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Wallis's Formula and the Arc Length of Clovers

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Abstract. A family of infinite product formulas that generalize Wallis's formula is presented in this note along with a geometric interpretation of the formulas in terms of arc length.

1. Introduction

One of the most beautiful equations in mathematical analysis is Wallis's formula,

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}. \tag{1.1}$$

Many methods of proving this result have been devised since John Wallis discovered the formula in 1655. Every proof of this famous assertion must be based on a formal definition the constant π. Wallis defined π as the area of the unit circle and his discovery of equation (1.1) started with the expression

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\, dx.$$

In this note we will define π as the arc length of the unit semi-circle. From this definition and the basic theorems on arc length one can prove that

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx. \tag{1.2}$$

The purpose of this note is to provide a simple, direct proof of Wallis's formula based on equation (1.2) and to show that this method leads to an interesting generalization of Wallis's formula. For each integer $m \geqslant 1$ we prove that

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$$\int_0^1 \frac{1}{\sqrt{1-x^m}}\,dx = \frac{2}{m}\prod_{n=1}^{\infty}\Big(\frac{2nm-m+2}{2nm-2m+2}\Big)\Big(\frac{2n}{2n+1}\Big). \tag{1.3}$$

But wait, there's more! There is a very nice geometric interpretation of this formula that also generalizes the geometric content of Wallis's formula. It turns out that

$$\frac{\varpi_m}{2} = \int_0^1 \frac{1}{\sqrt{1-x^m}}\,dx, \tag{1.4}$$

where ϖ_m is the arc length of a leaf in the m-clover defined by the polar equation

$$r^{m/2} = \cos(\tfrac{m}{2}\theta). \tag{1.5}$$

The points in the plane with polar coordinates (r,θ) satisfying (1.5) form a clover with m leaves when m is odd and $m/2$ leaves when m is even.

An equivalent form of formula (1.3) was discovered by Hyde [4] and the expression (1.4) for the arc length of the principal leaf in the m-clover is given by Cox and Shurman in [2]. The alternative approach to formula (1.3) presented in this note was inspired by the the classic analysis text of Goursat and Hedrick [3] published in 1904. This text contains an exercise which asks for a proof of formula (1.3) for the special case $m = 4$. Section 3 of this note is devoted to a review of arc length and its basic properties and formula (1.2) is derived as a key example. The arc length of the lemniscate is also computed in Section 3. In Section 4 we show that the arc length of the principal leaf in the m-clover satisfies formula (1.4).

2. A Generalization of Wallis's Formula

THEOREM 2.1. *For all integers $m \geqslant 1$ and $n \geqslant 0$ define*

$$I_m^n = \int_0^1 \frac{x^n}{\sqrt{1-x^m}}\,dx.$$

(a) *For all integers $m \geqslant 1$ and $n \geqslant 0$, the improper integral I_m^n exists.*

(b) *For all integers $m \geqslant 1$ and $n \geqslant 0$,*

$$I_m^{n+m} = \Big(\frac{2n+2}{2n+2+m}\Big)I_m^n.$$

(c) *For all integers $m \geqslant 1$, $I_m^{m-1} = \frac{2}{m}$.*

Proof. For all real numbers x with $0 \leqslant x < 1$ and all integers $m \geqslant 1$ and $n \geqslant 0$,

$$\frac{x^n}{\sqrt{1-x^m}} \leqslant \frac{1}{\sqrt{1-x}}.$$

Therefore, for any real number b such that $0 \leqslant b < 1$,

$$\int_0^b \frac{x^n}{\sqrt{1-x^m}}\,dx \leqslant \int_0^b \frac{1}{\sqrt{1-x}}\,dx = -2\sqrt{1-x}\Big]_0^b = 2 - 2\sqrt{1-b}.$$

Since the value of the integral on the left increases with b, it follows that the improper integral I_m^n exists and is bounded above by 2. This proves (a).

Now let b be any real number such that $0 < b < 1$. For any integers $m \geqslant 1$ and $n \geqslant 0$ consider the integral

$$\int_0^b x^n\sqrt{1-x^m}\,dx.$$

Recall the general formula for integration by parts:

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Evaluating the integral above using the integration by parts formula with

$$f(x) = \sqrt{1-x^m}, \quad f'(x) = \frac{-mx^{m-1}}{2\sqrt{1-x^m}}, \quad g(x) = \frac{x^{n+1}}{n+1}, \quad \text{and} \quad g'(x) = x^n$$

we obtain

$$\int_0^b x^n\sqrt{1-x^m}\,dx = \frac{b^{n+1}}{n+1}\sqrt{1-b^m} \quad + \quad \frac{m}{2(n+1)}\int_0^b \frac{x^{n+m}}{\sqrt{1-x^m}}\,dx.$$

Taking the limit as b approaches 1 from below shows that

$$\int_0^1 \frac{x^{n+m}}{\sqrt{1-x^m}}\,dx \;=\; \Big(\frac{2n+2}{m}\Big)\int_0^1 x^n\sqrt{1-x^m}\,dx.$$

Also, since $x^{n+m} = x^n - x^n(1-x^m)$, we have

$$\int_0^1 \frac{x^{n+m}}{\sqrt{1-x^m}}\,dx \;=\; \int_0^1 \frac{x^n}{\sqrt{1-x^m}}\,dx \;-\; \int_0^1 x^n\sqrt{1-x^m}\,dx.$$

Using the two previous equations we obtain:

$$\Big(\frac{2n+2+m}{m}\Big)\int_0^1 \frac{x^{n+m}}{\sqrt{1-x^m}}\,dx \;=\; \Big(\frac{2n+2}{m}\Big)\int_0^1 \frac{x^n}{\sqrt{1-x^m}}\,dx$$

and the reduction formula (b) now follows.

Finally, note that for any real number b with $0 < b < 1$ and any integer $m \geqslant 1$,

$$\int_0^b \frac{x^{m-1}}{\sqrt{1-x^m}}\,dx = -\frac{2}{m}\sqrt{1-x^m}\,\Big]_0^b = -\frac{2}{m}\sqrt{1-b^m} + \frac{2}{m}.$$

Taking the limit as b approaches 1 from below we obtain (c). □

For all integers $n \geqslant 0$ the substitution $x = \sin\theta$ can be used to show that

$$I_2^n = \int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/2} \sin^n\theta\,d\theta.$$

A proof of Wallis's formula using the sequence of integrals I_2^n in their trigonometric form as shown above is given as a series of exercises by Spivak [5]. These two equivalent expressions for I_2^n also reveal that the proof of formula (1.3) that is developed in the next theorem is generalization of the proof outlined by Spivak for $m = 2$. A nice summary of Wallis's original proof is presented by Spivak after the series of exercises on Wallis's formula.

THEOREM 2.2. *For all integers $m \geqslant 1$,*

$$\int_0^1 \frac{1}{\sqrt{1-x^m}}\,dx = \frac{2}{m}\prod_{n=1}^{\infty}\Big(\frac{2nm-m+2}{2nm-2m+2}\Big)\Big(\frac{2n}{2n+1}\Big).$$

PROOF. Let I_m^n denote the integral defined in the previous theorem. Using the reduction formula (b) and result (c) from the previous theorem we have for $n \geqslant 1$:

$$I_m^{nm} = \Big(\frac{2}{m+2}\Big)\Big(\frac{2m+2}{3m+2}\Big)\Big(\frac{4m+2}{5m+2}\Big)\cdots\Big(\frac{2nm-2m+2}{2nm-m+2}\Big)\,I_m^0,$$

$$I_m^{nm+m} = \Big(\frac{2}{m+2}\Big)\Big(\frac{2m+2}{3m+2}\Big)\Big(\frac{4m+2}{5m+2}\Big)\;\cdots\;\Big(\frac{2nm+2}{2nm+2+m}\Big)\,I_m^0,$$

and,

$$I_m^{nm+m-1} = \Big(\frac{2}{m}\Big)\Big(\frac{2}{3}\Big)\Big(\frac{4}{5}\Big)\cdots\Big(\frac{2n}{2n+1}\Big).$$

Now, for all integers $m \geqslant 1$ and $n \geqslant 0$, observe that

$$I_m^{nm+m} \;\leqslant\; I_m^{nm+m-1} \;\leqslant\; I_m^{nm}.$$

If we multiply each of the three integrals in this pair of inequalities by

$$\Big(\frac{m+2}{2}\Big)\Big(\frac{3m+2}{2m+2}\Big)\Big(\frac{5m+2}{4m+2}\Big)\cdots\Big(\frac{2nm-m+2}{2nm-2m+2}\Big)$$

we obtain

$$\Big(\frac{2nm+2}{2nm+2+m}\Big)\ I_m^0 \leqslant \Big(\frac{2}{m}\Big)\Big(\frac{2+m}{2}\Big)\Big(\frac{2}{3}\Big)\cdots\Big(\frac{2nm-m+2}{2nm-2m+2}\Big)\Big(\frac{2n}{2n+1}\Big) \leqslant I_m^0.$$

Computing the limit of each expression in this pair of inequalities as $n \to \infty$, the theorem now follows from the squeeze theorem for limits. □

3. Arc Length

There are two reasons why we take a little extra care in our review of arc length in this section. First, showing that the principal leaf in an m-clover has well-defined arc length and that formula (1.4) holds is slightly complicated by the fact that the integral in formula (1.4) is improper. Second, the usual formula for computing the arc length of a curve given in polar coordinates is not used here. The alternative formula that we need is not complicated, but it does not appear in most textbooks. So, it seems worthwhile to take the time to derive this formula.

For any vector v in $\mathbf{R}^n$, $v = (v_1, v_2, \ldots, v_n)$, the length of v is denoted by $\|v\|$,

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

A **path** is a continuous mapping $\gamma\colon [a,b] \to \mathbf{R}^n$, where $n \geqslant 1$ and a and b are real numbers with $a < b$. For any partition $P = (x_0, x_1, \ldots, x_m)$ of $[a,b]$, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$, let

$$A(\gamma, P) = \sum_{k=1}^{m} \|\gamma(x_k) - \gamma(x_{k-1})\|.$$

Let $S = \{A(f,P)\colon P \text{ is a partition of } [a,b]\}$. Now, consider the following two cases.

If $n \geqslant 2$ and the set S is bounded above then we say that γ has **well-defined arc length** on $[a,b]$ and the arc length of γ on $[a,b]$ is defined as the least upper bound of S. If γ has well-defined arc length on $[a,b]$ the arc length is denoted by $L(\gamma,[a,b])$.

If $n = 1$ and the set S is bounded above then we say that γ is a function of **bounded variation** on $[a,b]$ and the **total variation** of γ on $[a,b]$ is defined to be the least upper bound of S. The definition of bounded variation can be extended to real-valued functions that are not necessarily continuous, but we will not require the more general concept in this note.

There is a concept of arc length for real-valued functions as well. For a continuous function $f\colon [a,b] \to \mathbf{R}$ we define the arc length of f on $[a,b]$ as the arc length of the path $\gamma\colon [a,b] \to \mathbf{R}^2$, where $\gamma(t) = (t, f(t))$, provided that this path has well-defined arc length on $[a,b]$. In this case the arc length of f on $[a,b]$ is denoted by $L(f,[a,b])$.

There are a number of basic results about functions of bounded variation and arc length that we will state without proof. For proofs of these assertions see Apostol [1], Chapter 8.

(a) If $f\colon [a,b] \to \mathbf{R}$ is monotonic then f is of bounded variation on $[a,b]$ and the total variation of f on $[a,b]$ is $|f(b) - f(a)|$.

(b) If f and g are of bounded variation on $[a,b]$ then $f+g$, $f-g$, and fg are of bounded variation on $[a,b]$.

(c) A path $\gamma\colon [a,b] \to \mathbf{R}^n$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$, has well-defined arc length on $[a,b]$ if and only if for $k = 1,2,\ldots,n$, γ_k is of bounded variation on $[a,b]$.

(d) Let $\gamma\colon [a,b] \to \mathbf{R}^n$ be a path. For any point c such that $a < c < b$, γ has well-defined arc length on $[a,b]$ if and only if γ has well-defined arc length on $[a,c]$ and $[c,b]$ and in this case, $L(\gamma,[a,b]) = L(\gamma,[a,c]) + L(\gamma,[c,b])$.

(e) Let $\gamma\colon [a,b] \to \mathbf{R}^n$ be a path with well-defined arc length on $[a,b]$. Define a function $L\colon [a,b] \to \mathbf{R}$ by $L(a) = 0$ and $L(x) = L(\gamma,[a,x])$ if $a < x \leqslant b$. Then L is an increasing continuous function on $[a,b]$.

(f) If $\gamma\colon [a,b] \to \mathbf{R}^n$ is continuously differentiable on $[a,b]$ then γ has well-defined arc length on $[a,b]$ and

$$L(\gamma,[a,b]) = \int_a^b \|\gamma'(t)\|\,dt.$$

Theorem 3.1. *Suppose that $\gamma\colon [a,b] \to \mathbf{R}^n$ is a path with well-defined arc length on $[a,b]$ and that for all x such that $a < x < b$, γ is continuously differentiable on $[a,x]$. Assume that the integral*

$$\int_a^x \|\gamma'(t)\|\,dt$$

is improper when $x = b$ and that the improper integral obtained when $x = b$ exists. Then

$$L(\gamma,[a,b]) = \int_a^b \|\gamma'(t)\|\,dt.$$

PROOF. Let $L(a) = 0$ and $L(x) = L(\gamma, [a, x])$ if $a < x \leqslant b$. If $a < x < b$ then

$$L(\gamma, [a, x]) + L(\gamma, [x, b]) = L(\gamma, [a, b])$$

and so $L(\gamma, [x, b]) = L(b) - L(x)$. Since L is continuous on $[a, b]$,

$$\lim_{x \to b^-} L(\gamma, [x, b]) = 0.$$

Now,

$$L(\gamma, [a, x]) \leqslant L(\gamma, [a, b]) = L(\gamma, [a, x]) + L(\gamma, [x, b])$$

and so

$$\int_a^x \|\gamma'(t)\|\, dt \;\leqslant\; L(\gamma, [a, b]) \;=\; \int_a^x \|\gamma'(t)\|\, dt + L(\gamma, [x, b]).$$

The theorem now follows from the squeeze theorem for limits. □

The unit semi-circle is the path $\gamma\colon [-1, 1] \to \mathbf{R}^2$, where $\gamma(t) = (t, \sqrt{1 - t^2})$. Since the component functions of γ are monotonic on the subintervals $[-1, 0]$ and $[0, 1]$, γ has well-defined arc length on $[-1, 1]$. We define the real number π as the arc length of the unit semi-circle. By symmetry, $\pi/2$ is the arc length of γ on the interval $[0, 1]$. For all t such that $0 \leqslant t < 1$ we have

$$\gamma_1'(t) = 1 \quad \text{and} \quad \gamma_2'(t) = \frac{t}{\sqrt{1 - t^2}},$$

and so

$$\gamma_1'(t)^2 + \gamma_2'(t)^2 = \frac{1}{1 - t^2}.$$

Thus, for any x such that $0 < x < 1$, γ is continuously differentiable on $[0, x]$ and

$$L(\gamma, [0, x]) = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt.$$

Now, the improper integral obtained when $x = 1$ exists by Theorem 2.1 and so, by Theorem 3.1, we obtain

$$\frac{\pi}{2} = \int_0^1 \frac{1}{\sqrt{1 - t^2}}\, dt.$$

Wallis's formula now follows by taking $m = 2$ in Theorem 2.2. Notice that this proof of Wallis's formula does not depend on the theory of the trigonometric functions.

Another interesting arc length calculation involves the lemniscate of Bernoulli. In general, a lemniscate is the set of all points (x, y) in the plane such that the product of the distances from (x, y) to two fixed points is constant. The two fixed points that define a lemniscate are called the foci. If a is a positive real number and the foci are taken as $(-a,0)$ and $(a,0)$ then we can define a lemniscate as the set of all points (x,y) such that

$$[(x-a)^2+y^2]\;[(x+a)^2+y^2]=a^4.$$

This equation can be rewritten as

$$(x^2+y^2)^2=2a^2(x^2-y^2).$$

Taking $a=1/\sqrt{2}$ we obtain the equation of the particular lemniscate that we are interested in,

$$(x^2+y^2)^2=x^2-y^2.$$

Notice that the points $(-1,0)$ and $(1,0)$ satisfy this equation, along with $(0,0)$. Also, the point (x,y) is not on the lemniscate if $|x|>1$.

To obtain a parametrization of the section of the lemniscate in the first quadrant, consider the equation $x^2+y^2=t^2$ for $0\leqslant t\leqslant 1$. Solving for x and y in terms of t we obtain the parametrization $\gamma\colon[0,1]\to\mathbf{R}^2$,

$$\gamma_1(t)=t\sqrt{\frac{1+t^2}{2}}\quad\text{and}\quad\gamma_2(t)=t\sqrt{\frac{1-t^2}{2}}.$$

Since γ_1 is increasing on $[0,1]$ and γ_2 is the product of an increasing function and a decreasing function on $[0,1]$, γ_1 and γ_2 are functions of bounded variation on $[0,1]$. Therefore, γ has well-defined arc length on $[0,1]$. For all t such that $0\leqslant t<1$,

$$\gamma_1'(t)=\frac{1+2t^2}{\sqrt{2}\sqrt{1+t^2}}\quad\text{and}\quad\gamma_2'(t)=\frac{1-2t^2}{\sqrt{2}\sqrt{1-t^2}},$$

and so

$$\gamma_1'(t)^2+\gamma_2'(t)^2=\frac{1}{1-t^4}.$$

Thus, for any x such that $0<x<1$, γ is continuously differentiable on $[0,x]$ and

$$L(\gamma,[0,x])=\int_0^x\frac{1}{\sqrt{1-t^4}}\,dt.$$

The improper integral obtained when $x=1$ exists by Theorem 2.1. Therefore, if the arc length of the section of the lemniscate formed by points (x,y) in the first and fourth quadrants is denoted by ϖ_4 then Theorem 3.1 implies that

$$\frac{\varpi_4}{2}=\int_0^1\frac{1}{\sqrt{1-t^4}}\,dt.$$

The computations that we have done here for the lemniscate are interesting, but they also suggest that to make further progress we are going to need the power of the trigonometric functions and ability to describe a curve in polar coordinates. It will be helpful to review the formula for calculating the arc length of a path given in polar coordinate form when θ is a function of r.

THEOREM 3.2. *Suppose that $\gamma\colon [a,b] \to \mathbf{R}^2$, where $\gamma(r) = (r\cos\theta(r),\ r\sin\theta(r))$ and $\theta(r)$ is continuously differentiable on $[a,b]$. Then the arc length of γ on $[a,b]$ is given by*

$$\int_a^b \sqrt{1 + r^2\ \theta'(r)^2}\, dr.$$

PROOF. We have

$$\gamma_1(r) = r\cos\theta(r) \quad \text{and} \quad \gamma_2(r) = r\sin\theta(r),$$

and so

$$\gamma_1'(r) = -r\sin\theta(r)\ \theta'(r) + \cos\theta(r) \quad \text{and} \quad \gamma_2'(r) = r\cos\theta(r)\ \theta'(r) + \sin\theta(r).$$

Hence,

$$\gamma_1'(r)^2 + \gamma_2'(r)^2 = 1 + r^2\ \theta'(r)^2,$$

and the theorem follows. □

4. The Arc Length of a Clover Leaf

The m-clover is defined as the set of all points in the plane with polar coordinates (r,θ) satisfying

$$r^{m/2} = \cos(\tfrac{m}{2}\theta).$$

The m-clover has m leaves when m is odd and $m/2$ leaves when m is even. The set of points (r,θ) on the clover with r in $[0,1]$ and θ in $[-\pi/m,\ \pi/m]$ form the principal leaf. Since $\cos(-\frac{m}{2}\theta) = \cos(\frac{m}{2}\theta)$ we will restrict our attention to points (r,θ) on the clover where θ is in the interval $[0,\ \pi/m]$. This half of the principal leaf is the image of the path $\gamma\colon [0,1] \to \mathbf{R}^2$ where

$$\gamma(r) = (r\cos\theta(r), r\sin\theta(r))$$

and

$$\theta(r) = \tfrac{2}{m}\arccos(r^{m/2}).$$

We need to show that γ has well-defined arc length on $[0,1]$.

First, notice that for any $m \geqslant 1$, $r^{m/2}$ is a strictly increasing function on $[0,1]$. Since $\arccos(x)$ is strictly decreasing on $[0,1]$, it follows that $\theta(r)$ is also strictly decreasing on $[0,1]$ with $\theta(0) = \pi/m$ and $\theta(1) = 0$. For $m \geqslant 2$ we now see that each component function of γ is the product of monotonic functions on $[0,1]$ and therefore is of bounded variation. Hence, if $m \geqslant 2$ then γ has well-defined arc length on $[0,1]$.

For $m = 1$ we have a little more work to do because $\sin(x)$ is not monotonic on $[0,\pi]$. However, when $m = 1$,

$$\theta(\tfrac{1}{2}) = 2\arccos(\tfrac{1}{\sqrt{2}}) = \frac{\pi}{2}.$$

Therefore, each component function of γ is a product of monotonic functions on each subinterval $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Thus, the component functions are of bounded variation on each subinterval and so γ has well-defined arc length on $[0,1]$.

Now, for any integer $m \geqslant 1$ and all r such that $0 \leqslant r < 1$,

$$\theta'(r) = -\frac{r^{\frac{m}{2}-1}}{\sqrt{1-r^m}}.$$

Therefore, for any x such that $0 < x < 1$, Theorem 3.2 implies that the arc length of γ on $[0,x]$ is equal to

$$\int_0^x \sqrt{1 + r^2\left(\frac{r^{m-2}}{1-r^m}\right)}\, dr = \int_0^x \frac{1}{\sqrt{1-r^m}}\, dr.$$

The improper integral obtained when $x = 1$ exists by Theorem 2.1. Hence, if ϖ_m denotes the arc length of the principal leaf in the m-clover then

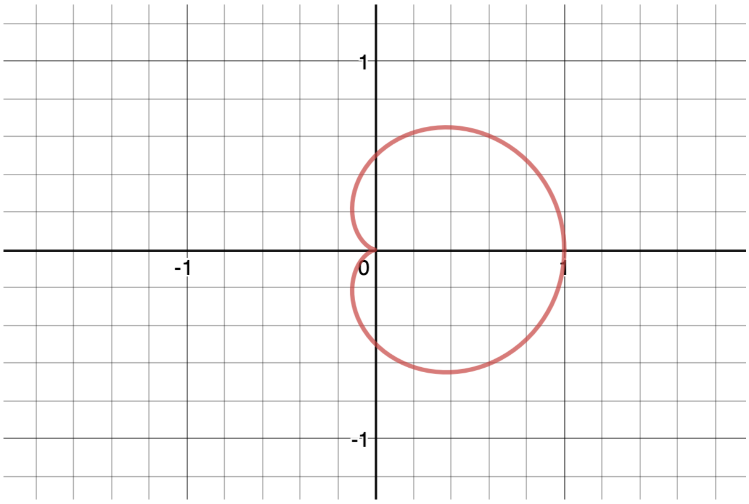
$$\frac{\varpi_m}{2} = \int_0^1 \frac{1}{\sqrt{1-r^m}}\, dr.$$

This proves formula (1.4) from the introduction and also provides the geometric significance of the generalization of Wallis's formula given by equation (1.3).

For $m = 1$ the 1-clover is a cardioid and the arc length can be computed exactly,

$$\frac{\varpi_1}{2} = \int_0^1 \frac{1}{\sqrt{1-r}}\, dr = 2.$$

That is, $\varpi_1 = 4$. It is easy to check that the right hand side of formula (1.3) is simply equal to 2 when $m = 1$.

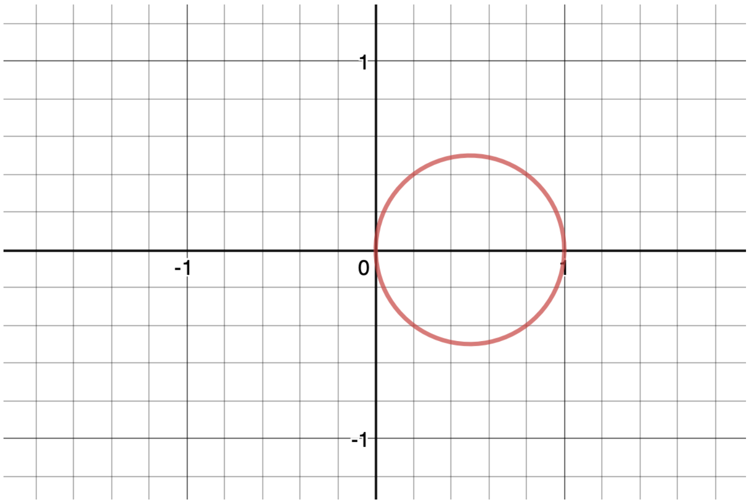


$m = 1$: cardioid

For $m = 2$ the 2-clover is given by the polar equation $r = \cos\theta$. The single leaf of the 2-clover is the circle with equation $x^2 + y^2 = x$. This is the circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. In this case we have

$$\frac{\varpi_2}{2} = \int_0^1 \frac{1}{\sqrt{1-r^2}}\, dr = \frac{\pi}{2},$$

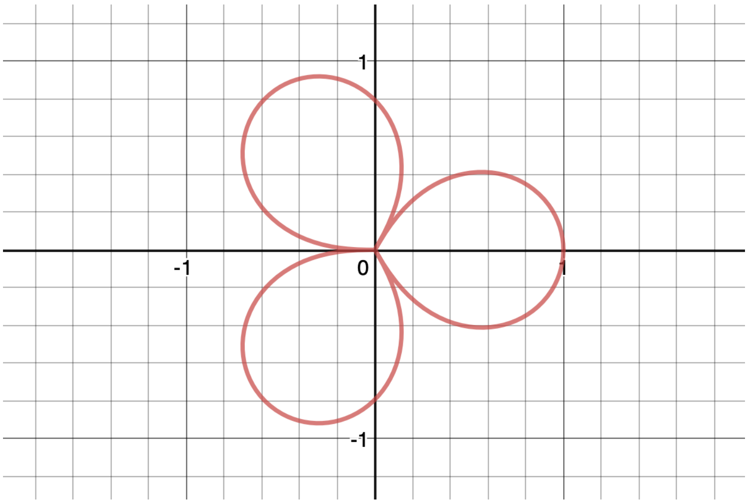
and so, of course, $\varpi_2 = \pi$.



$m = 2$: circle

For $m = 3$ the 3-clover is just that, a 3-leaf clover. In this case we have

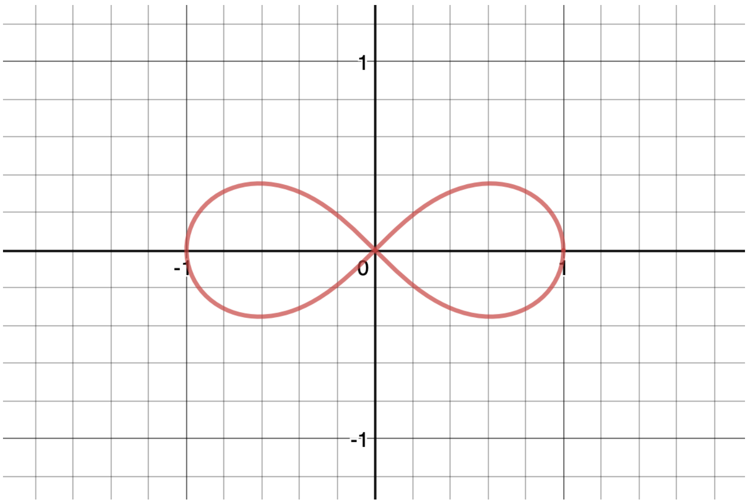
$$\frac{\varpi_3}{2} = \frac{2}{3} \prod_{n=1}^{\infty} \Big(\frac{6n-1}{6n-4}\Big)\Big(\frac{2n}{2n+1}\Big).$$



$m = 3$: 3-leaf clover

For $m = 4$ the 4-clover is simply the lemniscate we studied in the previous section, $(x^2+y^2)^2 = x^2 - y^2$. In this case our formulas show that

$$\varpi_4 = \prod_{n=1}^{\infty} \Big(\frac{4n-1}{4n-3}\Big)\Big(\frac{2n}{2n+1}\Big).$$



$m = 4$: lemniscate

At this point we have completed our exercises from Goursat, Hedrick, and Spivak!

Finally, we would like to thank Keith Conrad who kindly read an earlier draft of this note and suggested many improvements.

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