Clearly, the Hölder inequality holds:

$$|\langle A, B \rangle| \leqslant \|A\|_p \|B\|_q.$$

2000 *Mathematics Subject Classification.* 46E30.
Key words and phrases. l^p-space, Matrix operator, Linear operator.

The tensor product $l^{r_1} \otimes l^{r_2}$ of l^{r_1} and $l^{r_2}(1 < r_i < \infty, i = 1, 2)$ is defined as the closure of the linear span $\{x \otimes y : x \in l^{r_1}, y \in l^{r_2}\}$, where

$$x \otimes y =: x \cdot y^T = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \end{pmatrix} (s_1, s_2, \cdots) = \begin{pmatrix} t_1 s_1 & t_1 s_2 & \cdots \\ t_2 s_1 & t_2 s_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

LEMMA 1. *Let p, r_1, r_2 be in $(1, \infty)$, then the following are equivalent:*
(i) $p \geqslant \max(r_1, r_2)$;
(ii) $(uv)^p \leqslant u^{r_1} v^{r_2}$ for all $u, v \in [0, 1]$;
(iii) $\|x \otimes y\|_p \leqslant \|x\|_{r_1} \|y\|_{r_2}$ for all $x \in l^{r_1}$ and $y \in l^{r_2}$;
(iv) $l^{r_1} \otimes l^{r_2} \subset l^p(\mathcal{N} \times \mathcal{N})$.

PROOF. (i)$\Rightarrow$(ii):(i) implies that $u^{p-r_1} v^{p-r_2} \leqslant 1$ for all $u, v \in [0, 1]$, which is equivalent to (ii).
(ii)$\Rightarrow$(iii): For any $0 \neq x = (t_1, t_2 \cdots)^T \in l^{r_1}$ and $0 \neq y = (s_1, s_2 \cdots)^T \in l^{r_2}$, we have

$$\sum_{i=1}^{\infty} \left(\frac{|t_i|}{\|x\|_{r_1}} \right)^{r_1} = 1 \text{ and } \sum_{j=1}^{\infty} \left(\frac{|s_j|}{\|y\|_{r_2}} \right)^{r_2} = 1.$$

Therefore, (ii) implies (iii), since

$$\begin{aligned} \left\| \frac{x \otimes y}{\|x\|_{r_1} \|y\|_{r_2}} \right\|_p^p &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{|t_i s_j|}{\|x\|_{r_1} \|y\|_{r_2}} \right)^p \\ \leqslant & \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{|t_i|}{\|x\|_{r_1}} \right)^{r_1} \left(\frac{|s_j|}{\|y\|_{r_2}} \right)^{r_2} \\ = & \quad \sum_{i=1}^{\infty} \left[\left(\frac{|t_i|}{\|x\|_{r_1}} \right)^{r_1} \right] \left[\sum_{j=1}^{\infty} \left(\frac{|s_j|}{\|y\|_{r_2}} \right)^{r_2} \right] = 1. \end{aligned}$$

(iii)$\Rightarrow$(iv):Trivial.
(iv)$\Rightarrow$(i): Let us assume that (i) is not satisfied, then $p < \max(r_1, r_2)$. We claim that there exist $u_n \searrow 0$ and $v_n \searrow 0$ such that

$$\left(\frac{u_n v_n}{4^n} \right)^p > u_n^{r_1} v_n^{r_2}, n = 1, 2, \cdots. \tag{0.1}$$

In fact, if $p < r_1$ and $p < r_2$, we choose

$$u_n = v_n = \left(\frac{1}{4^{np} + 1} \right)^{\frac{1}{(r_1 - p) + (r_2 - p)}}, n = 1, 2, \cdots,$$

which satisfy (1). If $p < r_1$ and $p \geqslant r_2$, we may select a real number $K > 0$ such that $K(r_1 - p) > p - r_2$, then put

$$v_n = \left(\frac{1}{4^{np} + 1} \right)^{\frac{1}{K(r_1 - p) - (p - r_2)}} \text{ and } u_n = v_n^K.$$

Therefore $u_n \searrow 0$ and $v_n \searrow 0$ and hence (1) is satisfied. Similarly one may check the case of $p \geqslant r_1$ and $p < r_2$. Since $u_n \searrow 0$ and $v_n \searrow 0$, we assume that $u_n^{r_1} \leqslant 1/2^n$ and $v_n^{r_2} \leqslant 1/2^n$ for all $n \geqslant 1$, otherwise we pick up subsequences u_{m_n} and v_{m_n} satisfying

$$u_{m_n}^{r_1} \leqslant \frac{1}{2^n}, \qquad v_{m_n}^{r_1} \leqslant \frac{1}{2^n}.$$

Then we select integers $K_n \geqslant 1$ and $J_n \geqslant 1$ for each $n \geqslant 1$ such that

$$\frac{1}{2^{n+1}} < K_n u_n^{r_1} \leqslant \frac{1}{2^n}, \qquad \frac{1}{2^{n+1}} < J_n u_n^{r_2} \leqslant \frac{1}{2^n}. \tag{0.2}$$

Define

$$x_0 = (\overbrace{u_1, u_1, \cdots, u_1}^{K_1}, \cdots\cdots, \overbrace{u_n, u_n, \cdots, u_n}^{K_n}, \cdots\cdots)^T$$

and

$$y_0 = (\overbrace{v_1, v_1, \cdots, v_1}^{J_1}, \cdots\cdots, \overbrace{v_n, v_n, \cdots, v_n}^{J_n}, \cdots\cdots)^T.$$

Then by (2), we have $x_0 \in l^{r_1}$ and $y_0 \in l^{r_2}$ since

$$\|x_0\|_{r_1}^{r_1} = \sum_{n=1}^{\infty} K_n u_n^{r_1} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

and

$$\|y_0\|_{r_2}^{r_2} = \sum_{n=1}^{\infty} J_n v_n^{r_2} \leqslant \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Observe that

$$x_0 \otimes y_0 = \left(\begin{array}{cccc} \overbrace{u_1v_1, \cdots, u_1v_1}^{J_1} & \cdots\cdots & \overbrace{u_1v_n, \cdots, u_1v_n}^{J_n} & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_1v_1, \cdots, u_1v_1 & \cdots\cdots & u_1v_n, \cdots, u_1v_n & \cdots\cdots \\ u_2v_1, \cdots, u_2v_1 & \cdots\cdots & u_2v_n, \cdots, u_2v_n & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ u_2v_1, \cdots, u_2v_1 & \cdots\cdots & u_2v_n, \cdots, u_2v_n & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} K_1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} K_2 \\ \quad\vdots \end{array}.$$

It follows from (1) and (2) that

$$\begin{aligned} \|x_0 \otimes y_0\|_p^p &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} K_i J_j (u_i v_j)^p \geqslant \sum_{n=1}^{\infty} K_n J_n 4^{np} u_n^{r_1} v_n^{r_2} \\ &> \sum_{n=1}^{\infty} \frac{4^{np}}{4^{n+1}} = \frac{1}{4} \sum_{n=1}^{\infty} 4^{n(p-1)} = \infty. \end{aligned}$$

That is, $x_0 \otimes y_0 \notin l^p(\mathcal{N} \times \mathcal{N})$, which contradicts (i) finishes the proof. $\square$

THEOREM 1. *If $p_1, p_2, p \in (1, \infty)$ and $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/p + 1/q = 1$, then the following conditions are equivalent:*
(i) $p \geqslant \max(p_1, q_2)$;

(ii) $\mathbf{A} \in \mathcal{B}(l^{p_1}, l^{p_2})$, *for each* $A = (a_{ij}) \in l^q(\mathcal{N} \times \mathcal{N})$;
(iii) $\mathbf{A} \in \mathcal{B}_c(l^{p_1}, l^{p_2})$,*for each* $A = (a_{ij}) \in l^q(\mathcal{N} \times \mathcal{N})$.

PROOF. (i)⇒(ii): Let (i) be satisfied. In view of the Lemma we have

$$\|x \otimes y\|_p \leqslant \|x\|_{p_1} \|y\|_{q_2} \tag{0.3}$$

for all $x \in l^{p_1}$ and $y \in l^{q_2}$. Therefore, $A = (a_{ij}) \in l^q(\mathcal{N} \times \mathcal{N})$ implies that $\|A\|_q < \infty$ and we deduce from (3) that

$$\begin{aligned} & \|\mathbf{A}\|_{l^{p_1} \to l^{p_2}} = \sup\{\|Ax\|_{p_2} : \|x\|_{p_1} \leqslant 1\} \\ = \ & \sup\{|\langle Ax, y\rangle| : \|x\|_{p_1} \leqslant 1, \|y\|_{q_2} \leqslant 1\} \\ = \ & \sup\left\{\left|\langle A^T, x \otimes y\rangle\right| : \|x\|_{p_1} \leqslant 1, \|y\|_{q_2} \leqslant 1\right\} \\ \leqslant \ & \sup\left\{\|A^T\|_q \|x \otimes y\|_p : \|x\|_{p_1} \leqslant 1, \|y\|_{q_2} \leqslant 1\right\} \\ \leqslant \ & \|A\|_q. \end{aligned}$$

That is, $\mathbf{A}$ is bounded on l^{p_1} into l^{p_2}.
(ii)⇒(iii):For any given $A = (a_{ij}) \in l^q(\mathcal{N} \times \mathcal{N})$, put

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Then $\mathbf{A}_n$ is compact for each $n \geqslant 1$. Since

$$\|\mathbf{A} - \mathbf{A}_n\|_{l^{p_1} \to l^{p_2}} \leqslant \|A - A_n\|_q = \left(\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} |a_{ij}|^q\right)^{\frac{1}{q}} \to 0$$

as $n \to \infty$, we conclude that A is also compact and hence (iii) holds.
(iii)⇒(i): Suppose that (iii) holds and that (i) is not satisfied, then $p < \max(p_1, q_2)$. In virtue of (1), there exist $u_n \searrow 0$ and $v_n \searrow 0$ such that

$$\left(\frac{u_n v_n}{4^n}\right)^p > u_n^{p_1} v_n^{q_2}, \qquad n = 1, 2, \cdots.$$

Therefore, we have

$$\frac{u_n v_n}{4^n} > (u_n^{p_1} v_n^{q_2})^{\frac{1}{p}} = \frac{u_n^{p_1} v_n^{q_2}}{(u_n^{p_1} v_n^{q_2})^{\frac{1}{q}}},$$

or

$$\left(\frac{4^n u_n^{p_1} v_n^{q_2}}{u_n v_n}\right)^q < u_n^{p_1} v_n^{q_2}. \tag{0.4}$$

Without loss of generality, we may assume that $u_n^{p_1} < \frac{1}{2^n}$ and $v_n^{q_2} \leqslant \frac{1}{2^n}$ for every $n \geqslant 1$. Let us choose two integers $K_n \geqslant 1$ and $J_n \geqslant 1$ for each $n \geqslant 1$ such that

$$\frac{1}{2^{n+1}} < K_n u_n^{p_1} \leqslant \frac{1}{2^n}, \qquad \frac{1}{2^{n+1}} < J_n v_n^{q_2} \leqslant \frac{1}{2^n}. \tag{0.5}$$

Next we define

$$A_0 = \begin{pmatrix} A_1 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & A_n & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix},$$

Where

$$A_n = \frac{4^n u_n^{p_1} v_n^{q_2}}{u_n v_n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{J_n \times K_n} \qquad n \geqslant 1.$$

Therefore, $A_0 \in l^q(\mathcal{N} \times \mathcal{N})$. In fact, by (4) and (5) we have

$$\begin{aligned} \|A_0\|_q &= \left[\sum_{n=1}^{\infty} K_n J_n \left(\frac{4^n u_n^{p_1} v_n^{q_2}}{u_n v_n} \right)^q \right]^{\frac{1}{q}} \\ &< \left[\sum_{n=1}^{\infty} K_n J_n u_n^{p_1} v_n^{q_2} \right]^{\frac{1}{q}} \leqslant \left[\sum_{n=1}^{\infty} \frac{1}{4^n} \right]^{\frac{1}{q}} \\ &= \left(\frac{1}{3} \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Finally, we show that $\mathbf{A}_0 \notin \mathcal{B}_c(l^{p_1}, l^{p_2})$. Put

$$x_0 = (\overbrace{u_1, u_1, \cdots, u_1}^{K_1}, \cdots\cdots, \overbrace{u_n, u_n, \cdots, u_n}^{K_n}, \cdots\cdots)^T$$

and

$$y_0 = (\overbrace{v_1, v_1, \cdots, v_1}^{J_1}, \cdots\cdots, \overbrace{v_n, v_n, \cdots, v_n}^{J_n}, \cdots\cdots)^T.$$

Then by (5) we deduce that

$$\|x_0\|_{p_1} = \left(\sum_{n=1}^{\infty} K_n u_n^{p_1} \right)^{\frac{1}{p_1}} \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right)^{\frac{1}{p_1}} = 1$$

and

$$\|y_0\|_{q_2} = \left(\sum_{n=1}^{\infty} J_n v_n^{q_2} \right)^{\frac{1}{q_2}} \leqslant \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right)^{\frac{1}{q_2}} = 1.$$

Note that

$$A_0 x_0 = \left(\overbrace{\frac{K_1 4 u_1^{p_1} v_1^{q_2}}{v_1}, \cdots, \frac{K_1 4 u_1^{p_1} v_1^{q_2}}{v_1}}^{J_1}, \cdots, \overbrace{\frac{K_n 4^n u_n^{p_1} v_n^{q_2}}{v_n}, \cdots, \frac{K_n 4^n u_n^{p_1} v_n^{q_2}}{v_n}}^{J_n}, \cdots \right)^T,$$

by the notation $\langle\{\xi_i\},\{\eta_i\}\rangle = \sum\limits_{i=1}^{\infty}\xi_i\eta_i$ and (5), we conclude that

$$\begin{aligned} & \|\mathbf{A}\|_{p_1\to p_2} = \sup\{\|A_0x\|_{p_2} : \|x\|_{p_1} \leqslant 1\} \\ \geqslant \quad & \|A_0x_0\|_{p_2} = \sup\{|\langle Ax_0, y\rangle| : \|y\|_{q_2} \leqslant 1\} \\ \geqslant \quad & \langle A_0x_0, y_0\rangle = \sum_{n=1}^{\infty} K_nJ_n4^n u_n^{p_1} v_n^{q_2} \\ \geqslant \quad & \sum_{n=1}^{\infty}\frac{1}{4} = +\infty. \end{aligned}$$

Thus, we reach a contradiction to (iii). The proof is finished. □

Acknowledgement: The author would like to thank Prof. Z. D. Ren for his advices and help.

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Received 30 08 2004

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