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Abel summability and angular convergence

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Abstract. We construct an example of a series that is Abel summable but whose associated power series does not converge on angular sectors. We also give some related results.

1. Introduction

A basic result in complex variables is Abel's theorem [**10**], that says that if the numerical series $\sum_{n=0}^{\infty} a_n$ converges then the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$, the function $f(z)$ es analytic in $|z| < 1$ and

$$\lim_{\substack{z\to 1\\ \text{Ang.}}} f(z) = \sum_{n=0}^{\infty} a_n \,. \tag{1.1}$$

The notation Ang. in the limit means in an angular way, that is, the limit exists as $z \to 1$ in any angular sector, $\pi/2+\varepsilon < \arg(z-1) < 3\pi/2-\varepsilon$, for $\varepsilon > 0$. Some authors use the term non-tangential instead of angular.

Motivated by Abel's theorem, one can define the concept of Abel summability. Indeed, if $\sum_{n=0}^{\infty} a_n$ is a *divergent* series, one can consider the auxiliary series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. If this auxiliary series converges for $0 \leqslant x < 1$, and if

$$\lim_{x\to 1^-} f(x) = S\,, \tag{1.2}$$

then one says that the series $\sum_{n=0}^{\infty} a_n$ is Abel summable to the value S and writes

$$\sum_{n=0}^{\infty} a_n = S \quad (\mathrm{A})\,. \tag{1.3}$$

For instance, the series $\sum_{n=0}^{\infty} (-1)^n$ is divergent, but $\sum_{n=0}^{\infty}(-1)^n x^n$ converges for $0 \leqslant x < 1$ to the sum $1/(1+x)$, whose limit when $x \to 1^-$ is $1/2$; thus

$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2} \quad (\mathrm{A})\,. \tag{1.4}$$

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Many examples of sums in the Abel sense of divergent series can be found in [**5**].

Observe that according to Abel's theorem, every convergent series is (A) summable, to the same limit. The technical term is regularity: Abel summability is a regular summation method since it sums convergent series to their sum.

We now come to the question we would like to study. Suppose that $\sum_{n=0}^{\infty} a_n = S$ (A). Then it is easy to see that the auxiliary series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ not only converges for $0 \leqslant z < 1$ but also in the disc $|z| < 1$ and, in fact, $f(z)$ is analytic in the disc. Abel summability to S means that $f(z) \to S$ as $z \to 1^-$, when z is a *real* number, namely, along the radius of the disc that goes through $z = 1$. The question is, does $f(z)$ converge to S when $z \to 1$ in an angular fashion?

The aim of this article is to show how one can construct an extreme counterexample to this question. Namely, in Section 2 we show a procedure that produces a series $\sum_{n=0}^{\infty} a_n$ that is Abel summable to S but such that in *no* sector $\pi - \varepsilon < \arg(z-1) < \pi + \varepsilon$, no matter how small $\varepsilon > 0$ is, it is true that $f(z) \to S$ as $z \to 1$.

In Section 3, the last section, we consider several related results, that show that it is not always possible to construct counterexamples if f satisfies some additional conditions.

As suggested by a referee, it would be interesting to study if counterexamples of this kind are "exceptional" in the sense that they occur only in a small set of boundary points, whether a set of zero measure or of zero capacity; this is the usual situation in several questions related to the existence of angular limits [**9**].

2. The counterexample

Interestingly, to construct our counterexample it is enough to build a simpler example, namely, one of a series $\sum_{n=0}^{\infty} a_n$ that is Abel summable to S but such that there exists *some* sector $\frac{\pi}{2} + \varepsilon < \arg(z-1) < \frac{3\pi}{2} - \varepsilon$ where it is not true that $f(z) \to S$ as $z \to 1$. This simpler example is easy: take

$$f(z) = e^{-(1-z)^{-\alpha}}, \tag{2.1}$$

where $\alpha > 0$ and where the branch of $\omega^{-\alpha} = (1-z)^{-\alpha}$ is the one defined for $\omega \in \mathbb{C} \setminus (-\infty, 0]$ which has the value 1 at $\omega = 1$. The series $\sum_{n=0}^{\infty} a_n$ is the one obtained by setting $z = 1$ in the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then $f(z)$ is analytic in $|z| < 1$ and $f(z) \to 0$ as $z \to 1$ in the sector $\pi - \frac{\pi}{2\alpha} < \arg(z-1) < \pi + \frac{\pi}{2\alpha}$ but $f(z) \nrightarrow 0$ as $z \to 1$ in any sector $\pi - \varepsilon < \arg(z-1) < \pi + \varepsilon$ if $\varepsilon > \frac{\pi}{2\alpha}$. Hence, the bigger the value of α, the sector where $f(z) \to 0$ as $z \to 1$ gets smaller. In particular, if $\alpha > 1$, the series is (A) summable but there is no angular convergence.

Let us now go to the extreme counterexample. If $0 \leqslant \delta < \pi/2$ let $\mathcal{X}_\delta$ be the space of analytic functions in $|z| < 1$ whose limit as $z \to 1$ exists in the closed sector $\pi - \delta \leqslant \arg(z-1) \leqslant \pi + \delta$. Observe that $\mathcal{X}_0$ is the space of analytic functions in $|z| < 1$ whose limit as $z \to 1$ along the radius exists. The extreme counterexample is a function $f \in \mathcal{X}_0$ that does not belong to $\mathcal{X}_\delta$ for any $\delta > 0$.

In the space $\mathcal{X}_\delta$ we introduce a topology by giving the following family of seminorms (norms, actually)

$$\|f\|_{\delta,r} = \sup\left\{|f(z)| : z \in S_{\delta,r}\right\} , \tag{2.2}$$

where if $0 < r < 1$ the set $S_{\delta,r}$ is the union of the disc $\{z \in \mathbb{C} : |z| \leqslant r\}$ and the sectorial set $\{z \in \mathbb{C} : \pi - \delta \leqslant \arg(z-1) \leqslant \pi + \delta,\ |z| \geqslant r,\ \Re e\, z > 0\}$; sets of this type are called Stoltz angles.

With these seminorms $\mathcal{X}_\delta$ becomes a locally convex topological vector space. Since $\| \quad \|_{\delta,r} \leqslant \| \quad \|_{\delta,s}$ for $0 < r < s < 1$, it follows that if $\{r_n\}_{n=1}^\infty$ is an increasing sequence that converges to 1, $r_n \nearrow 1$, then the seminorms $\| \quad \|_{\delta,r_n}$ generate the topology of $\mathcal{X}_\delta$, thus $\mathcal{X}_\delta$ is a metrizable vector space [**11**]. It is easy to see that $\mathcal{X}_\delta$ is complete, since if $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence, then

$$\lim_{n,m\to\infty} \|f_n - f_m\|_{\delta,r} = 0 , \tag{2.3}$$

for all r, with $0 < r < 1$, and consequently the sequence $\{f_n(z)\}_{n=1}^\infty$ is a Cauchy sequence for all $z \in \{\omega \in \mathbb{C} : |\omega| < 1\} \cup \{1\}$. Let

$$f(z) = \lim_{n\to\infty} f_n(z) , \quad z \in \{\omega \in \mathbb{C} : |\omega| < 1\} \cup \{1\} . \tag{2.4}$$

Using (2.3) we deduce that $\{f_n\}$ converges uniformly to f in each set $S_{\delta,r}$ and therefore in each compact set contained in the disc $\{\omega \in \mathbb{C} : |\omega| < 1\}$. If we now employ the well-known Weierstrass theorem on the analyticity of a limit, uniform over compacts, of a sequence of analytic functions, we conclude that f is analytic in the disc $\{\omega \in \mathbb{C} : |\omega| < 1\}$. Finally, the uniform convergence of the f_n's to f in $S_{\delta,r}$ gives the continuity of f in $S_{\delta,r}$, and this, in turn, guarantees that $f \in \mathcal{X}_\delta$. Consequently, $\mathcal{X}_\delta$ is a Fréchet space, that is, a complete metric vector space.

Observe now that in any complete metric space $\mathcal{Y}$, Baire's theorem holds [**6**]. Baire's theorem says that $\mathcal{Y}$ is of the *second category*, i.e., if $\{\mathcal{Y}_n\}$ is a sequence of subsets of $\mathcal{Y}$ with $\bigcup_{n=1}^\infty \mathcal{Y}_n = \mathcal{Y}$ then at least one of the $\mathcal{Y}_n$'s is not nowhere-dense, that is, the interior of its closure is non-empty; those spaces that can be written as a countable union of nowhere-dense sets are called of the *first category.* Thus, in a Fréchet space $\mathcal{X}$, if $\{\mathcal{V}_n\}$ is a sequence of vector subspaces of $\mathcal{X}$ of the first category, then not only $\mathcal{V}_n \neq \mathcal{X}\ \forall n$, but also $\bigcup_{n=1}^\infty \mathcal{V}_n \neq \mathcal{X}$.

Let us apply these ideas to the space $\mathcal{X}_\delta$. If $\delta' > \delta$ then $\mathcal{X}_{\delta'}$ is a vector subspace of $\mathcal{X}_\delta$ and, as we shall see in a moment, of the first category in $\mathcal{X}_\delta$. Let $r \in (0,1)$ be fixed and let

$$V = \left\{f \in \mathcal{X}_{\delta'} : \|f\|_{\delta',r} \leqslant 1\right\} . \tag{2.5}$$

Let W be the closure of V in $\mathcal{X}_\delta$. Observe that convergence in the space $\mathcal{X}_\delta$ implies pointwise convergence in $|z| < 1$ and at $z = 1$. Therefore, if $f \in W$ then $|f(z)| \leqslant 1$ for $z \in S_{\delta',r}$. The simpler example shows that for each $s \in (0,1)$ and each $\varepsilon > 0$ there exists $g \in \mathcal{X}_\delta$ with $\|g\|_{\delta,s} < \varepsilon$ but with $\sup\{|f(z)| : z \in S_{\delta',r}\} > 1$, and, consequently, $g \notin W$. We deduce that 0 is not an interior point of W in $\mathcal{X}_\delta$, since no neighborhood of 0 in $\mathcal{X}_\delta$ is contained in W. By translation, we conclude that the interior of W is empty. In other words, V is a nowhere-dense subset of $\mathcal{X}_\delta$ and since $\mathcal{X}_{\delta'} = \bigcup_{n=1}^\infty nV$, we obtain that $\mathcal{X}_{\delta'}$ is of the first category in $\mathcal{X}_\delta$.

Moreover, if $\{\delta_n\}_{n=1}^{\infty}$ is a decreasing sequence with $\delta_n \searrow \delta$, then Baire's theorem allows us to conclude that $\bigcup_{\delta'>\delta} \mathcal{X}_{\delta'} = \bigcup_{n=1}^{\infty} \mathcal{X}_{\delta_n} \neq \mathcal{X}_{\delta}$. We have proved the ensuing result.

THEOREM 2.1. *There exist analytic functions in* $|z| < 1$ *such that* $f(z)$ *has a limit as* $z \to 1$ *in the sector* $\pi - \delta \leqslant \arg(z-1) \leqslant \pi + \delta$ *but not in any sector* $\pi - \delta' \leqslant \arg(z-1) \leqslant \pi + \delta'$ *for* $\delta' > \delta$.

The extreme counterexample corresponds to the case $\delta = 0$.

THEOREM 2.2. *There exist series* $\sum_{n=0}^{\infty} a_n$ *that are Abel summable but such that the function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *does not have a limit as* $z \to 1$ *in any sector* $\pi - \varepsilon \leqslant \arg(z-1) \leqslant \pi + \varepsilon$ *for any* $\varepsilon > 0$.

3. Other results

In this section we consider some related results.

We would like to point out, first of all, that it is not possible to construct a counterexample as the one of the previous section if we also ask the function f to be bounded. Indeed, according to a classical result, Montel's theorem [**8**] (see also [**10**, pg. 170]), if g is analytic and bounded in the semi-strip $\Re e\, z > 0$, $a < \Im m\, z < b$, and if there exists some value y_0 for which the limit

$$\lim_{x\to\infty} g(x + iy_0) = L\,, \tag{3.1}$$

exists, then

$$\lim_{x\to\infty} g(x + iy) = L\,, \tag{3.2}$$

for all $y \in (a, b)$ and, in fact, the convergence is uniform in $[a + \varepsilon, b - \varepsilon]$ for all $\varepsilon > 0$. Using a conformal mapping we deduce the impossibility of such a counterexample.

LEMMA 3.1. *Let* f *be analytic and bounded in* $S_{\delta,r}$ *for some* $r < 1$, $\delta > 0$. *Suppose* $f \in \mathcal{X}_{\varepsilon}$ *for some* $\varepsilon \geqslant 0$. *Then* $f \in \mathcal{X}_{\rho}$ *for all* $\rho < \delta$.

In particular, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic and bounded in* $S_{\delta,r}$ *and if*

$$\sum_{n=0}^{\infty} a_n = L \quad (\mathrm{A})\,, \tag{3.3}$$

then $f(z) \to L$ *as* $z \to 1$ *in any sector* $\pi - \rho \leqslant \arg(z-1) \leqslant \pi + \rho$ *for* $\rho < \delta$.

Notice that the Lemma guarantees that $f \in \mathcal{X}_{\rho}$ for $\rho < \delta$, but it does not say that f belongs to $\mathcal{X}_{\delta}$. In general this is not true: the function $f(z) = \exp\left(-(1-z)^{-\alpha}\right)$ introduced before is an example for $\delta = \frac{\pi}{2\alpha}$, since $f(z) \to 0$ as $z \to 1$ in $\pi - \rho \leqslant \arg(z-1) \leqslant \pi + \rho$ if $\rho < \delta$, but when $|\arg(z-1) - \pi| = \delta$ the function $f(z)$ oscillates as $z \to 1$.

Another classical result [**10**, pg. 179] says that if h is analytic and bounded in a semi-strip $\Omega : \Re e\, z > 0,\ a < \Im m\, z < b$, and if the limits

$$\lim_{x\to\infty} h(x+ia) = L_a\,, \quad \lim_{x\to\infty} h(x+ib) = L_b\,, \tag{3.4}$$

exist, then $L_a = L_b = L$ and

$$\lim_{x\to\infty} h(x+iy) = L\,, \quad \forall y \in [a,b]\,. \tag{3.5}$$

The values of h along the boundary lines, $x+ia$ and $x+ib$ used in (3.4) are defined as we now explain. Observe that since h is analytic and *bounded* in Ω, then [**1, 9**] the angular boundary values

$$h(\xi) = \lim_{\substack{z\to\xi,\, z\in\Omega \\ \text{Ang.}}} h(z)\ , \tag{3.6}$$

exist almost everywhere in the boundary $\partial\Omega$, and thus $h(x+ia)$ and $h(x+ib)$ are defined almost everywhere in $x>0$ as angular boundary values.

Using an appropriate conformal mapping we deduce that if f is analytic in $|z|<1$, bounded in $S_{\delta,r}$ for some $r<1$, $\delta>0$, and if $f(z)$ has a limit when $z\to 1$ along the lines $\arg(z-1) = \pi-\delta$ and $\arg(z-1) = \pi+\delta$, then $f \in \mathcal{X}_\delta$.

However, it is more interesting to use a conformal mapping that sends the unit disc onto the semi-strip, with the point $z = e^{i\theta_0}$ corresponding to $\omega = \infty$, so that the lines $\Im m\,\omega = a$, $\Im m\,\omega = b$ correspond to arcs of the unit circle that end to the right and left of $z = e^{i\theta}$, respectively: we then deduce that if f is analytic and bounded in $|z|<1$, and if $\varphi(\theta) = f\left(e^{i\theta}\right)$, then φ cannot have a jump discontinuity at $\theta = \theta_0$. Other types of discontinuity are possible, but not those of jump type, for functions like φ that are boundary values of bounded analytic functions.

The hypothesis that f is bounded cannot be eliminated. Consider for instance the function

$$f(z) = \int_0^{Tz} e^{-\omega^2}\, d\omega\,, \tag{3.7}$$

where T is a conformal mapping that sends the unit disc to the upper half-plane, in such a way that $T(-1) = 0$, $T(1) = \infty$, and additionally $\lim_{\theta\to 0^\pm} T\left(e^{i\theta}\right) = \mp\infty$. Then $\varphi(\theta) = f\left(e^{i\theta}\right)$ has a jump discontinuity at $\theta = 0$, since $\varphi(0^\pm) = \pm\sqrt{\pi}/2$.

Nevertheless, there is a more general hypothesis that permits one to conclude that there are no jump discontinuities. Instead of assuming f bounded, we just ask that the limit of $f\left(re^{i\theta}\right)$ as $r\to 1^-$ exists in the *distributional* sense. It is convenient to point out that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then f has distributional boundary values in $|z|=1$ if and only if $a_n = O\left(|n|^\alpha\right)$, $n\to\infty$ for some α and this, in turn, is equivalent to the estimate $f\left(re^{i\theta}\right) = O\left((1-r)^{-\beta}\right)$, $r\to 1^-$ uniformly in θ for some β [**2**]. Also, in this case $\varphi(\theta) = f\left(e^{i\theta}\right)$ is a distribution, so that the lateral limits $\varphi\left(\theta_0^\pm\right)$ can be considered in the distributional sense of Łojasiewicz [**7**] (see also [**4**]).

Theorem 3.1. *Let f be analytic in $|z|<1$, with distributional limits as $|z|\to 1$. Let $\varphi(\theta) = f\left(e^{i\theta}\right)$. Let $\theta_0 \in \mathbb{R}$ be such that the lateral limits $\varphi\left(\theta_0^\pm\right)$ exist distributionally. Then $\varphi\left(\theta_0^+\right) = \varphi\left(\theta_0^-\right)$ and the distributional point value $\varphi(\theta_0)$ exists.*

Proof. We may suppose that $\theta_0 = 0$. If the distributional lateral limits exist, then there exists $N \in \mathbb{N}$ and constants $a_0, \ldots, a_N \in \mathbb{C}$ such that

$$\varphi(\varepsilon\theta) = \varphi\left(\theta^+\right) H(\varepsilon) + \varphi\left(\theta^-\right) H(-\varepsilon) + \sum_{j=0}^{N} a_j \delta^{(j)}(\varepsilon\theta) + o(1) , \tag{3.8}$$

as $\varepsilon \to 0^+$, where $H(\varepsilon) = 1$, $\varepsilon > 0$, $H(\varepsilon) = 0$, $\varepsilon < 0$, is the Heaviside function.

Suppose first that $a_0 = \cdots = a_N = 0$. Since f is analytic, then $f\left(re^{i\theta}\right) = \langle \varphi(\tau), P_r(\theta - \tau) \rangle$, where P_r is the Poisson kernel [**3**, Seccin 3.11]. Hence, if l_α is the line $\arg(z-1) = \alpha$, $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$, (3.8) shows that $f(z) \to \lambda\varphi(\theta^+) + (1-\lambda)\varphi(\theta^-)$ as $z \to 1$ along l_α, where $\lambda = \frac{3}{2} - \frac{\alpha}{\pi}$. But f is bounded in the sector between the lines l_{α_1} and l_{α_2}, and tends to limits $\lambda_i\varphi(\theta^+) + (1-\lambda_i)\varphi(\theta^-)$, $i = 1, 2$, along them. However, these limits must coincide, and if $\lambda_1 \neq \lambda_2$ this is possible only if $\varphi(\theta^+) = \varphi(\theta^-)$.

In the general case, when (3.8) holds, we integrate N times, after substraction of the corresponding constant, to obtain a primitive of order N of φ which is also a distributional boundary value and whose jump is precisely a_N. From what we already proved, $a_N = 0$. The result follows. □

A somewhat weaker version of the theorem is the following.

Corollary 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in $|z| < 1$. Suppose that f is bounded in $S_{\delta,r}$ for some $r < 1$, $\delta > 0$. If $f(z) \to S_\pm$ distributionally as $z \to 1$ along the lines* $\arg(z-1) = \pi \pm \delta$, *then $S_+ = S_- = S$ and $\sum_{n=0}^{\infty} a_n$ is Abel summable to S.*

Using these ideas we can introduce an extended notion of Abel summability. Let $\sum_{n=0}^{\infty} a_n$ be a series and construct the auxiliary series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, which we will suppose convergent for $|z| < 1$. Suppose now that the lateral limit $f(1^-) = \lim_{x\to 1^-} f(x) = S$ exists in the distributional sense of Łojasiewicz. Then we say that $\sum_{n=0}^{\infty} a_n$ is Abel-distributionally summable to S, and write

$$\sum_{n=0}^{\infty} a_n = S \quad \text{(A-dist.)} \tag{3.9}$$

Naturally, Abel summability implies Abel-distributional summability . Nevertheless, the reciprocal does not hold: if $\sum_{n=0}^{\infty} a_n$ is the series obtained by setting $z = 1$ in the Taylor series $\sum_{n=0}^{\infty} a_n z^n = \operatorname{sen}(z-1)^{-1}$, then $\sum_{n=0}^{\infty} a_n = 0$ (A-dist.) since $\lim_{x\to 1^-} \operatorname{sen}(x-1)^{-1} = 0$ distributionally, but since the limit does not exist in the ordinary sense, the series is not (A) summable. However, we have:

Theorem 3.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < 1$. Suppose that f is bounded in $S_{\delta,r}$ for some $r < 1$, $\delta > 0$. If $\sum_{n=0}^{\infty} a_n$ is Abel-distributionally summable, then it is Abel summable.*

Proof. If $\sum_{n=0}^{\infty} a_n$ is (A-dist.) summable, there exists $n \in \mathbb{N}$ such that

$$F(x) = \frac{1}{n!\,(1-x)^n} \int_x^1 (t-x)^n f(t)\, dt \tag{3.10}$$

has an ordinary limit S as $x \to 1^-$. But $F(z)$ is also bounded in $S_{\delta,r}$, therefore Montel's theorem guarantees that $F(z) \to S$ as $z \to 1$ in any sector $\pi - \rho \leqslant \arg(z-1) \leqslant \pi + \rho$ para $\rho < \delta$. The Corollary permits us to conclude that $\sum_{n=0}^{\infty} a_n$ is Abel summable to S, since $F(z) \to S$ as $z \to 1$ along $\arg(z-1) = \pi \pm \rho$ means that $f(z) \to S$ distributionally as $z \to 1$ along these lines. $\square$

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