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Brackets: a method for definite integration

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Abstract. The method of brackets is an integration method with origins in the evaluation of Feynman diagrams. It consists of a small number of rules, some of which are still at the heuristic level. This work contains examples which illustrate this method. Elementary as well as more involved definite integrals are discussed.

The main advantage of this method over other classical procedures is that the method of brackets is easy to apply. All computations are reduced to linear systems of small order.

1. Introduction

The evaluation of definite integrals is one of the basic problems of Calculus. Many of the examples coming from scientific problems were incorporated into older textbooks such as [**12, 13**]. In the pre-digital times these evaluations were collected in Tables of Integrals, beginning with the work of D. Bierens de Haan [**5**] and including the volume by I. S. Gradshetyn and I. M. Ryzhik [**21**]. The goal of this note is to introduce a method of integration, called `the method of brackets`, that has been used by the authors to verify a large number of entries in [**21**]. K. van Deusen in her thesis at Tulane [**29**], presented a comparison of the method of brackets with the current methods appearing in the literature for the evaluation of Feynman diagrams. In particular she has shown that this method is an optimization and generalization of the so-called `negative dimensional integration method` (NDIM), see [**19**]. Details of this comparison appeared in [**30**].

The history of the second author interest in the evaluation of integrals began in 1991 when a first year graduate student, George Boros, approached him, saying that he was able to prove the formula

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1.1}$$

where $P_m(a)$ is a polynomial in a. He gave a formula for the coefficients of the polynomial in terms of a complicated triple sum with alternating signs, suggesting that some

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of the coefficients might be negative. Having no experience in these type of questions, my immediate reaction was to try to evaluate the integral using `Mathematica`. At that time, it was not possible to get an answer when a and m are parameters. `The same is true today`. In order to get some understanding of the problem, I was able to generate some data. My initial observation was that, in spite of his formula, the coefficients seem to be positive. I told him that he should prove that. Some time later he was able to do this, by first proving the formula

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1} + \frac{1}{\pi\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} N_{0,4}(a;k) c^{k+1} \tag{1.2}$$

and then using some results of Ramanujan. This work appeared in [**9**]. To be honest, I was never able to understand how he thought of this. We often talked about this. George Boros thesis [**7**] contains many similar results.

Intrigued by this, the second author searched in [**20**] for entries involving the double square root function and found entry 3.248.5 stating that

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{\frac{3}{2}} \sqrt{\varphi(x) + \sqrt{\varphi(x)}}} = \frac{\pi}{2\sqrt{6}}, \tag{1.3}$$

where $\varphi(x) = 1 + 4x^2/(3(1+x^2)^2)$. The reason for the failure of his many failed attempts to prove (1.3) is now clear: *the entry is incorrect.* (The later versions of the table do not contain this entry). Recently J. Arias de Reyna [**4**] has expressed the correct value as a difference of two elliptic integrals and P. Blaschke [**6**] has provided an argument to the effect that the correct entry should have been

$$\int_0^{\infty} \frac{dx}{(1+x^2)^{\frac{3}{2}} \sqrt{\varphi(x) + \sqrt{\varphi(x)^{\mathbf{3}}}}} = \frac{\pi}{2\sqrt{6}}, \tag{1.4}$$

`Typing formulas is complicated`.

Motivated by this, the second author began to provide proofs to all the entries in [**21**], generating a sequence of short papers beginning with [**22**]. During this process the author received an email from the first author stating that in his Ph. D. thesis in Physics, he had developed a procedure that optimized NDIM [**19**]. He used this method of integration to evaluate some complicated definite integrals coming from the study of Feynman diagrams. This is how two *sansanos*[1] joined efforts to develop the `method of brackets`.

Feynman developed a systematic procedure to convert the interaction of elementary particles into a diagram. A multi-dimensional integral is then attached to the diagram, from which some physical properties of the interaction can be obtained. See [**25, 26, 31**] for details. There is a wide variety of methods to evaluate these integrals, see [**26, 32**] for details. An approach based on Mellin-Barnes integrals, where the procedures are rigorously established appears in [**11**].

The method of brackets described here consists of a small number of rules, some of which are at the moment at a heuristic level. This method reduces the problem of evaluating integrals to the solution of a small size linear system. During the last

[1]This is the name given to students from Universidad Santa Maria in Valparaiso, Chile.

decade, the second author has involved a number of his graduate students in this problem:

(1) K. Kohl: *Algorithmic methods for definite integration*; 2011
(2) L. Jiu: *The method of brackets and the Bermoulli symbol*; 2016
(3) T. Ngo: *An analytic approach to the method of brackets*; 2018
(4) K. van Deusen: *A comparison of negative-dimensional integration techniques*; 2021
(5) Z. Bradshaw: *A rigorous treatment of the method of brackets*; 2023

These theses contain a variety of results of this method. The set of rules discussed below came as a modification of the so-called negative dimension integration method (NDIM) [**2, 3, 27, 28**]. The most important feature of the method of brackets is its simplicity and that it applies to a wide range of problems. One of the rules, see Step 3 below, is connected to the so-called Ramanujan's Master Theorem [**1, 16**]:

If a complex-valued function $f(x)$ has an expansion of the form $f(x) = \sum_{k=0}^{\infty} \frac{\varphi(k)}{k!}(-x)^k$, *then the Mellin transform of $f(x)$ is given by* $\int_0^\infty x^{s-1} f(x)\, dx = \Gamma(s)\varphi(-s)$.

The origin and analysis of this formulation are discussed in [**19**] and [**15**].

The next goal in the analysis of the method of brackets is to provide a rigorous foundation to the heuristic rules presented here. In his thesis, Z. Bradshaw [**10**] has provided rigorous proofs of some of them.

The evaluation of definite integrals has produced in the recent past some interesting connections to many mathematical topics; see [**23**] for some of them. The goal of this paper is to present to a general audience a variety of techniques used in this endevour. The next section illustrates these ideas with the example

$$I(a, b; \mu, \nu) = \int_0^\infty \frac{dx}{(ax^\mu + b)^\nu}. \tag{1.5}$$

The last method in this section presents some of the rules of the method of brackets. Section 3 illustrates how to evaluate integrals coming from Feynman diagrams. The main point to emphasize is the simplicity of the method.

2. The evaluations of an eulerian integral

Many of the entries in [**21**] can be reduced to special values of the eulerian integrals

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}\, dt \quad \textit{and} \quad B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt, \tag{2.1}$$

the classical gamma and beta function, respectively. This section presents a variety of proofs for the evaluation of (1.5). For simplicity of exposition, the parameters a, b; μ, ν are assumed to be real and positive.

Proof 1. In this electronic era, the first attempt to a proof should be via a symbolic language. In this case, using `Mathematica` directly gives

$$I(a,b;\mu,\nu) = \frac{1}{\mu} b^{1/\mu-\nu} a^{-1/\mu} B\left(\frac{1}{\mu}, \nu - \frac{1}{\mu}\right). \tag{2.2}$$

Requesting `Mathematica` to evaluate this integral provides also convergence conditions. The evaluation is faster if one includes in this request the condition

$$\textit{Assumptions}((a>0)\ \&\&\ (b>0)\ \&\&\ (\mu>0)\ \&\&\ (\nu>0))$$

as part of the input. The question of whether this is an acceptable proof could be the subject of an interesting discussion.

Proof 2. In the evaluation of an integral, a good change of variables usually is a good first step. Introduce here a new variable t by the relation $ax^\mu = bt$ to produce

$$I(a,b;\mu,\nu) = \frac{1}{\mu} b^{1/\mu-\nu} a^{-1/\mu} \int_0^\infty \frac{t^{1/\mu-1}\, dt}{(1+t)^\nu}\, dt, \tag{2.3}$$

and the expression (2.2) now follows from the classical formula

$$\int_0^\infty \frac{t^{x-1}\, dt}{(1+t)^{x+y}} = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{2.4}$$

appearing as entry 8.380.3 in [**21**].

Proof 3. Given that the table [**21**] is over 1000 pages long, it is reasonable for the beginner to miss a desired evaluation. See [**8**] for an illustration of this possibility. Indeed, in the case at hand, entry 3.251.11 states that

$$\int_0^\infty x^{\mu-1}(1+bx^p)^{-\nu}\, dx = \frac{1}{p} b^{-\mu/p} B\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right), \tag{2.5}$$

and with the right assignment of parameters it yields (2.2).

Proof 4. The next proof serves as an introduction to the method of brackets. Start with the expansion of the integrand in (2.3) in a series using the binomial theorem

$$(1-u)^{-\alpha} = \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} u^n \tag{2.6}$$

where $(\alpha)_n$ is the Pochhammer symbol

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{2.7}$$

This produces

$$\frac{1}{(ax^\mu+b)^\nu} = b^{-\nu} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \left(\frac{a}{b}\right)^n x^{n\mu}. \tag{2.8}$$

Integrating (2.8) leads to the (diverging) expression

$$\int_0^\infty \frac{dx}{(ax^\mu+b)^\nu} = b^{-\nu} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \left(\frac{a}{b}\right)^n \int_0^\infty x^{n\mu}\, dx. \tag{2.9}$$

This statement is now rewritten in terms of two new objects: [1] the *indicator* ϕ_n defined by

$$\phi_n = \frac{(-1)^n}{n!} \tag{2.10}$$

and [2] the `bracket` defined by

$$\langle r \rangle = \int_0^\infty x^{r-1}\,dx \quad \text{for } r \in \mathbb{C}. \tag{2.11}$$

Then (2.9) becomes

$$\int_0^\infty \frac{dx}{(ax^\mu + b)^\nu} = b^{-\nu} \sum_n \phi_n \frac{\Gamma(\nu+n)}{\Gamma(\nu)} \left(\frac{a}{b}\right)^n \langle n\mu + 1 \rangle. \tag{2.12}$$

The object on the right of (2.12) is called a *bracket series.* The change in notation: from $\sum_{n=0}^{\infty}$ to $\sum_n$ is our way to indicate to the reader that these bracket series will be assigned a value and that this will involve *non integer choices for* n.

The `method of brackets` for the evaluation of the integral of f over $(0, \infty)$ is presented next. It consists in a small number of rules described below. Details are presented in [**15**] and [**17**].

Definition 2.1. *This consists on three easy steps:*
`Step 1.` *Start with an expansion of the integrand in the form*

$$f(x) = \sum_{n=0}^\infty \phi_n a_n x^{\alpha n + \beta - 1} \tag{2.13}$$

with a_n, α, $\beta \in \mathbb{R}$ *and* ϕ_n *as in* (2.10).
`Step 2.` *Integrate to form the brackets series* $\sum_n \phi_n a_n \langle \alpha n + \beta \rangle$.
`Step 3.` *Assign the bracket series in Step 2 the number*

$$\frac{1}{|\alpha|} A(n^*)\Gamma(-n^*). \tag{2.14}$$

Here A is a complex-valued function interpolating the sequence $\{a_n\}$ at the integers; that is, $A(n) \to a_n$, for $n \in \mathbb{N}$. The number $n^* = -\beta/\alpha$[2] *is the solution of the linear equation obtained by requiring the vanishing of the brackets. Finally Γ is the classical gamma function. This step can be established rigorously: it is the so-called Ramanujan Master Theorem.*

Therefore the evaluation of (2.12) reduces to solving $n\mu + 1 = 0$. Doing so gives $n^* = -1/\mu$ and then use Step 3 to obtain

$$\int_0^\infty \frac{dx}{(ax^\mu + b)^\nu} = b^{-\nu} \frac{\Gamma(\nu - 1/\mu)}{\Gamma(\nu)} \left(\frac{a}{b}\right)^{-1/\mu}. \tag{2.15}$$

[2] The notation n^* was used in the earlier papers on the method of brackets. The notation is kept here for convenience.

This is (2.2). *It is a remarkable fact that this method gives the value of the integral.*

Proof 5. In the argument above one starts with an expansion of the integrand and then use the method of brackets to evaluate the integral. The proof is now simplified by introducing a new rule for brackets.

DEFINITION 2.2. ***Multinomial expansion rule.*** *For $\alpha \in \mathbb{R}$, the expression*

$$(a_1 + a_2 + \cdots + a_r)^{-\alpha} \tag{2.16}$$

is assigned the brackets series

$$\sum_{n_1 \cdots n_r} \phi_{1,2,\cdots,r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle \alpha + n_1 + \cdots + n_r \rangle}{\Gamma(\alpha)}, \tag{2.17}$$

where $\phi_{1,2,\cdots,r}$ is a short-hand notation for the product $\phi_{n_1} \cdots \phi_{n_r}$. Therefore the sum of r terms in (2.16) *is assigned an r-dimensional bracket series.*

An idea of the proof starts with the identity

$$\frac{1}{b^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-bt} \, dt \tag{2.18}$$

and with $b = a_1 + \cdots + a_r$ it follows that

$$\begin{aligned}
\frac{1}{(a_1 + \cdots + a_r)^\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(a_1 + \cdots + a_r)t} \, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-a_1 t} \cdots e^{-a_r t} \, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \left[\sum_{n_1, \cdots, n_r} \phi_{n_1 \cdots n_r} a_1^{n_1} \cdots a_r^{n_r} t^{n_1 + \cdots n_r} \right] dt \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \sum_{n_1, \cdots, n_r} \phi_{n_1 \cdots n_r} a_1^{n_1} \cdots a_r^{n_r} t^{\alpha + n_1 + \cdots n_r - 1} \, dt
\end{aligned}$$

and the last integral is the bracket $\langle \alpha + n_1 + \cdots + n_r \rangle$.

Then the integrand in (1.5) is assigned the bracket series expansion

$$(ax^\mu + b)^{-\nu} = \sum_{n_1, n_2} \phi_{1,2} \frac{a^{n_1} b^{n_2}}{\Gamma(\nu)} \langle \nu + n_1 + n_2 \rangle x^{\mu n_1}. \tag{2.19}$$

Integration now produces

$$\int_0^\infty (ax^\mu + b)^{-\nu} \, dx = \sum_{n_1, n_2} \phi_{1,2} \frac{a^{n_1} b^{n_2}}{\Gamma(\nu)} \langle \nu + n_1 + n_2 \rangle \langle \mu n_1 + 1 \rangle. \tag{2.20}$$

Therefore, without requiring an expansion of the integrand, the evaluation of (1.5) has been transformed into a 2-dimensional brackets series with 2 brackets.

Each representation of an integral by a bracket series has associated a *complexity index of the representation* via

$$\text{complexity index } = \text{number of sums } - \text{ number of brackets.} \tag{2.21}$$

It is important to observe that the complexity index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal complexity index* should be chosen. The level of difficulty in the analysis of the resulting bracket series increases with the complexity index.

The next rule states how to evaluate a multidimensional bracket series of index 0.

DEFINITION 2.3. ***Multi-dimensional brackets series of index*** 0. *Assume that $A = (a_{ij})$ is a non-singular real matrix. Then*

$$\sum_{n_1 \geqslant 0} \cdots \sum_{n_r \geqslant 0} \phi_{n_1 \cdots n_r} f(n_1, \cdots, n_r) \langle a_{11}n_1 + \cdots + a_{1r}n_r + c_1 \rangle \cdots \langle a_{r1}n_1 + \cdots + a_{rr}n_r + c_r \rangle$$

$$= \frac{1}{|det(A)|} f(n_1^*, \cdots n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*),$$

where $\{n_j^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

This rules states that to evaluate (2.20) one only need to solve the linear system

$$\begin{aligned} n_1 + n_2 &= -\nu \\ \mu n_1 &= -1. \end{aligned} \tag{2.22}$$

The solutions are $n_1^* = -1/\mu$ and $n_2^* = 1/\mu - \nu$ and since the matrix A has determinant $\det A = -\mu$, rule 2.3 now gives

$$\int_0^\infty (ax^\mu + b)^{-\nu} \, dx = \frac{1}{|-\mu|} \frac{a_{n_1^*} b_{n_2^*}}{\Gamma(\nu)} \times \Gamma(-n_1^*) \Gamma(-n_2^*). \tag{2.23}$$

Replacing the values of n_1^* and n_2^* gives (2.2). *Therefore, the evaluation of* (2.20) *has been reduced to the solution of a two-dimensional linear system of equations.* `Remarkable`!

In this section we have given a rule to evaluate multi-dimensional bracket series on index 0. The case of non-zero index will appear in the next section.

3. The original problem

The method of brackets described above was developed in [**19**] as a method to compute integrals coming from Feynman diagrams. For the present exposition, a `Feynmann diagram` is simply a graph G with $E+1$ `external lines` and N `internal lines` and L `loops`. See [**18**] for details. The internal lines are sometimes referred as `propagators`. All but one of these external lines are assumed to be independent. From its connection to Physics, the internal and external lines represent particles that transfer momentum among the vertices of the diagram. Each of these particles carries a `mass` $m_i \geqslant 0$ for $i = 1, \ldots, N$. The vertices represent the interaction of these particles and conservation of momentum at each vertex assigns the momentum corresponding to the internal lines. A Feynman diagram has an associated integral given by a so-called `parametrization` of the diagram.

EXAMPLE 3.1. The first diagram corresponds to the so-called `massless sunset diagram` indicated in Figure 1. The diagram is parametrized by

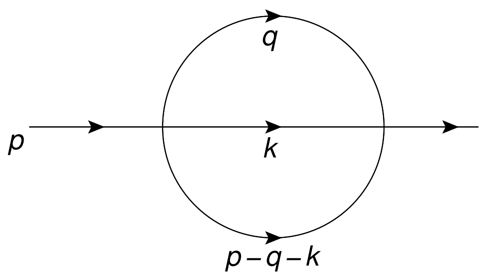

FIGURE 1. The massless sunset diagram

the integral (in momentum space)

$$G = \int \frac{d^D q}{i\pi^{D/2}} \frac{d^D k}{i\pi^{D/2}} \frac{1}{q^2 \cdot k^2 \cdot (p-q-k)^2} \tag{3.1}$$

where $D = 4 - 2\epsilon$ and $\epsilon \to 0^+$ is the so-called `dimensional regulator`. Schwinger parametrization is a systematic procedure to express the integral G in terms of some parameters (see [**24**]). In this example one finds out that G becomes

$$G = (-1)^{-D} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\exp\left(-\frac{xyz}{(xy+xz+yz)}p^2\right)}{(xy+xz+yz)^{D/2}} \, dx \, dy \, dz. \tag{3.2}$$

This triple integral is now evaluated by the method of brackets. Start by expanding the exponential to obtain

$$G = (-1)^{-D} \sum_{n_1} \phi_1 (p^2)^{n_1} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{n_1} y^{n_1} z^{n_1} \, dx \, dy \, dz}{(xy+xz+yz)^{D/2+n_1}}. \tag{3.3}$$

The expansion rule 2.2 now gives

$$\frac{1}{(xy+yz+xz)^{D/2+n_1}} = \sum_{n_2,n_3,n_4} \phi_{2,3,4} x^{n_2+n_3} y^{n_2+n_4} z^{n_3+n_4} \frac{\langle \frac{D}{2} + n_1 + n_2 + n_3 + n_4 \rangle}{\Gamma(\frac{D}{2} + n_1)} \tag{3.4}$$

and then G becomes

$$G = (-1)^{-D} \sum_{\vec{n}} \phi_{1,2,3,4} (p^2)^{n_1} \frac{\langle \frac{D}{2} + n_1 + n_2 + n_3 + n_4 \rangle}{\Gamma(\frac{D}{2} + n_1)} \times \langle n_1 + n_2 + n_3 + 1 \rangle \langle n_1 + n_2 + n_4 + 1 \rangle \langle n_1 + n_3 + n_4 + 1 \rangle. \tag{3.5}$$

This is a 4-dimensional bracket series with 4 brackets (and hence of index 0). The matrix of the system has determinant 1 with solution $n_1^* = D-3$, $n_2^* = n_3^* = n_4^* = 1 - D/2$. This gives the value

$$G = (-1)^{-D} \frac{\Gamma(3-D)\Gamma(D/2-1)^3}{\Gamma(3D/2-3)} (p^2)^{D-3}. \tag{3.6}$$

EXAMPLE 3.2. The second example considers the diagram depicted in Figure 2. It contains two external lines and three internal lines. One of them, marked by the momentum q, has mass M and the other are massless. As in the first example, $D = 4 - 2\epsilon$ and $\epsilon > 0$ is the dimensional regulator.

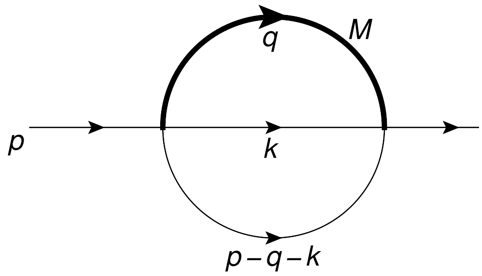

FIGURE 2. The sunset diagram with mass.

The integral associated to this Feynman diagram in terms of the Schwinger parameters is now given by

$$G = (-1)^{-D} \int_0^\infty \int_0^\infty \int_0^\infty \exp(xM^2) \frac{\exp\left(-\frac{xyz}{(xy+xz+yz)}p^2\right)}{(xy+xz+yz)^{D/2}} dx\,dy\,dz. \tag{3.7}$$

The evaluation begins with the expansion of the exponentials in Taylor series and then use the expansion rule 2.2 to write G in the form

$$G = (-1)^{-D} \sum_{\vec{n}} \phi_{\vec{n}} (-M^2)^{n_1} (p^2)^{n_2} \frac{1}{\Gamma(D/2+n_2)} \tag{3.8}$$

$$\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \rangle \langle n_1 + n_2 + n_3 + n_4 + 1 \rangle \langle n_2 + n_3 + n_5 + 1 \rangle \langle n_2 + n_4 + n_5 + 1 \rangle,$$

with $\vec{n} = (n_1, n_2, n_3, n_4, n_5)$. The vanishing of the brackets leads to the system

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix} = \begin{pmatrix} -D/2 \\ -1 \\ -1 \\ -1 \end{pmatrix}. \tag{3.9}$$

This is a 5-dimensional bracket series with 4 brackets, so it is a problem on non-zero index. The next rule describes how to deal with it.

DEFINITION 3.1. `Multi-dimensional brackets series of non-zero index`. *The value of a multi-dimensional bracket series of positive complexity index is obtained by computing all the contributions of maximal rank using Rule 2.3. These contributions to the integral appear as series in the free indices and depend on the parameters of the problem. Series with the same argument are added to combine them into a single sum.*

The matrix in (3.9) is of rank 4, so there are 5 systems to consider, one per free parameter $n_j,\ 1 \leqslant j \leqslant 5$. The details are given for one case.

n_1 `as a free parameter`. Then (3.9) becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix} = \begin{pmatrix} -D/2 \\ -n_1-1 \\ -1 \\ -1 \end{pmatrix} \tag{3.10}$$

with solution

$$n_2^* = D - n_1 - 3,\ n_3^* = -\frac{1}{2}D + 1,\ n_4^* = -\frac{1}{2}D + 1,\ n_5^* = -\frac{1}{2}D + n_1 + 1.$$

Rule 2.3 now gives

(3.11)

$$G_1 = (-1)^{-D}(p^2)^{D-3}\Gamma^2\left(\frac{D}{2}-1\right)\sum_{n_1=0}^{\infty}\frac{1}{n_1!}\frac{\Gamma(n_1+3-D)\Gamma(D/2-1-n_1)}{\Gamma(3D/2-3-n_1)}\left(\frac{M^2}{p^2}\right)^{n_1}.$$

a series in the free parameter. To identify this result, simplify the gamma terms using

$$\Gamma(u+n) = (u)_n\Gamma(u) \quad \text{and} \quad (u)_{-n} = \frac{(-1)^n}{(1-u)_n} \tag{3.12}$$

to obtain

$$G_1 = (-1)^{-D}(p^2)^{D-3}\frac{\Gamma^3(\frac{D}{2}-1)\,\Gamma(3-D)}{\Gamma(\frac{3D}{2}-3)}\,{}_2F_1\left(\begin{matrix} 3-D \ \ 4-\frac{3D}{2} \\ 2-\frac{D}{2} \end{matrix}\middle| \frac{M^2}{p^2}\right) \tag{3.13}$$

where ${}_2F_1$ is the hypergeometric function.

Two of the remaining four indices give

(3.14)

$$G_2 = (-1)^{-D}(-M^2)^{D-3}\frac{\Gamma^2(\frac{D}{2}-1)\Gamma(-\frac{D}{2}+2)\Gamma(3-D)}{\Gamma(\frac{D}{2})}\,{}_2F_1\left(\begin{matrix} 3-D \ \ 2-\frac{D}{2} \\ \frac{D}{2} \end{matrix}\middle| \frac{p^2}{M^2}\right)$$

and

(3.15)

$$G_3 = (-1)^{-D}(-M^2)^{D/2-1}(p^2)^{D/2-2}\frac{\Gamma^2(\frac{D}{2}-1)\,\Gamma(2-\frac{D}{2})\Gamma(1-\frac{D}{2})}{\Gamma(D-2)}\,{}_2F_1\left(\begin{matrix} 2-\frac{D}{2} \ \ 3-D \\ \frac{D}{2} \end{matrix}\middle| \frac{M^2}{p^2}\right)$$

while the remaining two produce no solutions. Taking into account that the hypergeometric series converges when the last argument has modulus strictly less than 1, it follows that

$$G = \begin{cases} G_1 + G_3 & \text{if } \left|\frac{M^2}{p^2}\right| < 1, \\ G_2 & \text{if } \left|\frac{M^2}{p^2}\right| > 1. \end{cases} \tag{3.16}$$

4. Conclusions

The method of brackets for definite integration converts the evaluation of an integral into the solution of a linear system of equations. It is based on a small list of heuristic rules. The authors have used this method to evaluate a large number of entries of classical tables of integrals [**14**]. There has never been an instance when an application of these heuristic rules produces an error.

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