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Some open problems related to 16b Hilbert problem

Javier Chavarriga and Maite Grau

Abstract. This article contains a brief summary of the topics and concepts related to 16*b* Hilbert problem which refers to the existence of a bound on the number of limit cycles of a polynomial system in function of its degree. Both algebraic and analytic perspectives are described. Moreover, collateral open questions related to the mentioned problem are given. Finally, the bibliography, which does not intend to be exhaustive, contains more than one hundred of references.

1. Introduction and historical precedents

The Algebraic Theory of Differential Equations on the Plane was born in the last third part of the XIX century with the ideas of Darboux [**40**], Poincaré [**82**], Painlevé [**77**] and Autonne [**3**]. The main goal of this theory is to characterize the differential equations which have a rational first integral or a first integral generated by rational functions. This study is initiated by Darboux [**40**] in its famous memoir of 1878 and, in a certain sense, has its epilogue in Poincaré's second article of 1897 [**82**], where the author writes:

> *. . . Je me suis occupé de nouveau de la même question dans ces derniers temps, dans l'espoir que je parviendrais à généraliser les résultats obtenus. Cet espoir a été déçu. J'ai obtenu cependant quelques resultats partiels, que je prends la liberté de publier, estimant qu'on pourra s'en servir plus tard pour obtenir, par un nouvel effort, une solution plus satisfaisante du problème.*

In the last years, this theory has been taken again and new and interesting results have been obtained.

J. Liouville showed, in the middle of the past century, the insolubility of certain differential equations by means of quadratures, that is, the impossibility of expressing its general solution by a combination of elementary functions or Liouville functions. An *elementary function* is an element of the mathematical closure

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of the set of polynomial functions, exponential functions, logarithmic functions, trigonometrical functions and inverse trigonometrical functions by the arithmetic operations and composition applied a finite number of times. A *Liouville function* or a function which can be expressed by means of quadratures is a function constructed from rational functions by using algebraic operations, composition, exponentials and integration. Liouville's works at the end of the XIX century gave a new point of view in the study of differential equations known as *Qualitative theory of ordinary differential equations.* The fundamental idea of this theory is the study and determination of the properties of the solution directly from the differential equation instead of seeking the solution itself since, in general, it is not known.

The qualitative theory is born from Poincaré [**80, 81**] and Liapunov's works [**66**] at the end of the XIX century and the beginning of the XX century. The main purpose consists on the qualitative description of the significant mathematical objects to well–understand the solutions' structure of a differential equation or of a system of differential equations. Poincaré, in [**80**], page 375, writes:

> *. . . Il est donc nécessaire d'étudier les fonctions définies par des equations différentielles en elles-mêmes et sans chercher à les ramener a des fonctions plus simples.*

These thoughts induced Poincaré to tackle the study of differential equations beyond an essentially different point of view from his predecessors. His study provokes a conceptual change on the understanding of differential equations. These equations, under Poincaré's point of view, are no more purely formal objects subject to calculus rules but they have a geometrical sense.

Moreover, Poincaré relies so much in these qualitative studies that he writes in the same memoir, page 377, about the applications of the results to be obtained:

> *Tel es le vaste champ de découvertes qui s'ouvre devant les géomètres. Je n'ai pas eu la prétention de le parcourir tout entier, mais j'ai voulu du moins en franchir les frontières.*

In our days, due to the great development of the computational science, the qualitative theory of differential equations has been reactivated not only in mathematics but also in affine sciences. This theory does not exclusively use methods of the classical theory of differential equations but also takes profit of the results and techniques of functional analysis, algebraic and differential geometry and algebraic topology.

In particular, the qualitative theory has a wide development for systems in the plane. From a topological point of view, the more important result may be the Poincaré–Bendixson Theorem which has no generalization to higher dimension. This theorem states that any bounded solution not tending to a singular point must necessarily be a periodic orbit, or tend to a limit cycle or tend to a graphic.

Both cited theories, algebraic and qualitative, are, in some sense, one complementary from the other. The algebraic one considers the differential equations like intrinsic objects to certain classes of functions and the qualitative one like independent objects by themselves.

2. Definitions and preliminary concepts

Let us consider a first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x,y)}{P(x,y)} \ , \tag{2.1}$$

where P and Q are functions $\mathcal{C}^k(\mathcal{U})$, $k \geqslant 1$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$. Following Poincaré's ideas, we introduce a parameter t, which is usually known as time, and which allows us expressing the former differential equation as a system of first order differential equations in the plane:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x,y) \ , \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x,y) \ . \tag{2.2}$$

If P and Q are polynomials with real coefficients, we define the *degree* of the polynomial system (2.2) by $m = \max\{\deg P, \deg Q\}$.

Let $\phi(t,(x_0,y_0))$ the solution curve of equation (2.2) passing through the point $(x_0,y_0) \in \mathcal{U}$. Let us recall that, by the existence and uniqueness Theorem, the solution curve of system (2.1) with initial condition $y(x_0) = y_0$ exists if $\frac{Q(x,y)}{P(x,y)}$ is a continuous function in a neighborhood of the point (x_0,y_0) and is unique if $\frac{Q(x,y)}{P(x,y)}$ satisfies the Lipschitz equation with respect to the variable y in a neighborhood of the point (x_0,y_0), that is, for any pair of points (x,y_1), (x,y_2) belonging to $\mathcal{U}$ $|\frac{Q(x,y_1)}{P(x,y_1)} - \frac{Q(x,y_2)}{P(x,y_2)}| \leqslant L|y_1 - y_2|$ is verified, where L is a positive constant.

We call *orbit* of system (2.2) the set

$$\Gamma = \{(x,y) \in \mathcal{U} \mid (x,y) = \phi(t,(x_0,y_0)), t \in \mathbb{R}\}.$$

We say that a point p is an *ω-limit point* of the solution curve $\phi(t,(x,y))$ of system (2.2) if there exists a succession $(t_n)_{n\in\mathbb{N}} \to \infty$ such that $\lim_{n\to\infty} \phi(t_n,(x,y)) = p$. Analogously, we say that a point q is an *α-limit point* of the solution curve $\phi(t,(x,y))$ of system (2.2) if there exists a succession $(t_n)_{n\in\mathbb{N}} \to -\infty$ such that $\lim_{n\to\infty} \phi(t_n,(x,y)) = q$. Let us notice that an α-limit point or an ω-limit point does not necessarily belong to the solution curve. The set of all ω-limit points of a trajectory is called *ω-limit set* of this curve and the set of all α-limit points of a trajectory is called *α-limit set* of this solution curve.

We say that a point (x_0,y_0) is *singular point*, also called critical point or equilibrium point, for system (2.2) if both $P(x_0,y_0) = 0$ and $Q(x_0,y_0) = 0$. We notice that, in general, for this point the existence and uniqueness theorem's hypothesis are not verified with respect to the variable x neither with respect to the variable y. This is because neither $\frac{Q(x,y)}{P(x,y)}$ nor $\frac{P(x,y)}{Q(x,y)}$ will be continuous functions at the point (x_0,y_0).

We call *periodic orbit* to any closed orbit which does not contain singular points. We call *homoclinic orbit* to an orbit such that its α-limit and its ω-limit meet in the same singular point. We call *heteroclinic orbit* to an orbit such that its α-limit and its ω-limit are two distinct singular points. We call *separatrix cycle* to a closed curve given by the union of two heteroclinic orbits and the singular points which connect them. The finite union of separatrix cycles with an adequate orientation is called a *graphic*. A *limit cycle* is an isolated periodic orbit.

In general, the solution curves and the orbits for system (2.2) do not need to be expressible by means of elementary functions. The first question to be asked about the solution curves of a polynomial system (2.2) of degree m is if these can be implicitly described by $f(x,y)=0$, where f is a polynomial with coefficients in $\mathbb{C}$. The equation $f(x,y)=0$, where $f(x,y)$ is a polynomial with complex coefficients of degree n, can be seen as an affine representation of an algebraic curve of degree n. Let us assume that (2.2) has a solution curve which is not a singular point, contained in an algebraic curve $f(x,y)=0$. It is obvious that the derivative of f with respect to t must vanish on the algebraic curve $f(x,y)=0$. This condition can be expressed by

$$\dot{f} = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = k f \ , \tag{2.3}$$

where $k(x,y) \in \mathbb{C}[x,y]$ is a polynomial of degree less or equal than $m-1$, where m is the degree of system (2.2). We call *invariant algebraic curve* for a polynomial system (2.2) to any algebraic curve given by $f(x,y)=0$, such that $f(x,y)$ verifies (2.3). It is easy to see that if $f(x,y)=0$ is an invariant algebraic curve and $f(x,y)=f_1^{s_1}(x,y)\cdots f_r^{s_r}(x,y)$ is a factorization in irreducible elements of $\mathbb{C}[x,y]$, then each of the factors $f_i(x,y)=0$ for $i=1,\cdots,r$ is also an invariant algebraic curve. We say that the algebraic curve $f(x,y)=0$ with $f \in \mathbb{C}[x,y]$ is an *algebraic solution* for system (2.2) if, and only if, it is an invariant algebraic curve and $f(x,y)$ is an irreducible polynomial in $\mathbb{C}[x,y]$.

Given a planar algebraic curve with degree n, we can always write it as $f(x,y)=f_r(x,y)+f_{r+1}(x,y)+\cdots+f_n(x,y)$ where $f_i(x,y)$ are homogeneous polynomials of degree i with $f_r(x,y)\neq 0$. We say that a point p belongs to the curve $f=0$ if $f(p)=0$. By means of a translation we can always assume that the point p is the origin. Therefore, if $r=0$ the curve does not contain the singular point. If $r=1$, we say that p is a *simple point* of f and if $r>1$ we have that $f_r=\prod_{i=1}^{k} l_i^{s_i}$, with $s_1+s_2+\cdots+s_k=r$, where each $l_i=a_i x+b_i y$ is a distinct factor of f_r. Each l_i is called a *tangent line of* $f=0$ *at the point* p and s_i is the *multiplicity of this tangent line*. We say that the origin is an *ordinary multiple point* if the multiplicity of all the tangents is 1. If not, it is called a *non ordinary multiple point*. In particular, for $r=2$, we say that the singular point is *double*; we call a *node* a double point with different tangent lines and we call a *cusp* a double point with equal tangent lines.

We say that a point p is a *dicritical* singular point of the vector field if there is an infinite number of solutions of the vector field which have the point p as α-limit or ω-limit.

Let U be an open set of $\mathbb{R}^2$, we say that the function $H \in C^k(U)$, not identically constant on U, is a *strong first integral* for system (2.2) if H is constant on each solution of system (2.2) defined on U. Here, $k \geqslant 0$ means that $k=0,1,2,\cdots,\infty,\omega$, that is, $k=0$ means that H is continuous, $k=1,2,\cdots,\infty$ means that H is of class

C^k and $k = \omega$ means that H is analytic. If $k \geqslant 1$ then the previous definition of integrability implies that the derivative of H in the direction given by the vector field $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ is zero, that is, $XH = 0$ in U.

This definition of strong first integral is the usual definition of first integral given in most part of the books about differential equations, see for instance Arnold [**2**] or Sotomayor [**93**]. We notice that with this definition, the linear system $\dot{x} = x$, $\dot{y} = y$ defined in $\mathbb{R}^2$ has no continuous strong first integral. If this system has a first integral H defined in an open neighborhood U of the origin, since this function has a constant value for each straight line through the origin, then H would be constant on U, in contradiction with the definition. To avoid this problem, we say that a function $H \in C^k(U)$, not constant on U, with $k \geqslant 0$, where U is an open set of $\mathbb{R}^2 \setminus \Sigma$ is a *weak first integral* of system (2.2) if H is constant on each solution of the system (2.2) contained in U. Now, with this definition, the function $H(x, y) = xy/(x^2 + y^2)$ is a weak first integral of the linear system $\dot{x} = x$, $\dot{y} = y$, in $\mathbb{R}^2 \setminus \{(0, 0)\}$, see [**11**].

We say that a function $R \in C^k(U)$ with $k \geqslant 1$, not identically null in U, is an *integrating factor* of system (2.2) in U if

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}\ , \tag{2.4}$$

in this case a first integral H is given by this integrating factor R

$$H(x, y) = \int R(x, y)P(x, y)\, dy + f(x), \tag{2.5}$$

where $\frac{\partial H}{\partial x} = -RQ$.

Let U be an open set of $\mathbb{R}^2$, we say that the function $V \in C^k(U)$ with $k \geqslant 1$, not identically null in U, is an *inverse integrating factor* of system (2.2) in U if it satisfies the following linear equation in partial derivatives:

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\ V. \tag{2.6}$$

We notice that this function V is a particular solution of system (2.2). The expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called *divergence* of system (2.2). This function V is very important because $R = 1/V$ defines in $U \backslash \{V = 0\}$ an integrating factor of system (2.2) which let us determine a first integral for system (2.2) in $U \backslash \{V = 0\}$. In [**56**] it is proved that this function V must be null over all the limit cycles contained in U.

3. Invariant algebraic curves

The knowledge of algebraic solutions of system (2.2) allows, in general, a greater comprehension of the dynamics of the system and can be used to show topological properties of the system. The algebraic solutions and the integrability are narrowly related as it can be seen in Darboux's works. The development of the qualitative theory, initiated and founded by Poincaré, has relegated during many years the works due to Darboux to a second position. Nowadays, this theory has been rediscovered and revised by several authors. Darboux's works give a different point of view based in algebraic solutions of differential equations extended to the complex projective plane $\mathbb{CP}^2$. Darboux shows in "*Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré* " (1878) the way of constructing first integrals of polynomial systems of differential equations with a number of algebraic solutions high enough. In particular, Darboux showed that a polynomial system of degree m with at least $m(m+1)/2+1$ invariant algebraic curves has a first integral which can be expressed by means of these algebraic curves. Darboux's idea consists on looking for a first integral of system (2.2) with the form

$$H(x,y) = \prod_{i=1}^{q} f_i^{\lambda_i}(x,y),$$

where $\lambda_i \in \mathbb{C}$ and $f_i(x,y)=0$ are invariant algebraic curves of system (2.2). The former first integral is called a *Darboux first integral*. In general, a Darboux first integral is a weak first integral.

Jouanoulou [**63**] showed in 1979 that if the number of algebraic solutions for a polynomial system of degree m is at least $m(m+1)/2+2$, then the system has a rational first integral and all the solutions of system (2.2) are algebraic. Prelle and Singer [**84**] showed in 1983 that if a polynomial system has an elementary first integral then this integral can be computed using the algebraic solutions of the system; in particular they showed that this polynomial system admits an integrating factor which is a rational function with coefficients in $\mathbb{C}$, see [**84**]. Later on, Singer [**91**] in 1992 showed that if a polynomial system has a Liovillian first integral then the system has an integrating factor of the form

$$R(x,y) = exp\left(\int_{(x_0,y_0)}^{(x,y)} U(x,y)\,dx + V(x,y)\,dy\right), \tag{3.1}$$

where U and V are rational functions which verify $\frac{\partial U}{\partial y} = \frac{\partial V}{\partial x}$. We call *generalized Darboux functions* to the functions of the form $H = \mathrm{e}^g \prod f_i^{\lambda_i}$ where g is a rational function, f_i are polynomials and $\lambda_i \in \mathbb{C}$. Colin Christopher [**30**] in 1994 showed that the integrating factor (3.1) can be written as a generalized Darboux integrating factor. Finally, the contribution of Chavarriga, Llibre and Sotomayor [**26**] must be noted since they show that if a polynomial system has certain singular points the number of algebraic solutions necessary to have a first integral expressible by means of them can be reduced. From all these works we have that given a polynomial system of degree m either the system has a finite number of algebraic solutions strictly lower than $m(m+1)/2+2$ either the system has an infinite number of algebraic solutions and admits a rational first integral of the form $H = \sum_{i=1}^{q} f_i^{n_i}$, where $n_i \in \mathbb{Z}$. Moreover, the degree of each algebraic solution is bounded by $N = max\{\sum_{n_i>0} n_i \deg f_i, \sum_{n_i<0} |n_i| \deg f_i\}$.

Other improvements and related results to the Theorem of Darboux integration have been published for several authors like Christopher and Kooij [**65**], Gasull [**55**] and Żołądek [**101**].

The Memoir of Darboux deals with differential equations in the complex projective plane $\mathbb{CP}^2$. Let us recall that the space $\mathbb{CP}^2$ is defined as $\{\mathbb{C}^3 - \{(0,0,0)\}\}/\sim$ where $[X,Y,Z] \sim [X',Y',Z']$ if, and only if, there exists a non null $k \in \mathbb{C}$ such that $[X',Y',Z'] = k[X,Y,Z]$. Similarly, we can define $\mathbb{RP}^2$.

A differential equation of first order and degree m in the projective plane is an expression

$$(MZ - NY)dX + (NX - LZ)dY + (LY - MX)dZ = 0, \tag{3.2}$$

where L, M, N are homogeneous polynomials of degree m in the variables X, Y, Z, and without common divisors.

A polynomial system (2.2) of degree m, with $(P,Q) = 1$, can be extended to a differential equation in the complex projective plane just defining $L(X,Y,Z) = Z^m P(X/Z, Y/Z)$, $M(X,Y,Z) = Z^m Q(X/Z, Y/Z)$ and $N(X,Y,Z) \equiv 0$.

Darboux showed that the number of singular points in $\mathbb{CP}^2$ of a differential equation (3.2) of degree m in the projective space is m^2+m+1 (counted with their multiplicity). To show this result and other ones of some importance he stated the following Lemma, written in its original form.

Darboux's Lemma. *Let A, A', B, B', C, C' be six homogeneous polynomials in the variables (X,Y,Z) such that $AA' + BB' + CC' = 0$. We denote by l, l', m, m', n, n' the degrees of A, A', B, B', C, C', respectively. Let h be the number of common points to A, B, C and h' the number of common points to A', B', C'. Assume that $h, h' < \infty$. Then*

$$h + h' = \frac{lmn + l'm'n'}{k} \tag{3.3}$$

where $k = l + l' = m + m' = n + n'$.

It is not difficult to see that this Lemma is false. Let us consider the following polynomials, which verify all the hypothesis of Darboux's Lemma,

$$A = y - x, A' = y + x, B = x, B' = x, C = y, C' = -y.$$

The common points to A, B, C are $P(0,0,1)$ and then $h = 1$. The common points to A', B', C' are $P(0,0,1)$ and then $h' = 1$. It is clear that Darboux's statement is not satisfied since $h + h' = 2$ and $(lmn + l'm'n')/(l + l') = (1+1)/(1+1) = 1$.

The mistake in the proof made by Darboux comes from a bad comprehension of the indexes h, h' given in the Lemma. In actual language "the number of common points" means the sum of the intersection indexes of the considered polynomials over common points.

We define the *local ring in the point* $P \in \mathbb{C}^2$, $\mathcal{O}_P(\mathbb{C}^2)$, as the subring of the quotient field on $\mathbb{C}$ formed by the set of rational functions on $\mathbb{C}^2$ which are defined on P. That is,

$$\mathcal{O}_P(\mathbb{C}^2) = \Big\{ \frac{f}{g} \,:\, f \text{ and } g \text{ are polynomials and } g(P) \neq 0 \Big\}.$$

We can define the intersection index of the polynomials p_1, p_2, …, p_s in P by

$$I_P(p_1, p_2, \ldots, p_s) = \dim_{\mathbb{C}} \mathcal{O}_P / \left((p_1, p_2, \ldots, p_s) \cap \mathcal{O}_P \right),$$

where $(p_1, p_2, \ldots, p_s)$ is the ideal of $\mathbb{C}^2$ generated by the polynomials p_i, $i = 1, 2, \ldots, s$.

Therefore, in actual notation, the indexes appearing in Darboux's Lemma read for $h = \sum_P I_P(A, B, C)$ and $h' = \sum_P I_P(A', B', C')$.

Recently, Chavarriga, Llibre and Moulin-Ollagnier [**24**] have proved a correct version of this Lemma: if there are no common points to the six polynomials A, A', B, B', C, C' the equality (3.3) is true. If there are common points, the following inequality is true:

$$h + h' \geqslant \frac{lmn + l'm'n'}{k}. \tag{3.4}$$

Problem 3.1. *Find h'' and its algebraic meaning in the given context which verifies $h + h' - h'' = \frac{lmn + l'm'n'}{k}$.*

The notion of algebraic particular solution of a given system (2.2) can be extended to a differential equation in the projective plane (3.2). Let $F(X, Y, Z) = 0$ be a homogeneous polynomial of degree n. In this case $F(X, Y, Z) = 0$ is an algebraic particular solution of system (3.2) if there exists a homogeneous polynomial $K(X, Y, Z)$ of degree $m - 1$ such that

$$\frac{\partial F}{\partial X} L + \frac{\partial F}{\partial Y} M + \frac{\partial F}{\partial Z} N = KF.$$

The former expression can be rewritten considering the homogeneity of the polynomial F by

$$\frac{\partial F}{\partial X}\left(L - \frac{KX}{n} \right) + \frac{\partial F}{\partial Y}\left(M - \frac{KY}{n} \right) + \frac{\partial F}{\partial Z}\left(N - \frac{KZ}{n} \right) \equiv 0. \tag{3.5}$$

We notice that we are under Darboux's conditions, where $A = \frac{\partial F}{\partial X}$, $B = \frac{\partial F}{\partial Y}$, $C = \frac{\partial F}{\partial Z}$, $A' = \left(L - \frac{KX}{n} \right)$, $B' = \left(M - \frac{KY}{n} \right)$, $C' = \left(N - \frac{KZ}{n} \right)$. We know that the number of singular points of the system is $m^2 + m + 1$. There are two types of singular points for the system: those which are on the invariant algebraic curve $F = 0$ and those which are on the cofactor K.

Let

$$h = \sum_P I_P \left(L - \frac{KX}{n}, M - \frac{KY}{n}, N - \frac{KZ}{n} \right)$$

and let

$$h' = \sum_P I_P \left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \right)$$

which is called the Milnor number of F. Applying Darboux's Lemma to equation (3.5) we have

$$h + h' \geqslant \frac{(n-1)^3 + m^3}{m+n-1} = (n-1)^2 - (n-1)m + m^2. \tag{3.6}$$

It is easy to see that $h \leqslant m^2$ from which we deduce $h' \geqslant (n-1)(n-m-1)$.
Let $f(x,y) = 0$ be an algebraic solution of system (2.2). Let $F(X,Y,Z) = 0$ be the projectivization of this curve which is an algebraic solution of system (3.2). Then, if $h = m^2$ and $\deg f > 1$ system (2.2) has a rational first integral. The proof is given in [**23**]. If the curve $F = 0$ is not singular then $h' = 0$ and in this case $n \leqslant m+1$, that is, a polynomial system of degree m only admits non singular algebraic particular solutions of degree at most $m+1$. If the curve $F = 0$ is singular then its degree can be greater than $m+1$.

A polynomial system (2.2) has a rational first integral when it has a first integral which is quotient of two polynomials $H(x,y) = f(x,y)/g(x,y)$. If the system has a rational first integral, then all its trajectories are algebraic and are described by $f(x,y) - cg(x,y) = 0$ where c is any constant. We define the degree p of H as the maximum of the degrees of f and g. Poincaré showed that if a system has a rational first integral it is always possible to find a *minimal integral*, that is, a rational function $H(x,y) = f(x,y)/g(x,y)$ verifying $(f,g) = 1$, $\deg f = \deg g = p$ with p minimum over all the degrees of the rational first integrals of the system. This notion can be naturally extended to differential equations in the projective plane. In this case $H(X,Y,Z) = F(X,Y,Z)/G(X,Y,Z)$ where F and G are homogeneous polynomials with the same degree p, that is H is a homogeneous function of degree 0.

Given a differential equation in the projective plane such that $m^2 + m + 1$ singular points are all distinct then it is easy to prove that the eigenvalues associated to a singular point are all different from zero. Under these assumptions, a necessary condition for the differential equation to have a rational first integral is that the quotient of the eigenvalues related to each singular point is a rational number and that there are no *logarithmic singular points*, that is points whose associated Jordan matrix is not diagonal. Under these hypothesis, Poincaré denotes by *node* a singular point whose quotient of eigenvalues is a positive rational number and *saddle* if this quotient is a negative rational number. In particular, if this quotient equals 1, then the node is called *dicritical*. We notice that if we consider a polynomial system (2.2) in the affine plane, the former conditions must be verified for all the singular points, real or complex, finite or not.

One open problem stated for first time by Poincaré [**82**] and afterwards by Prelle and Singer [**84**] is

PROBLEM 3.2. *Give a method to find an upper bound $N(m)$ to the degree of the algebraic solutions for a fixed polynomial system (2.2) of degree $m \geqslant 2$.*

The essential difficulty to solve this problem comes with the existence of saddle points and in particular the *critical remarkable values.* Given a minimal rational integral $f(x,y) - cg(x,y) = 0$ of a polynomial system (2.2), a value c_0 is called *remarkable* if $f(x,y) - c_0 g(x,y) = 0$ is reducible. Moreover, if the decomposition of $f(x,y) - c_0 g(x,y)$ has factors elevated to powers strictly higher than 1, the value c_0 is called *critical remarkable.* Poincaré shows that the number of saddles is the number of critical remarkable values minus 2.

One way to find algebraic solutions of a given polynomial system (2.2) consists on computing the s-extactic curve. Let $v_1, v_2, \ldots, v_l$ be a basis of the polynomials of $\mathbb{C}[x,y]$ with degree $\leqslant s$. We have that $l = \frac{(s+1)(s+2)}{2}$. The *s-extactic curve* associated to system (2.2) is

$$\varepsilon_s := \det \begin{bmatrix} v_1 & v_2 & \cdots & v_l \\ \dot{v_1} & \dot{v_2} & \cdots & \dot{v_l} \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{l-1} & v_2^{l-1} & \cdots & v_l^{l-1} \end{bmatrix},$$

where $\dot{v_i} = p\frac{\partial v_i}{\partial x} + q\frac{\partial v_i}{\partial y}$ for $i = 1, \ldots, l$ and $v_i^{(j)} = \dot{v_i}^{(j-1)}$ for $i = 1, \ldots, l$ and $j = 2, \ldots, l-1$.

The definition of extactic curve is independent from the chosen basis. It is easy to prove that if $f(x,y) = 0$ is an algebraic solution of degree lower or equal than s of system (2.2), then $f(x,y)$ must be a factor of the s-extactic curve. In case the system has a rational first integral, it is easy to see that from a certain order all the extactic curves are identically null. This gives an algorithmic method which allows the computation of invariant algebraic curves and which characterizes the polynomial systems with a rational first integral. We must notice that this method is computationally inefficient. This method has been recently rediscovered by Pereira [**78**], although historically Lagutinskii is the first to show it. We must recognize the excellent work due to Dobrovol'skii, Lokot and Strelcyn [**42**] about this Russian mathematician.

It is important to notice that for a fixed degree $m \geqslant 2$ there is no upper bound $N(m)$ for the degree of an irreducible invariant algebraic curve depending only on the degree m of the system. For $m = 1$, the system $\dot{x} = px$, $\dot{y} = qy$, where p and q are positive integers, with $p \neq q$, has the rational first integral $H(x,y) = x^q y^{-p}$ and has algebraic solutions of arbitrarily high degree of the form $y^p - x^q = 0$. Just considering different values for p and q, the degree of the algebraic solution may be any value. For $m = 2$, the system

$$\dot{x} = y + -\frac{2}{n}xy \ , \quad \dot{y} = -x + \frac{n+2}{n}(x^2 - y^2) \ ,$$

has the rational first integral

$$H(x,y) = \left[\left(1 - \frac{n+2}{n}x\right)^2 - \frac{n+2}{n}y^2\right]\left(1 - \frac{2}{n}x\right)^{-n-2},$$

and has algebraic solutions of degree $n+2$.

Another important problem related with the previous one and which has been recently solved, independently by Moulin-Ollagnier [**75**], Christopher and Llibre [**33**] and Chavarriga and Grau [**20**] is stating if the degree of the irreducible algebraic particular solutions is bounded only in function of the degree of the system when this system has no rational first integral. The answer is negative. Let us consider the family of systems

$$\dot{x} = 1, \ \dot{y} = 2n + 2xy + y^2, \tag{3.7}$$

depending only on the parameter n. For $n \in \mathbb{N}$, the system (3.7) has an unique irreducible invariant algebraic curve. This curve has degree $n+1$ and cofactor $2x+y$, and is

$$h(x,y) = H_n(x)y + 2nH_{n-1}(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n.

System (3.7) has no Darboux first integral nor Darboux integrating factor for any value of $n \in \mathbb{N}$. In particular, system (3.7) has no rational first integral for any $n \in \mathbb{N}$.

System (3.7) has no generalized Darboux first integral and has a generalized Darboux integrating factor

$$\frac{e^{x^2}}{h(x,y)^2}.$$

PROBLEM 3.3. *Are there polynomial systems of degree m with algebraic particular solutions of degree arbitrarily high without Darboux or generalized Darboux first integral and integrating factor ?*

Related results to this problem are given by Cerveau and Lins Neto [**7**] who show that if an algebraic solution only contains singular points of node type then $N(m) \leqslant m+2$; Carnicer [**6**] proved that if the system has no dicritical singularities then $N(m) \leqslant m+2$; Chavarriga and Llibre [**23**] have proved that if an algebraic solution has no multiple points then $N(m) \leqslant m+1$ and if $N(m) = m+1$ then the system has a rational first integral.

Darboux [**40**] studies the quadratic projective differential equations with one or several invariant algebraic curves with degree lower or equal than three. Druzhkova [**45**] in 1968 gives, in function of the coefficients of the quadratic system the sufficient and necessary conditions for the existence and uniqueness of an invariant conic. Evdokimenko [**50, 51, 52**] in 1970 makes the following step in the study of the quadratic systems with a particular solution. In these works the conditions for the existence of a cubic invariant are given, in the particular case that its cofactor has the form $K(x,y) = ax + by$. The works of Evdokimenco tend to establish the existence of algebraic limit cycles given by an invariant cubic. In fact, he proves, except for some errors posteriorly corrected, that there are no invariant cubics with a connex component being a limit cycle of a quadratic system. Chavarriga, Llibre and Moulin-Ollagnier [**24**] have proved the same result by a more simple reasoning.

4. Limit cycles and 16 Hilbert problem.

The study of limit cycles has been one of the first lines of investigation in the dynamical systems theory starting with Poincaré in 1882. From the obtained results in the first decades of the XX century we must highlight the work of Dulac [**44**] in 1923 on the finiteness of the number of limit cycles for a given polynomial system, which has been recently corrected by Ilyashenko [**61**] and by Ecalle [**49**]. The results of Ilyashenko and Ecalle are two independent works very difficult to understand by the mathematical community. In this context, the work due to B. van de Pol [**95**] in 1926 is also important. He shows the existence of limit cycles using graphical methods. Finally, we remark the work of Liénard [**68**] in 1928 about non linear oscillators. During the twenties, the Italian biologist U. D'Ancona, with the help of the mathematician V. Volterra, looks for a scientific explanation of the number of a certain type of fish captured in Fiume's harbor (Italy) between 1914 and 1923. The answer given by Volterra [**97**] is a planar differential equation giving the relationship between preys and predators. Afterwards, the works of A. Kolmogorov [**64**] in 1934 give an important impulse to these prey-predator models.

There is a relationship between the theory of limit cycles and invariant algebraic curves for planar polynomial systems as was suggested by Hilbert [**60**] when stating his famous sixteenth problem. This problem is divided in two parts: (a) about the topology of real algebraic curves and, in particular, their number and distribution of ovals, (b) about the maximum number of limit cycles $H(m)$ that a system (2.1) of degree m may have. This problem is restated by S. Smale in his proposal of problems for the XXI century [**92**].

13 Problem of Smale. 16 Problem of Hilbert part(b)
Consider the differential equation in $\mathbb{R}^2$

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y),$$

where P and Q are polynomials. Is there a bound K on the number of limit cycles of the form $K \leqslant d^q$ where d is the maximum of the degrees of P and Q, and q is a universal constant ?

This problem is one of the most difficult problems to solve of the list given by Hilbert at the beginning of the past century. Only two problems are to be solved in his list, the eighth (Riemman's hypothesis about the zeros of the zeta function) and the sixteenth.

Although the supposed-to-be correct proof of Dulac's Theorem about the finiteness of limit cycles given by Ecalle and Ilyashenko just described before, the existence of an upper bound for the number of limit cycles for a system (2.2) in function of m is still unknown. There is an extensive program developed by Dumortier, Roussarie and Rousseau among others, to show that this upper bound exists for quadratic systems. This program is based on the study of the bifurcations of limit cycles from all the graphics that can be realized by quadratic systems. By using different methods, inferior bounds on the number of limit cycles for a system (2.2) have been found. We remark the latest bound given by Christopher and Lloyd [**35**] who show that $H(m) \geqslant m^2 \log m$, where m is the degree of the system.

We notice that the study of invariant algebraic curves let us better understand the existence and uniqueness of limit cycles. For instance, a quadratic system $m = 2$ with two invariant straight lines has no limit cycles. A quadratic system with one invariant straight line has at most one limit cycles. The first result was given by Bautin using Bendixson-Dulac criterion to ensure the non-existence of limit cycles.

One way to weaken sixteenth Hilbert's problem, strengthening the hypothesis, is the study of the *algebraic limit cycles*, that is, algebraic curves $f(x, y) = 0$ which are particular solutions of (2.2) with a real closed oval which is a limit cycle. Several mathematicians have studied algebraic limit cycles getting results like that there are quadratic systems with algebraic limit cycles of degrees 2 and 4 but not of degree 3. In [**22**] a summary of these results can be found with a new and shorter proof of the non-existence of algebraic limit cycles of degree 3. Chavarriga, Llibre and Sorolla [**25**] have found all the cases of algebraic limit cycles of degree 4 for quadratic polynomial systems and Chavarriga, Giacomini and Llibre [**12**] have proved their uniqueness. Christopher, Llibre and Świrszcz have given examples of quadratic systems with an algebraic limit cycle of degree 5 and 6. A related open problem is:

PROBLEM 4.1. *Find, if they exist, polynomial systems of a fixed degree m with algebraic limit cycles of arbitrarily high degree.*

On the other hand, only few mathematicians have worked with algebraicity. Odani [**76**] initiated in 1995 the study of the existence of invariant algebraic curves for Liénard systems

$$\frac{dx}{dt} = \dot{x} = y, \qquad \frac{dy}{dt} = \dot{y} = -f(x)y - g(x), \tag{4.1}$$

where the polynomials f and g have degrees m and n respectively. He showed that these systems do not have invariant algebraic curves if the following conditions are satisfied: (i) f , $g \neq 0$, (ii) $\deg f \geqslant \deg g$ and (iii) $g(x)/f(x)$. In particular, these conditions imply that the famous limit cycle of Van der Pol's equation

$$\frac{dx}{dt} = \dot{x} = y, \qquad \frac{dy}{dt} = \dot{y} = -x + k(x^2 - 1)y, \tag{4.2}$$

where k is a parameter, is not algebraic. One year later, in 1996, Hayashi [**59**] has studied the invariant algebraic curves of system (4.1) when $n = m+1$. An excellent study of all these curves is due to Żołądek [**105**], who studies the cases $n > m$. He proves that for each pair (m, n) with $n > m$ there is always a system with an invariant algebraic curve. Moreover, he shows that in the case $n > m+1 > 2$ and $(m, n) \neq (2, 4)$ there are always systems with an algebraic limit cycle. However, for the case $(m, n) = (1, 3)$, as Żołądek states, the problem of existence of algebraic limit cycles is still open. Dumortier, Li and Rousseau in [**47**] and [**48**] show that these systems have, at most, one limit cycle and, in case it exists, it is hyperbolic.

Problem 4.2. *Let us consider a Liénard system* (4.1) *with* $f(x) = a_0 + a_1 x$, $g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$, $a_1 b_3 \neq 0$, *and* $x, y, t, a_i, b_j \in \mathbb{R}$. *Study if it has algebraic limit cycles.*

Last but not least, Giacomini and Neukirch [**57**] have found a powerful method to approximate non-algebraic limit cycles using algebraic curves with ovals tending to the limit cycle. This is an empirical method and, at the moment, there is no rigorous foundation for it.

5. The center problem

Poincaré in [**80**] defined the notion of *center* for a real system of differential equations in the plane. We consider a system of differential equations of class $\mathcal{C}^k(U)$, with $k \geqslant 1$ and U an open set of $\mathbb{R}^2$

$$\dot{x} = P(x, y) \ , \quad \dot{y} = Q(x, y) \ . \tag{5.1}$$

A center of the system (5.1) is an isolated singular point $O \in \mathcal{U}$, that is, $P(O) = Q(O) = 0$ with a neighborhood $\mathcal{V} \subset \mathcal{U}$ such that any $p \in \mathcal{V}\backslash\{O\}$ verifies $P^2(p) + Q^2(p) \neq 0$ and the integral curve which passes through p is closed around O.

Let us consider system (2.2) where P and Q are polynomials of degree m with an isolated singular point which we translate to the origin. Let us consider the case where the linear part of (2.2) has pure imaginary eigenvalues $\lambda = \pm \mathrm{i}\omega$ with $\omega \in \mathbb{R}$ and $\omega \neq 0$. In this case, we can consider a linear change of variable and a reescalation of time and system (2.2) reads for

$$\begin{aligned} \dot{x} &= -y + \sum_{s=2}^{m} X_s(x, y) \ , \\ \dot{y} &= x + \sum_{s=2}^{m} Y_s(x, y) \ , \end{aligned} \tag{5.2}$$

with X_s and Y_s homogeneous polynomials of degree s. Poincaré showed in [**81**] that the origin of system (5.2) is a center if, and only if, there exists an open neighborhood of the origin of system (5.2) which has an analytic first integral. Liapunov [**66**] extended the previous result to analytic systems.

By using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in (5.2), we call $r(t, r_0, \theta_0)$ and $\theta(t, r_0, \theta_0)$ the solution of the non linear system (5.2) which passes through the point (r_0, θ_0). We say that the origin is a *stable focus* if there exists $\delta > 0$ such that for each $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$ we have that $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ when $t \to \infty$. We say that the origin is an *unstable focus* if there exists $\delta > 0$ such that for each $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$ we have that $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ when $t \to -\infty$. Any trajectory verifying the former conditions is said to *spiral* around the origin. The *center problem* consists on distinguishing when the origin of system (5.2) is a center or a focus.

The first analytic technique to tackle this problem was introduced by Poincaré in [**81**]. This technique consists on finding a formal power series $H(x, y) = (x^2 + y^2)/2 + H_3 + H_4 + \cdots$ such that each H_i is an homogeneous polynomial of degree i, with $i \geqslant 2$, verifying

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \sum_{i=1}^{\infty} V_i (x^2 + y^2)^{i+1}$$

where $\dot{H}$ is the derivative of the function H with respect to the parameter t. Developing the former expression, we have

$$(x + \frac{\partial H_3}{\partial x} + \frac{\partial H_4}{\partial x} + \cdots)(-y + \sum_{s=2}^{m} X_s(x, y))$$

$$+(y + \frac{\partial H_3}{\partial y} + \frac{\partial H_4}{\partial y} + \cdots)(x + \sum_{s=2}^{m} Y_s(x, y)) = V_1(x^2 + y^2)^2 + V_2(x^2 + y^2)^3 + \cdots$$

We group together the terms by the powers of x and y, then to each order we get successive linear systems whose unknowns are the coefficients of H_i. These systems can be compatible or can be solved by choosing a certain constant V_i. These V_i are called *Poincaré-Liapunov constants*. An open problem is

Problem 5.1. *Determine the convergence radius of the series $H(x, y)$.*

This problem is far from being solved in general. This radius is thought to be limited by the solution curves which have as α-limit set or ω-limit set other singular points of system (5.2). Leaving out the convergence, the construction of the series $H(x, y)$ is very useful since we can compute H until encountering the first term with $V_i \neq 0$ and the sign of V_i gives the stability of the origin.

Taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in (5.2) we get

$$\begin{aligned} \dot{r} &= O(r^2)\ , \\ \dot{\theta} &= 1 + O(r)\ . \end{aligned} \tag{5.3}$$

Let $r(t, r_0, \theta_0)$, $\theta(t, r_0, \theta_0)$ be the solution with initial conditions $r(0, r_0, \theta_0) = r_0$ and $\theta(0, r_0, \theta_0) = \theta_0$. Then, for $r_0 > 0$ sufficiently small $\theta(t, r_0, \theta_0)$ is an strictly increasing function of t. Let $t(\theta, r_0, \theta_0)$ be the inverse of the former strictly increasing function. For a certain θ_0 we define the function $L(r_0) = r(t(\theta_0 + 2\pi, r_0, \theta_0), r_0, \theta_0)$.

This function is an analytic function of r_0, for r_0 sufficiently small and it is known as *Poincaré map.* We consider the Poincaré map associated to system (5.2) over the points on the axe OX^+ and $r < r_0$, we develop it in power series of $r = \sqrt{x^2+y^2}$ and we subtract r. We obtain a power series in r such as

$$L(r) - r = L_1 r + L_2 r^2 + L_3 r^3 + \cdots$$

Frommer in 1934 showed that if $V_1 = V_2 = \cdots = V_{p-1} = 0$ and $V_p \neq 0$ then $L_1 = L_2 = \cdots = L_{2p} = 0$ and $L_{2p+1} \neq 0$ and, therefore, we have that if $V_i = 0$ for all i, then the origin is a center, that is, there is a neighborhood foliated by periodic orbits because the function $L(r) - r \equiv 0$ for all $r < r_0$ and, therefore, the Poincaré map is the identity. In fact, it can be proved that $L_{2p+1} = 2\pi V_p$ modulus the annulation of the previous constants. On the other hand these V_i obtained by this process are polynomials on the coefficients of system (5.2). The infinite number of conditions $V_i = 0$ to ensure the existence of a center instead of a focus has a finite number of generators by Hilbert's basis Theorem since the ring of the polynomials with variables the coefficients of the system is noetherian. Therefore, the number of conditions to show the existence of a center instead of a focus for a given polynomial system (5.2) is finite. A way of obtaining such generators is still an open problem. This problem is a problem of computational algebra since the Poincaré-Liapunov constants are polynomials with rational coefficients and there are recursive methods to obtain them. The development of environments for technical computing has allowed the computation of the first constants. One of the more important difficulties is that there are no efficient algorithms to determinate simple sets of generators. In the basis, the greatest difficulty comes from the decomposition of integers in prime numbers.

From the study and annulation of the Poincaré-Liapunov constants we get distinct families with a center at the origin. Thanks to the works of Dulac [**43**] in 1908, Kapteyn [**62**] in 1911, Frommer [**54**] in 1934 and Bautin [**4**] in 1952 we know that the systems (5.2) with $m = 2$ have three independent Poincaré-Liapunov constants. The first integrals for the four center cases where given by Lunkevich and Sibirskii [**72**] in 1965. For the homogeneous cubic systems there are five independent constants since it has been proved by Żołądek [**102**] in 1994 and the first integrals for the three center cases where given by Chavarriga [**8**] in 1985. Some center cases are known for families of systems (5.2) as for instance, the systems of the form

$$\dot{x} = -y + P_s(x,y), \quad \dot{x} = x + Q_s(x,y) \tag{5.4}$$

where P_s and Q_s are homogeneous polynomials of degree s, see [**13, 14**]; and the characterization of the centers for a cubic system, that is, $\dot{x} = -y + P_2 + P_3$, $\dot{x} = x + Q_2 + Q_3$ is not still complete. We remark the works due to Żołądek [**104**] in this direction. Only some subfamilies have been completely characterized, as for instance, the cubic systems with *degenerate infinity*, see [**69, 15**].

Problem 5.2. **(a)** *Complete the characterization of centers for systems with the form (5.4) in case $s = 4$ and $s = 5$.* **(b)** *Complete the characterization of centers for cubic systems.*

Frommer observed the relationship between Poincaré-Liapunov constants and local limit cycles around the origin, which are also called *infinitesimal limit cycles*. Bautin [**4**] in 1952 showed the existence of at most three infinitesimal limit cycles for a quadratic system when the origin is a focus point. Petrovskii and Landis [**79**] in 1955 wrongly stated that the maximum number of limit cycles for a quadratic system is three, that is $H(2) = 3$. Petrovskii and Landis themselves in 1967 recognized an error in a Lemma which invalidated their result. In 1979, Shi Songling [**88**] and simultaneously Chen L. and Wang M. [**29**] found examples which showed that $H(2) \geqslant 4$. Shi Songling [**89**] proved in 1981 that for a system (5.2) the maximum number of infinitesimal limit cycles is $M(n)$, where $M(n)$ is the minimum number of generators of the ideal generated by Poincaré-Liapunov constants when these ones verify certain hypothesis. The proof basically consists in truncating the formal power series given by $H(x, y)$, and applying Poincaré-Bendixson's theory to the rings formed by their level curves. In 1987 Li J. and Huang Q. [**67**] found a cubic system with eleven limit cycles and Żołądek [**103**] in 1995 has shown that the perturbations of a given cubic system under certain hypothesis may give eleven infinitesimal limit cycles.

Another way to tackle the center problem is by using normal forms. Using polar coordinates it can be shown that for any order ℓ there is a change of coordinates which transforms system (5.2) to its polar form

$$\dot{r} = v_3 r^3 + v_5 r^5 + \cdots + O(r^\ell) \ , \quad \dot{\theta} = 1 + p_2 r^2 + p_4 r^4 + \cdots + O(r^\ell) \ , \tag{5.5}$$

where v_{2k+1} is the *k-Poincaré-Liapunov constant*. If $v_3 = v_5 = \cdots = v_{2k-1} = 0$ and $v_{2k+1} \neq 0$, system (5.2) is said to have a *weak focus of order k*. If all the Poincaré-Liapunov constants are null, then the origin is a center, since Poincaré [**81**] showed that in this case there is an analytic change of variables which transforms system (5.2) in

$$\dot{r} = 0 \ , \quad \dot{\theta} = 1 + p_2 r^2 + p_4 r^4 + \cdots \ .$$

whose solution curves verify that the radius is constant. The constants p_i have a fundamental role in the problem explained in the next section.

Poincaré showed in [**81**] that the system (5.2) has a center at the origin if, and only if, it has an analytic first integral defined in a neighborhood of the origin. Therefore, if we can find an analytic first integral defined in a neighborhood of the origin or an integrating factor well defined and different from zero in a neighborhood of the origin for a system (5.2), then the origin is a center. Recently, and due to the relations between integrability and invariant algebraic curves it is clear the important role of these invariant algebraic curves in the center-focus problem, see [**5**]. These last years, interesting results which relate algebraic solutions and Poincaré-Liapunov constants have been published. For instance, Cozma and Şubă in [**39**] show that a weak focus of a polynomial system of degree $m \geqslant 3$ with the first Poincaré-Liapunov constant equal to zero and $m(m+1)/2 - 2$ algebraic solutions has a Darboux integral or a Darboux integrating factor. Shubé [**90**], when generalizing the previous result, has proved that if $v_{2k+1} = 0$ for $k = 1, \ldots, [(m-1)/2]$

and the system has $m(m+1)/2-[(m+1)/2]$ algebraic solutions, then the origin is a center. In a recent work [**10**] it is shown that if a polynomial system of degree m with an arbitrary linear part has a center and admits $m(m+1)/2-[(m+1)/2]$ algebraic solutions, then this polynomial system has a Darboux integrating factor.

6. Isochronous centers

We observe periodic movements in any science field or in real life, for instance, the movement of the Moon around the Earth, the clock pendulum movement, the alternating current, the human brain waves, the periodic movements in economy, ... The periodic movements are characterized by a periodic variation in the space of coordinates x, y, called *phase space*.

If a planar differential system has a singular point O of center type, in a neighborhood $\mathcal{V}$ foliated by closed orbits around this singular point we can define a function $T:\mathcal{V}\backslash\{O\}\to\mathbb{R}^+$ called *the period function* of the center and its value in each point is the period of the orbit passing through it. We are interested in the case when this period function is constant; in this case the center is called an *isochronous center*. The isochrony problem was considered by Galileo Galilei in the XVI century when he studied the classical pendulum. Probably, the study of isochronous systems started before the development of the differential calculus. The basic problem was finding which curve in the vertical plane has the following property

> *a body under the gravity action and starting from rest such that from any point of the curve and moving through it uses the same time to get the inferior point (tautochronous property).*

The curve verifying this property is the cycloid. This problem was first solved by Huygens and published in 1673 in his treatise about the theory of pendulum clocks (the first non linear example of isochronous oscillation such that the particle is moving over the cycloid and the period of the movement is independent from the amplitude) getting isochronous oscillations in opposition to the period monotony of the usual pendulum. Huygens applied this result to the construction of clocks. The cycloid curve is also known as *braquistochronous* because among all the possible curves is the one with the quickest descensus of the material particle under the gravity law.

Poincaré [**81**] showed that for a non degenerated center of an analytic system there is always a local analytic change of coordinates of the form $u=x+o(|(x,y)|)$, $v=y+o(|(x,y)|)$ and an analytic function ψ which transforms system (5.2) to

$$\begin{aligned}\dot{u} &= -v(1+\psi(u^2+v^2))\ ,\\ \dot{v} &= u(1+\psi(u^2+v^2))\ .\end{aligned}$$

It is known that the isochrony problem only appears for degenerated centers, that is, centers whose linear part has pure imaginary eigenvalues, see [**31**]. There is always a local analytic change of coordinates for an isochronous center of an analytic system of the form $u = x + o(|(x, y)|)$, $v = y + o(|(x, y)|)$ and a constant k which transforms system (5.2) to

$$\dot{u} = -kv \ , \quad \dot{v} = ku \ . \tag{6.1}$$

By a change to polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, system (5.2) writes as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{R(r,\theta)}{1 + \Theta(r,\theta)} \ , \tag{6.2}$$

where $R(r,\theta) = \sum_{s=2}^{m} P_s(\theta) r^s$, $\Theta(r,\theta) = \sum_{s=2}^{m} Q_s(\theta) r^{s-1}$ being P_s and Q_s trigonometrical homogeneous polynomials of degree $s+1$. If we define $r(\theta, \rho)$ the periodic solution of (6.2) such that $r(0, \rho) = \rho$ with ρ sufficiently small, the period function is given by

$$T(\rho) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\dot{\theta}} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + \Theta\Big(r(\theta,\rho), \theta\Big)}.$$

It is known that the period function is analytic and, therefore, can be expressed by a convergent power series

$$T(\rho) = 2\pi + \sum_{k \geqslant 1} T_k \ \rho^k \ , \tag{6.3}$$

where the coefficients T_k are called the *period constants.*

A center is isochronous if, and only if, it is locally linearizable, that is, there is a local analytic change of variable which transforms system (5.2) to its linearization $(-y, x)$ around the singular point. This property gives an algorithm to find the conditions that imply the isochrony of a center. This algorithm is based in the iterative construction of a change of variable in system (6.1). This process is used in many works to compute the *isochrony constants.*

Writing system (5.2) as $\dot{x} = -y + f(x, y)$, $\dot{y} = x + g(x, y)$, where $f(x, y)$ and $g(x, y)$ are polynomials representing the non linear terms, we have that, $x\dot{y} - y\dot{x} = x^2 + y^2 + xg(x, y) - yf(x, y)$. On the other hand, if we use polar coordinates we see that $x\dot{y} - y\dot{x} = r^2\dot{\theta}$. From the previous expressions we deduce that if $xg(x, y) - yf(x, y) \equiv 0$ then, when the origin of the system is a center, it is isochronous. These centers are called *trivial isochronous centers* because in polar coordinates they verify the condition of constant angular speed $\dot{\theta} = 1$. The geometrical sense of the previous condition is that the higher than 1 order terms of the vector field (f, g) are perpendicular to the linear part $(-y, x)$.

We notice that the isochrony does not necessarily imply that the flux generated by the autonomous system in the plane behaves like the planar rotation of a rigid body, that is, the angular speed $\dot{\theta}$ does not need to be constant. The condition to be verified is

$$\int_0^{T} \dot{\theta}\Big(r(t), \theta(t)\Big) \ \mathrm{d}t = 2\pi \ ,$$

where $(r(t), \theta(t))$ is an arbitrary periodic solution of the center and T is the common period to all the solutions in a neighborhood of the origin.

There is a simple method to generate isochronous centers for hamiltonian vector fields. This method is based in the so-called *Jacobian couples* which are two polynomials $P(x,y)$ and $Q(x,y)$ such that $P(0,0) = Q(0,0) = 0$ and which verify that the determinant of the Jacobian of the application $(x,y) \to \big(P(x,y), Q(x,y)\big)$ is constant and different from zero. In this case, the hamiltonian system associated to $H(x,y) = \frac{1}{2}\big(P^2 + Q^2\big)$ is linearizable by the canonic change of coordinates $u = P(x,y)$, $v = Q(x,y)$.

The behavior of the period function around a center has been studied in several works, particularly the connection with certain specific classes of vector fields. For instance, the planar systems which come from a differential equation $\ddot{x} + g(x) = 0$ have been studied by Urabe in [**94**], where necessary and sufficient conditions for isochrony on the oscillations of these systems are given. Chicone and Jacobs [**36**] showed that for a complete comprehension of the period function properties in a certain class of systems, it is necessary to consider the bifurcations of the critical points for the period. The isochronous centers are used in [**37**] since the method to study the bifurcations of limit cycles needs of the knowledge of the period function. The period function has been studied by different authors and many works in these last years have been devoted to it.

The problem of characterization of isochronous centers has been treated by several authors. However, there is a low number of families of polynomial systems for which there is a complete classification of their isochronous centers. The quadratic systems with an isochronous center were classified by Loud [**70**] and the homogeneous cubic systems by Pleshkan [**83**]. The isochronous centers of a Kukles system , $\dot{x} = y$, $\dot{y} = Q(x,y)$, were studied in [**31**]. The class of complex systems of the form $\dot{z} = \mathrm{i}P(z)$ where $z = x + \mathrm{i}y$ and the reversible cubic systems with $\dot{\theta} = 1$ have been studied in [**74**]. The isochronous centers of certain cubic systems with four complex invariant straight lines are found in [**73**]. The isochronous centers with degenerate infinity are studied in [**69**] and in [**19**]. The isochronous centers for systems with a center-type linear part perturbed by homogeneous polynomials have been studied in [**17, 18**]. The study of isochronous centers and non isochronous centers given in [**41**] must also be noticed.
A system (5.1) is said to be *reversible with respect to time* if its expression does not vary by the change $X = -x, Y = y, T = -t$. A well-known result states that if a system (5.1) is reversible with respect to time, is analytic in a neighborhood of a singular point and its linear part in this point is of center-type, then this point is a center.

Problem 6.1. *Find all the reversible isochronous centers for cubic systems.*

Another interesting property is the coexistence of isochronous centers and invariant straight lines. For cubic polynomial systems there are examples of coexistence of isochronous centers with two invariant straight lines, [**73, 21**]. Chavarriga, Sáez and Szántó show in [**27**] a family of cubic systems with an isochronous center and three real invariant straight lines.

PROBLEM 6.2. **(a)** *Find, if it exists, an example of cubic polynomial system where an isochronous center and four real invariant straight lines coexist.* **(b)** *Given a polynomial system, determine the maximum number of isochronous centers and invariant straight lines which can coexist.*

Recently, Chavarriga, Sáez, Szántó and Grau [**28**] have studied the Kukles systems. The main goal is to determine the maximum number of limit cycles which may appear in function of the degree of the system and the number of invariant straight lines. Moreover, the possible coexistence of invariant algebraic curves of higher degree and limit cycles is studied.

On the other hand, Lukashevich in [**71**] shows that if $P(x, y)$ and $Q(x, y)$ are harmonic conjugate functions, that is, they satisfy Cauchy-Riemann equations $P_x = Q_y$ and $P_y = -Q_x$, then any center of the vector field (P, Q) is isochronous. Using the previous result, Villarini in [**96**] shows that if $P(x, y)$ and $Q(x, y)$ are harmonic conjugated functions, then the local flux of the vector field $X = (P, Q)$ and his orthogonal vector field $X^{\perp} = (Q, -P)$ commute, in the sense that the Lie commutator $[X, X^{\perp}]$ is null. In a later work, Sabatini [**86**] states that Villarini's proof works without great changes when changing orthogonal condition to transversal condition $[X, X_T] = 0$ getting, in this way, an important method for the study of isochrony via the dynamic interpretation of the Lie product. In recent works [**9**] the importance of invariant algebraic curves when looking for transversal vector fields is studied. A method which uses an algebraic solution $f = 0$ of the vector field X to find a transversal field X_T which commutes with X is developed. An excellent survey of all the results about isochrony is given in [**21**].

7. Some fundamental problems

The most important questions which are still open for planar differential equations are firstly: Under which conditions the original and the linearized system have the same qualitative behavior and the same topological structure around a singular point? This problem was first studied by Poincaré and Liapunov at the end of the XIX century. A partial answer is given by the Hartman-Grobman Theorem [**58**], the Andreev Theorem [**1**] for nilpotent singular points and Dumortier Theorems [**46**] for degenerated singular points. Only the cases not included in these theorems must be studied. Particularly, the problem of distinguishing a center from a focus for all kind of singular points is still unsolved.

In second place: Which are the conditions for a system of differential equations to have a first integral defined in a neighborhood of a singular point with a certain type (C^k, $k = 0, 1, 2, ...\infty, \omega$)? This problem is known as the *local integrability* *problem.*

In third place: Which are the conditions for a system of differential equations to have a first integral defined in the whole plane with a certain type (C^k, $k = 0, 1, 2, ...\infty, \omega$)? This problem is known as the *global integrability problem.*

In forth place: Which are the conditions for a system of differential equations to have a first integral of a certain type (polynomial, rational, Darboux, elementary, Liouville, etc) defined in a neighborhood of a singular point? This problem is called *formal integrability problem* and the role of the invariant algebraic curves is very important as we have previously seen.

These problems seem to be very related one another and some partial results confirm this intuition.

In fact, many similar relations can be established between the problem of the study of periodic solutions and the study of algebraic solutions for a polynomial system. If the system admits an infinite number of periodic solutions then it has a weak C^∞ first integral and the results known until the moment, point that this first integral would be analytic when the linear part is not completely degenerated. In case the system admits an infinite number of algebraic solutions, then the system has a rational first integral. Finally, in case the number of periodic solutions is finite (limit cycles) we get part (b) of Hilbert's problem, and if the number of algebraic solutions is finite we have Jouanoulou's result but we do not have an upper bound for the degree of these solutions.

References

[1] Andreev, A. *Investigation of the behavior of the integral curves of a system of two differential equations in the neighborhood of a singular points*, Translation of AMS **8**(1958), 187–207.

[2] Arnold, V. *Équations Différentielles Ordinaires.* Éditions Mir, Moscou (1974).

[3] Autonne, L. *Sur la théorie des équations différentielles du premier ordre et du premier degré*, Journal de l'École Polytechnique, **61** (1891), 35-122; **62** (1892), 47-180.

[4] Bautin, N.N. *On the number of limit cycles which appears with the variation of coefficients from an equilibrium position of focus or center type.* Mat. Sb. **30 (72)** (1952), 181–196; Amer. Math. Soc. Transl. **100** (1954), 397–413.

[5] Cairó, L; Feix, M.R. and Llibre, J., *Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems*, Resenhas **4,2** (1999), 127–161.

[6] Carnicer, M.M. *The Poincaré problem in the nondicritical case*, Annals of Math. **140** (1994), 289-294.

[7] Cerveau, D. and Lins Neto, A. *Holomorphic foliations in CP(2) having an invariant algebraic curve*, Ann. Inst. Fourier **41** (1991), 883-903.

[8] Chavarriga, J. *Integrable systems in the plane with a center type linear part*, Applicationes Mathematicae **22** (1994), 285-309.

[9] Chavarriga, J., García, I. and Giné, J. *Isochronous centers of a linear center perturbed by homogeneous polynomials.* Proceedings of the 3th Catalan Days on Applied Mathematics

(1996) edited by J. Chavarriga and J. Giné. Fundació Publica Institut d'Estudis Ilerdencs. pp. 65-80.

[10] Chavarriga, J., Giacomini, H. and Giné, J. *An improvement to Darboux integrability theorem for systems having a center.* Applied Mathematics Letters **12** (1999), 85-89.

[11] Chavarriga, J., Giacomini, H. Giné, J. and Llibre, J. *On the integrability of two-dimensional flows.* Journal of Differential Equations **157** (1999), 163-182.

[12] Chavarriga, J., Giacomini, H and Llibre, J. *Uniqueness of algebraic limit cycles for quadratic systems.*, Journal of Mathematical Analysis and Applications, **261** (2001), 85-99.

[13] Chavarriga, J. and Giné, J. *Integrability of a linear center perturbed by fourth degree homogeneous polynomial.* Publicacions Matemàtiques. **40** (1996), 21–39.

[14] Chavarriga, J. and Giné, J. *Integrability of a linear center perturbed by fifth degree homogeneous polynomial.* Publicacions Matemàtiques. **41** (1997), 335-356.

[15] Chavarriga, J., Giné, J. *Integrability of cubic systems with degenerate infinity*, Differential Equations and Dynamical Systems, **6, 4** (1998), 425-438.

[16] Chavarriga, J. and Giné, J. *El 16b problema de Hilbert: Estado de la cuestión.* Actas del IV Congreso Galdeano (1999). Por aparecer.

[17] Chavarriga, J., Giné, J. and García, I.A., *Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomial.* Bulletin des Sciences Mathematiques, **123** (1999), 77-96.

[18] Chavarriga, J., Giné, J. and García, I.A. *Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomial.* Journal of Computational and Applied Mathematics **126** (2001), 351-368.

[19] Chavarriga, J., Giné, J. and García, I. *Isochronous centers of cubic systems with degenerate infinity*, Differential Equations and Dynamical Systems, **7, 1** (1999), 49-66.

[20] Chavarriga, J., Grau, M., *A family of quadratic polynomial differential systems non–integrables Darboux*, Preprint, 2001.

[21] Chavarriga, J. and Sabatini, M. *A survey of isochronous centers.* Qualitative theory of Dynamical Systems **1** (1999), 1-70.

[22] Chavarriga, J. and Llibre, J. *On the algebraic limit cycles of quadratic systems*, Proceedings of the IV Catalan Days of Applied Mathematics, February 11-13 (1998). Edited by C. García, C. Olivé and M. Sanromà. 17-24.

[23] Chavarriga, J., Llibre, J. *Invariant algebraic curves and rational first integral for planar polynomial vector fields.* Journal of Differential Equations **196** (2001), 1-16

[24] Chavarriga, J., Llibre, J., Moulin-Ollagnier, J. *On a result of Darboux*, Journal of Computation and Mathematics **4** (2001) 197-210.

[25] Chavarriga, J., Llibre, J. and Sorolla, J. *Algebraic limit cicles of quadratic systems.* Preprint, Universidad de Lleida (1999)

[26] Chavarriga, J., Llibre, J. and Sotomayor J. *Algebraic solutions for polynomial systems with emphasis in the quadratic case*, Expositiones Mathematicae **15** (1997), 161-173.

[27] Chavarriga, J., Saez, E. and Szanto, I. *Coexistence of isochronous center with three invariant straight lines in a uniparametrical family of cubic fields.* Radovi Matematicki, **11**, 2 (2003), p.112–119.

[28] Chavarriga, J., Saez, E., Szanto, I. and Grau, M. *Coexistence of limit cycles and invariant algebraic curves on a Kukles system*, to appear in Nonlinear Analysis: Theory, Methods and Applications.

[29] Chen Lansun and Wang Mingshu *Relative position and number of limit cycles of a quadratic differential system*, Acta Math. Sinica **22** (1979), 751-758 (in Chinese).

[30] Christopher, C.J. *Liouvillian first integrals of second order polynomials differential equations.* Electron J. Differential Equations **49** (1999), 7pp (electronic)

[31] Christopher, C.J. and Devlin, J. *Isochronous centres in planar polynomial systems*, SIAM Jour. Math. Anal. **28** (1997), 162–177.

[32] Christopher, C.J. and Llibre, J. *Algebraic aspects of integrability for polynomial systems.* Qualitative theory of Dynamical Systems, to appear.

[33] Christopher C., Llibre, J., *A Family of Quadratic Polynomial Differential Systems with Invariant Algebraic Curves of Arbitrarily High Degree without Rational First Integrals*, to appear in Proceedings of the American Mathematical Society, 2000.

[34] Christopher, C.; Llibre, J.; Świrszcz, G., *Invariant algebraic curves of large degree for quadratic systems.* Preprint 2002.

[35] Christopher, C.J. and Lloyd, N.G. *Polynomial systems: a lower bound for the Hilbert numbers*, Proc. R. Soc. Lond. A **450** (1995), 219–224.

[36] Chicone, C. and Jacobs, M. *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc. **312** (1989), 433–486.

[37] Chicone, C. and Jacobs, M. *Limit cycle bifurcations from quadratic isochrones*, J. Differential Equations **91** (1991), 268-327.

[38] Coppel, W.A. *A survey of Quadratic Systems.* J. Differential Equations. **2** (1966), 293–304.

[39] Cozma, D. and Şubă, A. *Partial integrals and the first focal value in the problem of the centre.* NoDEA **2** (1995), 21–34.

[40] Darboux, G. *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges)*, Bull.Sci.Math. 32 (1878), 60-96; 123-144; 151-200.

[41] Devlin, J. *Coexisting isochronous and non isochronous centers.* Bulletin London

[42] Dobrovol'skii, V.A., Lokot, N.V. and Strelcyn, J.M. *Mikhail Nikolaevich Lagutinskii (1871-1915): Un Mathématicien Méconnu* Historia Mathematica **25** (1998), 245–264.

[43] Dulac, H. *Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un cente.* Bulletin de Sciences Mathematiques (2) **32** (1908), 230-252.

[44] Dulac, H. *Sur les cycles limites*, Bull. Sci. Math. France **51** (1923), 45–188.

[45] Druzhova, T.A.*The algebraic integrals of certain differential equations*, Differential Equations, **4** (1968), 736-739.

[46] Dumortier, F. *Singularities of vector fields on the plane.* J. Differential Equations. **23**(1977), 53–106.

[47] Dumortier, F. and Rousseau, C., *Cubic Liénard equations with linear damping*, Nonlinearity **3** (1990), 1015–1039.

[48] Dumortier, F. and Li, C., *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1500.

[49] Ecalle, J., *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac.* Actualités Mathématiques. Hermann, Paris (1992)

[50] Evdokimenco, R.M., *Construction of algebraic paths and the qualitative investigation in the large of the properties of integral curves of a system of differential equations*, Differential Equations **6** (1970), 1349–1358.

[51] Evdokimenco, R.M., *Behavior of integral curves of a dynamic system*, Differential Equations **9** (1974), 1095–1103.

[52] Evdokimenco, R.M., *Investigation in the large of a dynamic system with a given integral curve*, Differential Equations **15** (1979), 215–221.

[53] Filiptsov, V.F., *Algebraic limit cycles*, Differential Equations **9** (1973), 983-988.

[54] Frommer, M., *Uber das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umgebung Rationalre Unbestimmtkeitssellen*, Math. Annalen **109** (1934), 395-424.

[55] Gasull, A. *On polynomial systems with invariant algebraic curves*, Equadiff 91, International Conference on Differential Equations, Vol. 2, World Scientific, (1993), 531-537.

[56] Giacomini, H., Llibre, J. and Viano, M. *On the nonexistence, existence, and uniqueness of limit cycles*, Nonlinearity **9** (1996), 501-516.

[57] Giacomini, H., Neukirch, S. *Number of limit cycles for the Liénard equation*, Phys. Rev. E **56, 4** (1997), 3809-3813.

[58] Hartman, P. *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.

[59] Hayashi, M., *On polynomial Liénard systems which have invariant algebraic curves*, Funkcial. Ekvac. **39** (1996), 403–408.

[60] Hilbert, D. *Mathematische Problem (lecture)*, Second Internat. Congress Math. Paris 1900, Nachr. Ges. Wiss. Gttingen Math.-Phys. Kl. 1900, 253-297.

[61] Ilyashenko, Yu.S. *Finiteness theorems for limit cycles*, Russian Math. Surveys **40** (1990) 143-200.

[62] Kapteyn, W. *On the centre of the integral curves which satisfy differential equations of the first order and the first degree.* Konikl, Akademie von Wetenschappen te Amsterdam, Proceedings of the Section of Science, **13,2** (1911), 1241-1252; **14,2** (1911), 1185-1195.

[63] Jouanoulou, J.P. *Equations de Pfaff algébriques*, Lectures Notes in Mathematics, **708**, Springer Verlag, 1979.

[64] Kolmogorov, A.N. *Sulla teoria di Volterra della lutta per l'esistenza*, Giorn. Inst. Ital. Attuari. **7** (1936), 74-80.

[65] Kooij, R.E. and Christopher, C.J. *Algebraic invariant curves and the integrability of polynomial systems*, Appl. Math. Lett. **6** (1993), 51-53.

[66] Liapunov, M.A. *Problème général de la stabilité du mouvement*, Ann. of Math. Stud. **17**, Princeton University Press, 1947.

[67] Li Jibin and Huang Qiming *Bifurcations of limit cycles forming compound eyes in the cubic system*, Chi. Ann. of Math., ser B, **8,4** (1987), 391-403.

[68] Liénard, A. *Étude des oscilations entretenues*, Revue Général de l'Électricité **XXIII-21** (1928) 901-912; **XXIII-22** (1928) 946-954.

[69] Lloyd, N.G., Christopher, J., Devlin, J., Pearson, J.M. and Yasmin, N. *Quadratic like cubic systems.* Differential Equations and Dynamical Systems **5, 3-4** (1997), 329-345.

[70] Loud, W.S. *Behavior of the period of solutions of certain plane autonomous systems near centers.* Contributions to Differential Equations **3** (1964), 21–36.

[71] Lunkashevitch, N.A. *Isochronicity of center for certain systems of differential equations.* Diff. Uravn. **1** (1965), 220-226.

[72] Lunkevich, V.A. and Sibirskii, K.S. *Integrals of a general quadratic differential system in cases of a center.* Diff. Equations **18** (1982), 563–568.

[73] Mardesic, P., Moser-Jauslin, L. and Rousseau, C. *Darboux linearization and isochronous centers with a rational first integral.* J. Differential Equations **134** (1997), 216–268.

[74] Mardesic, P., Rousseau, C. and Toni, B. *Linearization of isochronous centers.* J. Differential Equations **121** (1995), 67–108.

[75] Moulin-Ollagnier, J., *About a Conjecture on Quadratic Vector Fields*, Journal of Pure and Applied Algebra **165, 2** (2001), 227–234.

[76] Odani, K. *The limit cycle of the van der Pol equation is not algebraic*, Journal of Differential Equations **115** (1995), 146-152.

[77] Painlevé, P. *Mémoire sur les équations différentielles du premier ordre*, Annales Scientifiques de l'École Normale Superieure, 3 serie, **8** (1891), 9-59, 103-140, 201-226 y 276-284; **9** (1892), 9-30, 101-144 y 283-308.

[78] Pereira, J.V. *Vector Fields, Invariant Curves and Linear Systems.* Preprint (2000). IMPA

[79] Petrovskii, I.G. and Landis, E.M. *On the number of limit cycles of the equation $dy/dx = P(x,y)/Q(x,y)$, where P and Q are polynomials of the second degree*, Amer. Math. Soc. Transl. Ser. 2 **10** (1958), 177-221; "Corrections", Mat. Sb. **48** (1959), 253-255.

[80] Poincaré, H. *Mémoire sur les courbes définies par les équations différentielles*, J. de Mathématiques Pures et Appliquées (3) **7** (1881), 375-422; **8** (1882), 251-296; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris 1951, 3-84.

[81] Poincaré, H. *Mémoire sur les courbes définies par les équations différentielles*, J. de Mathématiques Pures et Appliquées (4) **1** (1885), 167-244; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris 1951, 95-114.

[82] Poincaré, H. *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré* (I and II), Rendiconti del circolo matematico di Palermo **5** (1891), 161-191; **11** (1897), 193-239.

[83] Pleshkan, I. *A new method of investigating the isochronicity of a system of two differential equations.* Differential Equations **5** (1969), 796–802.
[84] Prelle, M.J. and Singer, M.F. *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. **279** (1983), 215-229.
[85] Qin Yuan-Xun. *On the algebraic limit cycles of second degree of the differential equation* $dy/dx = \sum_{0\leqslant i+j\leqslant 2} a_{ij}x^i y^j / \sum_{0\leqslant i+j\leqslant 2} b_{ij}x^i y^j$, Chinese Math. Acta **8** (1966), 608-619.
[86] Sabatini, M. *Characterizing isochronous centres by Lie brackets*, Differential Equations and Dynamical Systems **5, 1** (1997), 91-99.
[87] Shen Boqian. *A sufficient and necessary condition of the existence of quartic curve limit cycle and separatrix cycle in a certain quadratic system*, Annals of Differential Equations **7** (1991), 282-288.
[88] Shi Songling *A Concrete example of the existence of four limit cycles for plane quadratic systems*, sci. sinica **23** (1980), 153-158.
[89] Shi Songling *A method of constructing Cycles without contact around a weak focus*, Journal of Differential Equations **41** (1981), 301-312.
[90] Shubé, A.S. *Partial integrals, integrability and the center problem.* Differential Equations **32** (1996), 884-892.
[91] Singer, M.F. *Liouvillian first integrals of differential equations.* Trans. Amer. Math. Soc. **333** (1992), 673-688.
[92] Smale, S. *Mathematical problems for the next century.* Math. Intelligencer. **20** (1998), 7-15.
[93] Sotomayor, J. *Liçóes de ecuaçóes diferenciais ordinárias.* IMPA, Rio de Janeiro (1979).
[94] Urabe, M. *The potential force yielding a periodic motion whose period is an arbitrary continuous function of the amplitude of the velocity.* Archs. ration. Mech. Analysis **11** (1962), 26-33.
[95] Van der Pol, B. *Sur les oscillations de relaxation* Revue Général de l'Électricité **XXII** (1927) 489-490.
[96] Villarini, M. *Regularity properties of the period function near a centre of planar vector fields.* Nonlinear Analysis, T.M.A. 19 **8** (1992), 787–803.
[97] Volterra, V. *Leons sur la théorie mathématique de la lutte pour la vie* Paris, 1931.
[98] Yablonskii, A.I. *On limit cycles of certain differential equations*, Differential Equations **2** (1966), 164-168.
[99] Yablonskii, A.I. *Algebraic integrals of a differential-equation system*, Differential Equations **6** (1970), 1326-1333.
[100] Ye Yan-Quian. *Theory of limit cycles*, Translations of Math. Monographs **66**, Amer. Math. Soc., Providence (1986).
[101] Żołądek, H. *On algebraic solutions of algebraic Pfaff equations*, Studia Mathematica **114** (1995), 117-126.
[102] Żołądek, H. *On certain generalization of the Bautin's Theorem.* Nonlinearity **7** (1994), 233–279.
[103] Żołądek, H. *Eleven small limit cycles in a cubic vector field*, Nonlinearity **8** (1995), 843–860.
[104] Żołądek, H. *The classification of reversible cubic systems with center*, Topol. Meth. in Nonlin. Analysis **4** (1994), 79–136.
[105] Żołądek, H., *Algebraic invariant curves for the Liénard equation*, Trans. Amer. Math. Soc. **350** (1998), 1681–1701.

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Departament de Matemàtica
Universitat de Lleida
Avda. Jaume II, 69
Lleida, Spain

E-mail address: chava@eup.udl.es
E-mail address: mtgrau@matematica.udl.es