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A note on series of hyperbolic functions

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Abstract. By means of an elementary re-summation any convergent hyperbolic series can be associated with one of more ‘alleles’, i.e. similar series which has the same value. A number of examples are treated here, most of which stem from Fourier series for elliptic functions introduced by K. Jacobi.

1. Introduction

It is clear from examining any text on boundary value problems that functional series

$$\sum_{n=0}^{\infty} \frac{n^{\nu}\,\mathrm{hyp}(nx)^a}{\mathrm{hyp}(ny)^b} \tag{1.1}$$

or Fourier series with hyperbolic coefficients, such as

$$\sum_{n} \frac{\sinh(n\pi c)}{\sinh(n\pi d)} \sin(n\pi x), \tag{1.2}$$

where "`hyp`" denotes sinh, cosh, their powers, or simply a finite sum of real exponentials, occur abundantly in many branches of Science and Engineering. Accordingly, they have received a good deal of attention, as well for purely theoretical reasons, from a host of mathematicians including [**11, 12, 13, 8, 9, 21, 22, 4, 5, 6, 20, 18, 19, 10, 1, 2, 14**]. In this note I aim to show that by exploiting an underutilized elementary expedient for transforming such series, one finds that they are, in many cases, expressible in terms of series known in closed form. In particular, it is shown that, in many cases, the factor n in (1.1) can be eliminated. Two methods used for evaluating the classes (1.1) and (1.2) that have been most successful are contour integration [**12**] and the manipulation of Elliptic functions [**9, 21**]. This note will be based on the latter approach. The principal formulations of elliptic function theory are those due to Weierstrass, based on Eisenstein series, and Jacobi, based on the inversion of definite integrals. The reader can find a brief, clear and concise exposition

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of the entire subject in Chapters 21-22 of Whittaker and Watson's classic text [**17**]. It introduces the notation used here.

1.1. Jacobi's Fourier series. By exploiting periodicity along the imaginary axis, Jacobi constructed Fourier series for the functions: sn, cd, dn, nd, sd, cn, dc, nc, and cs, which are listed in [**18**]. For example,

$$\mathtt{sn}\left(\frac{2}{\pi}Kx, k\right) = \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin[(2n+1)x], \tag{1.3}$$

where

$$q = e^{\pi K'/K},\ K' = \mathbf{K}(k'),\ k' = \sqrt{1-k^2},\ 0 \leqslant k < 1. \tag{1.4}$$

If one expands the denominator of (1.3) in powers of q, performs the n-sum and expresses the result in hyperbolic form, (1.3) becomes

$$\mathtt{sn}\left(\frac{2}{\pi}Kx, k\right) = \frac{2\pi}{kK}\sum_{m=0}^{\infty}\frac{\cosh[\frac{1}{2}(2m+1)\pi K'/k]}{\cosh[(2m+1)\pi K'/K]-\cos(2x)}. \tag{1.5}$$

It is clear that for any series (1.1) the procedure just described produces an `allele` - usually a different series having the same value, but which may be simpler in that, for positive ν, it may lack the factor n in the numerator. As an example, consider Ling and Zucker's series [**4, 5**]

$$\mathbf{V}_3(c) = \sum_{n=1}^{\infty}\frac{n^3}{\sinh(n\pi c)}. \tag{1.6}$$

By expressing the denominator in exponential form and expanding in powers of $e^{-n\pi c}$, then summing the derivative-geometric series and expressing the resulting ratio of exponentials in hyperbolic form, one finds

$$V_3(c) = \frac{1}{4}\sum_{m=0}^{\infty}\frac{3+2\sinh^2[\frac{\pi}{2}(2m+1)c]}{\sinh^4[\frac{\pi}{2}(2m+1)c]} = \frac{3}{4}\mathbf{III}_4(c) + \frac{1}{4}\mathbf{III}_2(c). \tag{1.7}$$

In this way, it appears that the Ling- Zucker classes $\mathbf{V}-\mathbf{VIII}$ are redundant. We will return to other examples in the following section.

Returning to Jacobi's Fourier series [**3**], we find the following transformed expressions:

THEOREM 1.1.

$$\mathit{cn}\left(\frac{2Kx}{\pi},k\right)=\frac{2\pi\cos x}{kK}\sum_{m=0}^{\infty}(-1)^m\frac{\sinh[\frac{\pi}{2}(2m+1)\frac{K'}{K}]}{\cosh[\pi(2m+1)\frac{K'}{K}]-\cos^2 x}. \tag{1.8}$$

$$\mathit{cd}\left(\frac{2Kx}{\pi},k\right)=\frac{\pi\cos x}{kK}\sum_{m=0}^{\infty}\frac{\cosh[\frac{\pi}{2}(2m+1)\frac{K'}{K}]}{\sinh^2\left[\frac{\pi}{2}(2m+1)\frac{K'}{K}\right]+\cos^2 x}. \tag{1.9}$$

$$\mathit{sd}\left(\frac{2Kx}{\pi},k\right)=\frac{\pi\sin x}{kk'K}\sum_{m=0}^{\infty}\frac{\sinh[\frac{\pi}{2}(2m+1)\frac{K'}{K}]}{\sinh^2[\frac{\pi}{2}(2m+1)\frac{K'}{K}]+\cos^2 x}. \tag{1.10}$$

(1.11)
$$\mathit{nd}\left(\frac{2Kx}{\pi},k\right)=\frac{\pi}{2k'K}\left[1+2\sum_{m=0}^{\infty}(-1)^m\left(\frac{\sinh[(2m+1)\pi\frac{K'}{K}]}{\cosh[\pi(2m+1)\frac{K'}{K}]+\cos(2x)}-1\right)\right].$$

$$\mathit{ns}\left(\frac{2Kx}{\pi},k\right)=\frac{\pi}{2K}\left[\csc x+4\sin x\sum_{m=1}^{\infty}\frac{\cosh[(\pi m\frac{K'}{K})]}{\cosh[(2\pi m\frac{K'}{K})-\cos(2x)}\right], \tag{1.12}$$

$$\mathtt{sn}\left(\frac{2}{Kx},k\right)=\frac{\pi}{kK}\sin x\sum_{m=0}^{\infty}\frac{\cosh[\frac{1}{2}(2m+1)\pi\frac{K'}{K}]}{\cosh[(2m+1)\pi\frac{K'}{K}]-\cos^2 x}. \tag{1.13}$$

$$\mathit{ds}\left(\frac{2}{\pi}Kx,k\right)=\frac{\pi}{2K}\left[\csc x-4\sin x\sum_{m=1}^{\infty}\frac{\cosh(m\pi)\frac{K'}{K}}{\cosh(2m\pi\frac{K'}{K})-\cos(2x)}\right]. \tag{1.14}$$

$$\mathit{nc}\left(\frac{2}{\pi}Kx,k\right)=\frac{\pi}{2k'K}\left[\sec x-4\cos x\sum_{m=1}^{\infty}\frac{(-1)^m\cosh(m\pi)\frac{K'}{K}}{\cosh(2m\pi\frac{K'}{K})+\cos(2x)}\right]. \tag{1.15}$$

$$\left(\frac{2}{\pi}Kx,k\right)=\frac{\pi}{2k'K}\left[\tan x-2\sin(2x)\sum_{m=1}^{\infty}\frac{(-1)^m}{\cosh(2m\pi\frac{K'}{K})+\cos(2x)}\right]. \tag{1.16}$$

By comparison with Jacobi's expressions, one obtains

Corollary 1.1.

$$\sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{\cosh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]}=2\cos x\sum_{n=0}^{\infty}\frac{\sinh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]}{\cosh[(2n+1)\pi\frac{K'}{K}]-\cos(2x)}. \tag{1.17}$$

$$\sum_{n=0}^{\infty}\sin[(2n+1)x]\left(1-\tanh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]\right)=2\sin x\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\cosh[n\pi\frac{K'}{K}]}{\cosh[2n\pi\frac{K'}{K}]-\cos(2x)}. \tag{1.18}$$

$$2\sin x\sum_{n=0}^{\infty}\frac{\cosh[(n+\frac{1}{2})\frac{\pi}{2}\frac{K'}{K}]}{\cosh[(n+\frac{1}{2})\pi\frac{K'}{K}]-\cos(2x)}=\sum_{n=0}^{\infty}\frac{\sin[(2n+1)x]}{\sinh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]}. \tag{1.19}$$

$$\sum_{n=0}^{\infty}(-1)^n\left\{1-\tanh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]\right\}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\cosh[n\pi\frac{K'}{K}]}. \tag{1.20}$$

$$\sum_{n=1}^{\infty}(-1)^n\left[1-\frac{\sinh[n\pi\frac{K'}{K}}{\cosh[n\pi\frac{K'}{K}]+\cos(2\pi)}\right]=\sum_{n=1}^{\infty}(-1)^n\frac{\cos(2nx)}{\cosh(n\pi\frac{K'}{K})}. \tag{1.21}$$

$$2\cos x\sum_{n=0}^{\infty}\frac{\cosh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]}{\cosh[(2n+1)\pi\frac{K'}{K}]+\cos(2x)}=\sum_{n=0}^{\infty}(-1)^n\frac{\cos[(2n+1)x]}{\sinh[(2n+1)\frac{\pi}{2}\frac{K'}{K}]}. \tag{1.22}$$

$$\sum_{n=0}^{\infty}(-1)^n\cos[(2n+1)x](1-\tanh[(n+\tfrac{1}{2})\pi\frac{K'}{K}])=2\cos x\sum_{n=1}^{\infty}\frac{\cosh[m\pi\frac{K'}{K}]}{\cosh[2m\pi\frac{K'}{K}]+\cos(2x)}. \tag{1.23}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\sin[(2n+1)x]}{\cosh[(n+\frac{1}{2})\pi\frac{K'}{K}]}=\sin x\sum_{n=0}^{\infty}(-1)^n\frac{\sinh[(n+\frac{1}{2})\pi\frac{K'}{K}]}{\cosh[(2n+1)\pi\frac{K'}{K}]+\cos(2x)}. \tag{1.24}$$

$$\sin x\sum_{n=1}^{\infty}\frac{\cosh[n\pi\frac{K'}{K}]}{\cosh[2n\pi\frac{K'}{K}-\cos(2x)}=\sum_{n=0}^{\infty}\sin[(2n+1)x]\left\{1-\tanh[(n+\tfrac{1}{2})\pi\frac{K'}{K}]\right\}. \tag{1.25}$$

$$\sin(2x)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\cosh(n\pi\frac{K'}{K})-\cos(2x)}=\sum_{n=1}^{\infty}\sin(2nx)\left\{1-\tanh(n\pi\frac{K'}{K})\right\}. \tag{1.26}$$

$$\sin(2x)\sum_{n=1}^{\infty}\frac{1}{\cosh(n\pi\frac{K'}{K})+\cos(2x)}=\sum_{n=1}^{\infty}(-1)^n\sin(2nx)\left\{1-\tanh(n\pi\frac{K'}{K})\right\}. \tag{1.27}$$

1.2. Discussion. Numerous specific hyperbolic series can now be found by inserting values of x and k for which the elliptic values are known. Thus, for $x = 0$, since $sn(0;k) = 0$, $cn(0;k) = 1$; and $dn(0;k) = 1$, Theorem 1.1 gives (setting $\pi K'/K = r$) the specific examples listed in Table 1.

Table 1

$x = 0$

$$\sum_{n=0}^{\infty} \operatorname{csch}[(2n+1)r/2] = \frac{kK}{2} \tag{1.28}$$

$$\sum_{n=0}^{\infty} \frac{\cosh[(2n+1)r/2]}{\sinh^2\{[(n+1/2)r]-1\}} = \frac{k\pi}{\pi} \quad \textbf{FIX} \tag{1.29}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sinh[(2n+1)r]+1} = \frac{2k'K}{\pi} - 1. \tag{1.30}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\cosh(nr)} = \frac{1}{2}\left(1 - \frac{2k'K}{\pi}\right). \tag{1.31}$$

$$\sum_{n=1}^{\infty} \left(1 - \frac{\sinh(nr)}{2\cosh^2(nr/2)}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh(nr)}. \tag{1.32}$$

$$\sum_{n=0}^{\infty} \frac{\cosh[(2n+1)r/2]}{\sinh^2[(2n+1)r/2]+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sinh[(n+\frac{1}{2})r]}. \tag{1.33}$$

$x = \frac{\pi}{4}$

$$\sum_{n=0}^{\infty} \frac{\sinh[(2n+1)r/2]}{\cosh^2[(2n+1)r/2]-\frac{1}{2}} = \frac{k\sqrt{2k'}kK}{\sqrt{1+k'}}. \tag{1.34}$$

$$\sum_{n=0}^{\infty} \frac{\sinh[(2n+1)r/2]}{\sinh^2[(2n+1)r/2]+\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+\lfloor \frac{n}{2} \rfloor}}{\sinh[(2n+1)r/2]}. \tag{1.35}$$

Hundreds of additional hypergeometric series can be quickly generated by taking advantage of singular value compilations. A singular modulus k_c is the solution to the equation

$$\frac{K'(k_c)}{K(k_c)} = c, \tag{1.36}$$

where c is a positive integer. In this case, (1.36) reduces to a polynomial equation, so k_c is an algebraic number and, as proven by Selberg and Chowla [**16**], the elliptic integrals K, K' and E are expressible as products of Gamma functions. For example, for $c = 1$

$$k_1 = \frac{1}{\sqrt{2}}, \text{ and } K = K_1 = K_1' = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}. \tag{1.37}$$

The book [**7**] contains tables of these values for $1 \leqslant c \leqslant 100$. (The function $b(x)$ that appears is $\Gamma^2(x)/\Gamma(2x)$.) For example, from the Fourier series for $\mathtt{sn}(2Kx/\pi, k)$, with $x = \pi/2$, one gets the infinite series of identities whose first three members are listed in Table 2 below.

Table 2

$$\sum_{n=0}^{\infty}(-1)^n \mathrm{csch}[(2n+1)\frac{\pi}{2}] = \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{sech}[(2n+1)\frac{\pi}{2}] = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi^{3/2}\sqrt{32}}$$

$$\sum_{n=0}^{\infty}(-1)^n \mathrm{csch}[(2n+1)\frac{\pi}{\sqrt{2}}] = \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{sech}[(2n+1)\frac{\pi}{\sqrt{2}}] = 2\sqrt{(\sqrt{2}-1)(\sqrt{3}+1)}\frac{\Gamma^2(1/8)}{\pi\Gamma(1/4)}$$

$$\sum_{n=0}^{\infty}(-1)^n \mathrm{csch}[(2n+1)\frac{\pi\sqrt{3}}{2}] = \frac{1}{2}\sum_{n=0}^{\infty}\mathrm{sech}[(2n+1)\frac{\pi\sqrt{3}}{2}] = 2^{-1/3}\sqrt{3(2+\sqrt{3}}\frac{\Gamma^2(1/3)}{\pi\Gamma(2/3)}.$$

The formation of an allele to any hyperbolic series may yield something of interest or simply something unexpected. One example is the case $n=1$ of Ramanujan's series [**3**]:

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^{4n-1}}{\cosh[(2k+1)\frac{\pi}{2}]} = 0 \tag{1.38}$$

which is equivalent to

$$\sum_{n=0}^{\infty}\frac{\sinh[(2n+1)\frac{3\pi}{2}]}{\cosh^4[(2n+1)\frac{\pi}{2}]} = \frac{23}{5}\sum_{n=0}^{\infty}\frac{\sinh[(2n+1)\frac{\pi}{2}]}{\cosh^4[(2n+1)\frac{\pi}{2}]}. \tag{1.39}$$

A more interesting family of formulas stem from modular identities, such as Ramanujan's modular identity [**3**]: For $ab=\pi^2$,

$$a^2\sum_{n=1}^{\infty}\frac{n^3}{e^{\pi an}-1} - b^2\sum_{n=1}^{\infty}\frac{n^3}{e^{\pi bn}-1} = \frac{a^2-b^2}{320}, \tag{1.40}$$

whose allele is of the form

$$a^2\left[3\mathbf{I_2}(\mathbf{a/2})+\mathbf{I_4}(\mathbf{a/2})\right] - b^2\left[3\mathbf{I_2}(\mathbf{b/2})+\mathbf{I_4}(\mathbf{b/2})\right] = \frac{a^2-b^2}{40}. \tag{1.41}$$

in Ling-Zucker notation [4]. The series in (1.41) were probably familiar to Laplace and are evaluated in the papers of Ling [**13**] and Zucker [**22**] yielding

$$\begin{aligned} &a^2\left[\left(\frac{2K_{a/2}}{\pi}\right)(k_{a/2}^4-k_{a/2}^2+1) - \frac{479}{576}\left(\frac{2K_{a/2}}{\pi}\right)\left(\frac{2E_{a/2}}{K_{a/2}}+k_{a/2}^2-2\right)\right] - \\ &b^2\left[\left(\frac{2K_{b/2}}{\pi}\right)(k_{b/2}^4-k_{b/2}^2+1) - \frac{479}{576}\left(\frac{2K_{b/2}}{\pi}\right)\left(\frac{3E_{b/2}}{K_{b/2}}+k_{b/2}^2-2\right)\right] \\ &= \frac{1}{2}(a^2-b^2). \end{aligned} \tag{1.42}$$

Related to this is Schlomilch's formula [**3**]

$$a\sum_{n=1}^{\infty}\frac{n}{e^{2an}-1} + b\sum_{n=1}^{\infty}\frac{n}{e^{2bn}-1} = \frac{a+b}{24} - \frac{1}{4}, \quad \text{with } ab=\pi^2 \tag{1.43}$$

whose allele is Laplace's formula (in Ling-Zucker notation)

$$a\mathbf{I}_2(a/\pi) + b\mathbf{I}_2(b/\pi) = \frac{a+b}{6} - 1 \tag{1.44}$$

and evaluates to

$$a\left(\frac{2K_{a/\pi}}{\pi}\right)^2\left(\frac{3E_{a/\pi}}{K_{a/\pi}} + k_{a/\pi}^2 - 2\right) + b\left(\frac{2K_{b/\pi}}{\pi}\right)^2\left(\frac{3E_{b/'\pi}}{K_{b/\pi}} + k_{b/\pi}^2 - 2\right) = 6. \tag{1.45}$$

Turning next to the work of Yakubovich [**20**], it appears that the hyperbolic series considered there can also be evaluated in terms of Ling's basic series [**13**]. For example one has the alleles

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{\sinh(\pi n x)} &= \frac{1}{2}\mathbf{III}_2(x) \\ \sum_{n=1}^{\infty} \frac{n^2\cosh(n\pi x)}{\sinh^2(n\pi x)} &= \frac{1}{\pi}\frac{\partial}{\partial x}\mathbf{III}_2(x). \end{aligned} \tag{1.46}$$

It should be pointed out that this procedure is unproductive if a series and its allele are identical, which is the case for the series

$$\sum_{n=0}^{\infty} \frac{(2n+1)\cosh[(2n+1)\pi x/2]}{\sinh^2[(2n+1)\pi x/2]} \tag{1.47}$$

the allele of the second series in (1.47). Another useful consequence is that the alleles of relations such as Sayer's [**15**]

$$\sum_{r=1}^{\infty} \frac{(-1)^r r^{4p+1}}{\cosh(r\pi)} = 0, \quad p = 1,\, 2,\, 3,\, \ldots \tag{1.48}$$

and the like lead to reduction identities for the Ling-Zucker classes. Thus $p = 1$ in (1.48) is equivalent to

$$\mathbf{IV}_6(1) = \frac{30}{31}\mathbf{IV}_4(1) - \frac{4}{31}\mathbf{IV}_2(1). \tag{1.49}$$

Returning to (1.1), when ν is a negative integer, use of this technique gives

$$\sum_{n=1}^{\infty} \frac{n^{-k}}{e^{2\pi n} - 1} = \sum_{n=1}^{\infty} \mathtt{Li}_k(e^{-2\pi n}). \tag{1.50}$$

When $k = 4m + 3$ we have (Glasser-unplublished)

$$\begin{aligned} \sum_{k=0}^{\infty} \mathtt{Li}_{4n+3}(e^{-2\pi k}) &= \frac{1}{2}\zeta(4n+3) \\ &\quad - \frac{1}{2}4^{2n+1}\pi^{2n+3}\sum_{k=0}^{2n+3}(-1)^k\frac{B_{2k}B_{4n+4-2k}}{(2k)!(4n+4-2k)!} \end{aligned} \tag{1.51}$$

which gives for $m = 0, 1, 2, \ldots$

$$(1.52)\quad \sum_{n=1}^{\infty} \frac{\coth(n\pi)}{n^{4m+3}} = \frac{1}{2}\Big[\zeta(4m+3)+ +4^{2m+1}\pi^{2m+3}\sum_{k=0}^{2m+3}(-k)^k \frac{B_{2k}B_{4m+4-2k}}{(2k)!(4m+4-2k)!}\Big].$$

Finally, Table 3 contains a few hyperbolic series obtained for $x = \pi/4$, where $k = k_r$, $K = K_r$ are the singular values for $r = 1, 2, 3, \ldots$

Table 3

$x = \frac{\pi}{4}$

$$(1.53)\quad \sum_{n=0}^{\infty}(-1)^n \frac{\sinh[(n+1/2)\pi\sqrt{r}]}{\cosh[(2n+1)\pi\sqrt{r}]} = \frac{kk'K}{\pi\sqrt{2(1+k')}}$$

$$(1.54)\quad \sum_{n=0}^{\infty} \frac{\cosh[(n+1/2)\pi\sqrt{r}]}{\sinh^2[(n+1/2)\pi\sqrt{r}]} = \frac{kK\sqrt{2}}{\pi\sqrt{1+k'}}.$$

$$(1.55)\quad \sum_{n=1}^{\infty} \frac{\cosh[n\pi\sqrt{r}]}{\cosh[2n\pi\sqrt{r}]} = \frac{1}{2}\left(1 - \frac{sqrt2k'(1+k')K}{\pi}\right).$$

$$(1.56)\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh[2n\pi\sqrt{r}]} = \frac{1}{2}\left(1 - \frac{2K}{\pi}\right).$$

$$(1.57)\quad \sum_{n=0}^{\infty}(-1)^n[\tanh[(2n+1)\pi\sqrt{r}]-1] = \frac{1}{2}\left(\frac{2\sqrt{k'}K}{\pi} - 1\right).$$

$$(1.58)\quad \sum_{n=1}^{\infty} \frac{\cosh(n\pi\frac{K'}{K})}{\cosh(2n\pi\frac{K'}{K})} = \frac{1}{2}\left(\frac{\sqrt{2(1+k')}K}{\pi} - 1\right).$$

$$(1.59)\quad \sum_{n=0}^{\infty} \frac{\cosh[(n+1/2)\pi\sqrt{r}]}{\cosh[(2n+1)\pi\sqrt{r}]} = \frac{kK}{\pi\sqrt{2(1+k')}}.$$

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