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Four Famous Formulas are Equivalent

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Abstract. In this note we prove that Wallis's formula, the Probability Integral formula, Stirling's formula, and the Central Binomial Coefficient formula are in fact pairwise equivalent statements.

1. Introduction

The mathematical literature devoted to the following four assertions is vast:

Wallis's Formula:

(1.1)
$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}$$

Probability Integral Formula:

(1.2)
$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

Stirling's Formula:

(1.3)
$$n! \sim n^n\, e^{-n} \sqrt{2\pi n}$$

Central Binomial Coefficient Formula:

(1.4)
$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

Our goal is to prove the following theorem in the most elementary way possible.

Theorem 1. *Statements* (1.1), (1.2), (1.3), *and* (1.4) *are pairwise equivalent.*

Yes, the beautiful formula discovered by John Wallis in 1655 has been running around in various disguised forms for hundreds of years!

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Many readers will be familiar with the fact that Wallis's formula is often used in proofs of the Probability Integral formula and in proofs of Stirling's formula, so it is not a surprise to assert the formula (1.1) implies both formulas (1.2) and (1.3). However, it does not seem to be common knowledge that starting with either formula (1.2) or formula (1.3) we can derive Wallis's formula. As for the Central Binomial Coefficient formula, that comes along for free.

We will not take time in this article to review the amazing history surrounding these four formulas. However, the interested reader will be happy to learn that Paul Nahin recently published a wonderful book called *The Probability Integral, Its Origin, Its Importance, and Its Calculation* [1]. That's a great place to start learning the backstory!

For the discussion that follows, it will be useful to restate Wallis's formula in an equivalent form.

Theorem 2. *Wallis's formula as expressed in equation* (1.1) *is equivalent to the assertion:*

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \left(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\right). \tag{1.5}$$

Proof. First note that if we define a_n as follows,

$$a_n = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1},$$

then formula (1.1) is the assertion that the sequence $\{a_n\}$ converges to $\pi/2$. Taking square roots we have

$$\sqrt{a_n} = \left(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\right)\cdot\frac{1}{\sqrt{2n+1}}.$$

Therefore,

$$\frac{1}{\sqrt{n}}\cdot\left(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\right) = \sqrt{a_n}\cdot\sqrt{\frac{2n+1}{2n}}\cdot\sqrt{2}.$$

Computing limits we see that Wallis's formula (1.1) implies formula (1.5). The proof of the converse is similar. □

Let's take a closer look at assertion (1.5). Since

$$2\cdot 4\cdot 6\cdot 8\cdots(2n) = 2^n\, n! \quad\text{and}\quad 1\cdot 3\cdot 5\cdot 7\cdots(2n-1) = \frac{(2n)!}{2^n\, n!}$$

we can rewrite (1.5) as follows:

$$\sqrt{\pi} = \lim_{n\to\infty}\frac{2^{2n}\,(n!)^2}{(2n)!}\cdot\frac{1}{\sqrt{n}}. \tag{1.6}$$

Now, two important observations follow from assertion (1.6). First, if we apply the logarithm function to both sides of (1.6) we see that Wallis's formula is equivalent to the assertion:

$$\log\sqrt{\pi} = \lim_{n\to\infty}\left(2n\log 2 + 2\log n! - \log(2n)! - \tfrac{1}{2}\log n\right). \tag{1.7}$$

This version of Wallis's formula will be essential later in this paper. The second point about assertion (1.6) is that if we divide both sides of the equation by $\sqrt{\pi}$ and then take the reciprocal of each term we see that (1.6) is equivalent to the assertion:

$$\lim_{n\to\infty} \frac{(2n)!}{(n!)^2} \cdot \frac{\sqrt{\pi n}}{2^{2n}} = 1. \tag{1.8}$$

However, formula (1.8) is exactly equal to the Central Binomial Coefficient formula (1.4). Therefore, famous formulas (1.1) and (1.4) are equivalent. This is what we meant when we said formula (1.4) comes along for free.

2. Formulas (1.1) and (1.2) are Equivalent

The following theorem about a sequence of integrals closely connected to the moments of the normal distribution is the key to understanding the close connection between Wallis's formula and the Probability Integral formula.

Theorem 3. *For each integer $n \geqslant 0$ define*

$$E_n = \int_0^\infty x^n e^{-x^2}\, dx.$$

(a) *For all $n \geqslant 0$ the improper integral E_n converges.*

(b) *For all $n \geqslant 0$,*

$$E_{n+2} = \frac{n+1}{2} E_n.$$

(c) *For all $n \geqslant 1$,*

$$E_{2n} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots (2n-1)}{2^n} E_0.$$

(d) *For all $n \geqslant 0$,*

$$E_{2n+1} = \frac{n!}{2}.$$

(e) *For all $n \geqslant 1$, $E_n^2 \leqslant E_{n+1}E_{n-1}$.*

Proof. First notice that for any real number $b > 0$

$$\int_0^b x\, e^{-x^2}\, dx \;=\; \frac{e^{-x^2}}{-2}\Big]_0^b \;=\; \frac{1}{2} - \frac{e^{-b^2}}{2}.$$

Therefore, the improper integral E_1 converges to $1/2$. Moreover, the function e^{-x^2} is continuous on $[0,1]$ and, for all $x \geqslant 1$, we have $0 \leqslant e^{-x^2} \leqslant xe^{-x^2}$. So, by the Comparison Test for improper integrals, the probability integral E_0 also converges.

Now, recall the general formula for integration by parts:

$$\int_a^b f(x)g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\, dx.$$

For $n \geqslant 0$, apply the integration by parts formula with

$$f(x) = e^{-x^2}, \quad f'(x) = (-2x)e^{-x^2}, \quad g(x) = \frac{x^{n+1}}{n+1}, \quad \text{and} \quad g'(x) = x^n$$

to see that for all real numbers $b > 0$,

$$\int_0^b x^n e^{-x^2}\, dx = \frac{x^{n+1}}{n+1} e^{-x^2}\Big]_0^b \; - \; \int_0^b (-2x) \frac{x^{n+1}}{n+1} e^{-x^2}\, dx.$$

Rearranging this equation gives us

$$\int_0^b x^{n+2} e^{-x^2}\, dx = \frac{n+1}{2} \int_0^b x^n e^{-x^2}\, dx - \Big(\frac{1}{2}\Big) b^{n+1} e^{-b^2}.$$

Since we know that E_0 and E_1 converge, the formula above shows that E_n converges for all $n \geqslant 0$ and that we have the reduction formula (b). The evaluations of E_{2n} and E_{2n+1} given in formulas (c) and (d) are easily obtained from the reduction formula by induction.

Now for the tricky step! For any real number λ and all integers $n \geqslant 1$,

$$\begin{aligned} E_{n+1} + 2\lambda E_n + \lambda^2 E_{n-1} &= \int_0^\infty e^{-x^2} (x^{n+1} + 2\lambda x^n + \lambda^2 x^{n-1})\, dx \\ &= \int_0^\infty e^{-x^2} x^{n-1} (x+\lambda)^2\, dx \\ &> 0. \end{aligned}$$

Therefore, the quadratic polynomial $E_{n+1} + 2\lambda E_n + \lambda^2 E_{n-1}$ in λ has no real roots and its discriminant must be negative. That is,

$$4\, E_n^2 - 4\, E_{n+1}\, E_{n-1} < 0.$$

Thus, for all integers $n \geqslant 1$ we have $E_n^2 < E_{n+1}\, E_{n-1}$ and (e) is proved. This proof of (e) is due to Stieltjes, 1890. ☐

Theorem 4. *Statements* (1.2) *and* (1.5) *are equivalent.*

Proof. Define E_n as in the previous theorem. For $n \geqslant 1$ we have $E_{2n}^2 < E_{2n+1} E_{2n-1}$. Using the formulas for E_{2n-1}, E_{2n} and E_{2n+1} we see that

$$\frac{(1 \cdot 3 \cdot 5 \ldots (2n-1))^2}{2^{2n}}\; E_0^2 \;\leqslant\; \frac{n!(n-1)!}{4}.$$

Thus,

$$4E_0^2 \;\leqslant\; \frac{2^{2n}\; n!\; n!}{n\; (1 \cdot 3 \cdot 5 \ldots (2n-1))^2},$$

or

$$2E_0 \leqslant \frac{1}{\sqrt{n}}\Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big).$$

For $n \geqslant 0$ we have $E_{2n+1}^2 < E_{2n+2}E_{2n}$. Therefore,

$$\frac{n!\ n!}{4} \leqslant \Big(\frac{1\cdot 3\cdot 5\ldots(2n+1)}{2^{n+1}}\Big)\Big(\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n}\Big)E_0^2,$$

or

$$\frac{2^{n+1}\ 2^n\ n!\ n!}{(1\cdot 3\cdot 5\ldots(2n-1))^2(2n+1)} \leqslant 4\ E_0^2.$$

Therefore,

$$\sqrt{\frac{2}{2n+1}}\ \Big(\frac{n!\ 2^n}{1\cdot 3\cdot 5\ldots(2n-1)}\Big) \leqslant 2\ E_0,$$

or

$$\sqrt{\frac{2n}{2n+1}}\ \frac{1}{\sqrt{n}}\ \Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big) \leqslant 2\ E_0.$$

The last inequality can also be rewritten as:

$$\frac{1}{\sqrt{n}}\ \Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big) \leqslant 2\ E_0\sqrt{\frac{2n+1}{2n}}.$$

Thus, we have now shown that

$$\sqrt{\frac{2n}{2n+1}}\ \frac{1}{\sqrt{n}}\ \Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big) \leqslant 2\ E_0 \leqslant \frac{1}{\sqrt{n}}\Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big)$$

and that

$$2E_0 \leqslant \frac{1}{\sqrt{n}}\ \Big(\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\frac{2n}{2n-1}\Big) \leqslant 2\ E_0\ \sqrt{\frac{2n+1}{2n}}.$$

By two applications of the squeeze theorem for limits we see that assertion (1.5) is true if and only if the probability integral E_0 has the value $\sqrt{\pi}/2$. □

Theorem 2 and Theorem 4 together show that the formulas (1.1) and (1.2) are equivalent. In [2] Uspensky develops the ideas of Theorems 2 and 3 to show that Wallis's formula implies the Probability Integral formula. However, he does not discuss the converse.

3. Formulas (1.1) and (1.3) are Equivalent

In this section we need Wallis's formula as shown in assertion (1.7). Let's rewrite that here for convenience:

$$\log\sqrt{\pi} = \lim_{n\to\infty}(2n\log 2 + 2\log n! - \log(2n)! - \tfrac{1}{2}\log n). \tag{3.1}$$

Stirling's formula (1.3) can be expressed in the form:

$$\sqrt{2\pi} = \lim_{n\to\infty} \frac{n!}{n^n\, e^{-n}\sqrt{n}}.$$

Taking logarithms shows that Stirling's formula is equivalent to the assertion:

$$\log\sqrt{2\pi} = \lim_{n\to\infty}(\log n! - (n+\tfrac{1}{2})\log n + n). \tag{3.2}$$

So, the following theorem will establish the equivalence of formulas (1.1) and (1.3).

Theorem 5. *Statements* (3.1) *and* (3.2) *are equivalent.*

Proof. Let's first assume that (3.2) is true. Replacing n by $2n$ in (3.2) shows that:

$$\log\sqrt{2\pi} = \lim_{n\to\infty}(\log(2n)! - (2n+\tfrac{1}{2})\log 2n + 2n), \tag{3.3}$$

and multiplying both sides (3.2) by 2 shows that:

$$2\log\sqrt{2\pi} = \lim_{n\to\infty}(2\log n! - 2(n+\tfrac{1}{2})\log n + 2n). \tag{3.4}$$

Now, subtract (3.3) from (3.4) and simplify terms to find that:

$$\log\sqrt{2\pi} = \lim_{n\to\infty}(2n\log 2 + 2\log n! - \log(2n)! - (\tfrac{1}{2})\log n + (\tfrac{1}{2})\log 2). \tag{3.5}$$

If we now subtract $\log\sqrt{2}$ from each term in (3.5) and from the limit $\log\sqrt{2\pi}$ we immediately obtain assertion (3.1). Therefore, (3.2) implies (3.1).

Now, assume that assertion (3.1) is true, and let $d_n = \log n! - (n+\frac{1}{2})\log n + n$ for all $n \geqslant 1$. If we can just prove that $\{d_n\}$ converges then the proof that (3.1) implies (3.2) is almost identical the proof we just carried out to show that (3.2) implies (3.1). Feller provides the following beautiful argument that $\{d_n\}$ converges in [3]. First notice that

$$d_n - d_{n+1} = (n+\tfrac{1}{2})\log\left(\tfrac{n+1}{n}\right) - 1.$$

Since

$$\frac{n+1}{n} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}},$$

and the following series expansion is valid for $|t| < 1$,

$$\log\left(\frac{1+t}{1-t}\right) = 2\Big(t + \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + \tfrac{1}{7}t^7 + \cdots\Big),$$

we have:

$$d_n - d_{n+1} = \tfrac{1}{3}\left(\tfrac{1}{2n+1}\right)^2 + \tfrac{1}{5}\left(\tfrac{1}{2n+1}\right)^4 + \tfrac{1}{7}\left(\tfrac{1}{2n+1}\right)^6 + \cdots. \tag{3.6}$$

Comparing the right side of (3.6) to the geometric series with ratio $(2n+1)^{-2}$ yields:

$$0 < d_n - d_{n+1} < \tfrac{1}{3[(2n+1)^2-1]} = \tfrac{1}{12n} - \tfrac{1}{12(n+1)}.$$

It now follows that the sequence $\{d_n\}$ is decreasing while the sequence $\{d_n - \frac{1}{12n}\}$ is increasing. Thus, $\{d_n\}$ is a decreasing sequence bounded below by $\frac{11}{12}$ and it must converge to some positive number C. Therefore,

$$C = \lim_{n\to\infty} (\log n! - (n + \tfrac{1}{2}) \log n + n). \tag{3.7}$$

Now, just as we did above, first multiply both by sides of (3.7) by 2, then subtract the formula obtained by replacing n by $2n$ in (3.7) to obtain:

$$C = 2C - C = \lim_{n\to\infty} (2n \log 2 + 2 \log n! - \log(2n)! - \tfrac{1}{2} \log n + \tfrac{1}{2} \log 2).$$

Therefore,

$$C - \tfrac{1}{2} \log 2 = \lim_{n\to\infty} (2n \log 2 + 2 \log n! - \log(2n)! - \tfrac{1}{2} \log n).$$

Since we assume that (3.1) is true we obtain $C - \frac{1}{2} \log 2 = \log \sqrt{\pi}$, or $C = \log \sqrt{2\pi}$. With this value of C in equation (3.7) we now see that (3.1) implies (3.2). □

Theorems 2, 4, and 5, together with the observation that the Central Binomial Coefficient formula is equivalent to (1.5), complete the proof of Theorem 1.

4. An Easy Proof of Wallis's Formula

Since we now know that Theorem 1 is true, it would be nice to prove at least one of the four famous formulas! For completeness, we include an easy proof of Wallis's formula (1.1) that is similar to the proofs found in [4] and [5].

Theorem 6. *For each integer $n \geqslant 0$ define*

$$I_n = \int_0^{\pi/2} \cos^n x \; dx.$$

(a) *For all $n \geqslant 2$,*

$$I_n = \frac{n-1}{n} I_{n-2}.$$

(b) *For all $n \geqslant 1$,*

$$I_{2n} = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}.$$

(c) *For all $n \geqslant 1$,*

$$I_{2n+1} = 1 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdots \frac{2n}{2n+1}.$$

(d) *For all $n \geqslant 1$,*

$$I_{n+1} \leqslant I_n.$$

PROOF. For $n \geqslant 2$, the integration by parts formula with $g(x) = \sin x$, $g'(x) = \cos x$, $f(x) = \cos^{n-1} x$, and $f'(x) = -(n-1)(\cos^{n-2} x)(\sin x)$, yields

$$\begin{aligned}\int_0^{\pi/2} \cos^n x \, dx &= (n-1)\int_0^{\pi/2} (\cos^{n-2} x)(\sin^2 x)\, dx \\ &= (n-1)\int_0^{\pi/2} (\cos^{n-2} x)(1-\cos^2 x)\, dx \\ &= (n-1)\int_0^{\pi/2} \cos^{n-2} x \, dx - (n-1)\int_0^{\pi/2} \cos^n x \, dx.\end{aligned}$$

Thus, $nI_n = (n-1)I_{n-2}$ and the reduction formula (a) now follows. Since $I_0 = \pi/2$ and $I_1 = 1$, the evaluations of I_{2n} and I_{2n+1} given in (b) and (c) follow by induction. For all x such that $0 \leqslant x \leqslant \pi/2$ we have $0 \leqslant \cos x \leqslant 1$. Therefore, if $0 \leqslant x \leqslant \pi/2$ and $n \geqslant 1$, then $0 \leqslant \cos^{n+1} x \leqslant \cos^n x$. The inequality $I_{n+1} \leqslant I_n$ now follows. □

Theorem 7. *(Wallis's Formula)*

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}.$$

PROOF. Let I_n be the integral defined in the previous theorem. Then, for all $n \geqslant 1$, the inequality $I_{2n+1} \leqslant I_{2n}$ and the formulas for I_{2n} and I_{2n+1} imply that

$$\Big(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{2n}{2n+1}\Big) \leqslant \frac{\pi}{2}\Big(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2n-1}{2n}\Big).$$

Equivalently, $$\Big(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\Big) \leqslant \frac{\pi}{2}.$$

For all integers $n \geqslant 1$ we also have $I_{2n} \leqslant I_{2n-1}$ and so

$$\frac{\pi}{2}\Big(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\frac{2n-1}{2n}\Big) \leqslant \Big(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}\cdots\frac{2n-2}{2n-1}\Big).$$

Equivalently, $$\Big(\frac{2n}{2n+1}\Big)\frac{\pi}{2} \leqslant \Big(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\Big).$$

Therefore,

$$\Big(\frac{2n}{2n+1}\Big)\frac{\pi}{2} \leqslant \Big(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\Big) \leqslant \frac{\pi}{2}.$$

By the squeeze theorem for limits the proof is complete. □

5. The Numerical Accuracy of Stirling's Formula

The four famous formulas expressed in statements (1.1), (1.2), (1.3), and (1.4) are limit theorems which in themselves do not tell us much about how fast the relevant limits are approached. In [6] N.G. De Bruijn refers to our statement of Stirling's formula (1.3) as formula (1.1.1) and observes that:

> *For no single special value of n can we draw any conclusion from (1.1.1) about $n!$. It is a statement about infinitely many values of n, which, remarkably enough, does not state anything about any special value of n.*

The good news, however, is that since formula (1.3) has now been certified as true by Theorems 1 and 7, we can complete a line of thought that Feller begins on page 54 of [3]. Let's go back to the sequence $d_n = \log n! - (n + \frac{1}{2}) \log n + n$ for all $n \geqslant 1$ that was discussed in Section 3. Using equation (3.6) of the present paper, Feller notes that

$$d_n - d_{n+1} > \tfrac{1}{3[(2n+1)^2-1]} > \tfrac{1}{12n+1} - \tfrac{1}{12(n+1)+1}.$$

Therefore, the sequence $\{d_n - \frac{1}{12n+1}\}$ is decreasing. In Section 3 we observed that $\{d_n - \frac{1}{12n}\}$ is increasing. Since $\{d_n\}$ converges to $\log \sqrt{2\pi}$, it now follows that

$$\tfrac{1}{12n+1} < d_n - \log \sqrt{2\pi} < \tfrac{1}{12n}.$$

Thus, the following very useful estimate holds for all $n \geqslant 1$:

$$e^{\frac{1}{12n+1}} < \frac{n!}{n^n\, e^{-n} \sqrt{2\pi n}} < e^{\frac{1}{12n}}.$$

6. Two More Famous Formulas

There are actually two additional famous formulas that are equivalent to the four formulas discussed in Theorem 1. However, the equivalence proofs for these two formulas require differentiation under the integral sign, so they are not quite as elementary as the proofs given in the previous sections.

Theorem 8. *The Probability integral formula (1.2) is equivalent to the assertion:*

$$\int_0^1 \frac{1}{1+x^2}\, dx = \frac{\pi}{4}. \tag{6.1}$$

Proof. The proof is a very minor variation on an exercise in *Irresistible Integrals* by George Boros and Victor Moll [8], page 64. Boros and Moll attribute the proof to Borwein and Borwein. Let

$$F(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\, dx.$$

We will leave it as an exercise for the reader to verify that differentiation under the integral sign is valid in this case. Therefore,

$$F'(t) = \int_0^1 (-2t) e^{-t^2(1+x^2)}\, dx \;=\; -2e^{-t^2} \int_0^1 t\; e^{-(tx)^2} dx.$$

Now carry out the substitution $u = tx$, $du = t\,dx$, to obtain

$$F'(t) = -2e^{-t^2} \int_0^t e^{-u^2} du = -2e^{-t^2} \int_0^t e^{-x^2} dx.$$

Since the real number t is fixed but arbitrary this formula for $F'(t)$ is valid for all $t \geqslant 0$. There is one additional fact about $F(t)$ that we need. For any real numbers t and x, $e^{-t^2(1+x^2)} \leqslant e^{-t^2}$. Therefore

$$0 \leqslant \; F(t) \leqslant \; e^{-t^2} \int_0^1 \frac{1}{1+x^2}\,dx$$

and so

$$\lim_{t\to\infty} F(t) = 0.$$

Now define

$$G(t) = \Big(\int_0^t e^{-x^2} dx \Big)^2.$$

By the Fundamental Theorem of Calculus,

$$G'(t) = 2e^{-t^2} \int_0^t e^{-x^2} dx.$$

Therefore, for all $t \geqslant 0$, $F'(t) + G'(t) = 0$ and so $F(t) + G(t)$ is a constant. Since

$$F(0) = \int_0^1 \frac{1}{1+x^2}\,dx$$

and $G(0) = 0$ we see that for all $t \geqslant 0$,

$$F(t) + G(t) = \int_0^1 \frac{1}{1+x^2}\,dx.$$

Computing the limit as $t \to \infty$ of both sides of the preceding equation shows that

$$\Big(\int_0^\infty e^{-x^2} dx \Big)^2 = \int_0^1 \frac{1}{1+x^2}\,dx.$$

The theorem now follows. □

Of course, $\arctan'(x) = 1/(1+x^2)$ and

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

Therefore, Theorem 6.1 provides another path to proving all of the famous formulas.

Our final famous formula is similar to formula (6.1)

Theorem 9. *The Probability integral formula (1.2) is equivalent to the assertion:*

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}. \tag{6.2}$$

PROOF. The proof is a very minor variation on a proof that appears in the expository paper *The Gaussian Integral* by Keith Conrad [9]. That proof is attributed to Michael Rozman and an idea posted on math.stackexchange. Let

$$F(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

We will leave it as an exercise for the reader to verify that the improper integral exists and that differentiation under the integral sign is valid in this case. After computing $F'(t)$ and carrying out a substitution similar to the substitution in the previous theorem we find that

$$F'(t) = -2e^{-t^2}\int_0^\infty e^{-x^2}dx.$$

Also, a proof similar to the proof in the previous theorem shows that:

$$\lim_{t\to\infty} F(t) = 0.$$

For any real number $b > 0$, integrate the equation above for $F'(t)$ to obtain

$$F(b) - F(0) = \int_0^b F'(t)\,dt = -2\Big(\int_0^\infty e^{-x^2}dx\Big)\Big(\int_0^b e^{-t^2}dt\Big).$$

Computing the limit as $b \to \infty$ of both sides of the preceding equation shows that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = 2\Big(\int_0^\infty e^{-x^2}dx\Big)^2.$$

The theorem now follows. □

7. One Final Note

In this paper we have taken Wallis's formula as a primary result and shown that a number of other famous formulas are equivalent to Wallis's formula. To understand how Wallis originally discovered his formula, see the exciting discussion by David Bressoud in [7].

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