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New closed forms for a dilogarithmic integral, related integrals, and series

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Abstract. In this study, we present a new closed form for the generalized integral

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z,$$

where $a \in \mathbb{C} \setminus (-\infty, -1)$ and $\mathrm{Li}_2(z)$ is the dilogarithm function. This generalization is achieved by leveraging our established findings in conjunction with Vălean's results. Furthermore, we provide explicit closed forms for associated integrals, prove a transformation formula for double infinite series, expressing them as the sum of the square of an infinite series and another infinite series. We utilize this relationship to derive a novel closed form for the generalized series

$$\sum_{k=1}^{\infty} \frac{\zeta\left(m, \frac{rk-s}{r}\right)}{(rk-s)^m},$$

for $\Re(m) > 1$, $r, s \in \mathbb{C}$, where $r \neq 0$, $rk \neq s$, for any positive integer k, and $\zeta(s,z)$ denotes the Hurwitz zeta function. Utilizing Hermite's integral representation for $\zeta(s,z)$, we derive a family of integrals from this series.

1. Introduction

In this paper, we provide a new closed form for the integral

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z, \tag{1.1}$$

where $\mathrm{Li}_2(z)$ denotes the dilogarithm function, defined as [1, (25.12.1)]

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad |z| \leqslant 1.$$

Our approach entails the transformation of integrals into infinite series involving harmonic numbers, followed by the subsequent evaluation of these resulting series.

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Through this method, we not only determine a closed form for the aforementioned integral but also discover a closed form for

$$\sum_{k=1}^{\infty} \frac{\zeta\left(m, \frac{rk+r-s}{r}\right)}{(rk-s)^m}, \quad \Re m > 1 \land r, s \in \mathbb{C}, r \neq 0, rk \neq s, \forall\ k \in \mathbb{N}.$$

Utilizing Hermite's integral representation for $\zeta(s, z)$, we derive a family of integrals from this series. Examples of such integrals are:

$$\begin{aligned}
&\int_0^\infty \frac{x}{(x^2+1)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \frac{1}{2}\int_0^\infty \frac{x}{(x^2+4)^2(e^{2\pi x}-1)}\,\mathrm{d}x \\
&\qquad + \frac{1}{3}\int_0^\infty \frac{x}{(x^2+9)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \frac{1}{4}\int_0^\infty \frac{x}{(x^2+16)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \cdots \\
&= \frac{\pi^4}{288} - \frac{\zeta(3)}{4},
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \frac{x}{(4x^2+1)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \frac{1}{3}\int_0^\infty \frac{x}{(4x^2+9)^2(e^{2\pi x}-1)}\,\mathrm{d}x \\
&\qquad + \frac{1}{5}\int_0^\infty \frac{x}{(4x^2+25)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \frac{1}{7}\int_0^\infty \frac{x}{(4x^2+49)^2(e^{2\pi x}-1)}\,\mathrm{d}x + \cdots \\
&= \frac{\pi^4}{1024} - \frac{7\zeta(3)}{128}.
\end{aligned}$$

These integrals do not appear in existing literature. Throughout this work, H_n represents the nth harmonic number, defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N},$$

$\psi_{m-1}(z)$ represents the polygamma function, defined as [1, §5.15],

$$\psi_{m-1}(z) = (-1)^m (m-1)! \sum_{k=0}^{\infty} \frac{1}{(k+z)^m}, \quad m \geqslant 2,\, m \in \mathbb{N}, z \notin -\mathbb{N}_0,$$

and $\zeta(s, z)$ represents the Hurwitz zeta function, defined as [1, §25.11]

$$\zeta(s, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}, \quad z \notin -\mathbb{N}_0, \Re s > 1.$$

By incorporating Vălean's closed form for

$$\int_0^1 \frac{\operatorname{Li}_2(z)\ln(z)}{1-az}\,\mathrm{d}z, \quad a \in \mathbb{C} \smallsetminus (1, \infty) \cup \{0\},$$

alongside our derived closed forms for

$$\int_0^1 \frac{\ln z \ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z, \quad \int_0^1 \frac{\ln z \ln^2(1+az)}{z}\,\mathrm{d}z, \quad a \in \mathbb{C} \smallsetminus (-\infty, -1), \tag{1.2}$$

we present a generalized version of (1.1) in the form

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z, \quad a \in \mathbb{C} \smallsetminus (-\infty,-1). \tag{1.3}$$

The two integrals in (1.2) are equal when $a=-1$. The simplest evaluation of (1.3) occurs when $a=-1$. In this case, we have

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1-z)}{z}\,\mathrm{d}z = -\frac{\pi^4}{72}. \tag{1.4}$$

The exclusion of (1.3) for $a \in (-\infty,-1)$ stems from the observation that if $a \in (-\infty,-1)$, the integral diverges at a certain point within the integration domain. To illustrate this, set $a=-b$, where $b \in (1,\infty)$, and notice that for all values of b in this interval, there exists a unique $z=\frac{1}{b} \in (0,1)$ such that

$$\frac{\mathrm{Li}_2(z)\ln(1-bz)}{z} \to -\infty. \tag{1.5}$$

In summary, the results established in this article are outlined as follows. In Section 3.1, we present a closed form for (1.1) using known results. We also establish the relationship $2\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k)^3} = \int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z$. We begin our exploration by introducing a novel generalization of (1.1). Additionally, we provide generalizations for related integrals. Furthermore, we offer a representation for the series $\sum_{k=1}^{\infty}\frac{(-1)^k H_k a^k}{k^3}$ that allows us to provide the generalization (1.3) for $a \in \mathbb{C} \smallsetminus (-\infty,0]$ while avoiding logarithm of negative real numbers. Theorems 10 and 12 are not new, as we use our established results to provide a new proof of Jonquière's inversion formula for order 4 and arguments $-\frac{1}{z}$ and $\frac{z}{z-1}$. The closed forms presented in Theorems 3–9 and 13–20 are new and have not been presented elsewhere in the literature. In Section 3.2, we introduce a transformative approach for double infinite series, enabling us to express them as sums of the square of an infinite series and another infinite series. We apply this theorem to derive novel generalized identities. The Computer Algebra System (CAS) software employed for result verification throughout this paper is Mathematica 13.

2. Notations and Definitions

In this manuscript, we employ the following abbreviated notations: B_n represents the Bernoulli numbers [**1**, §24.2(i)], E_n represents the Euler numbers [**1**, §24.2(ii)], $\gamma \approx 0.5772156649$ represents Euler's constant, $\mathbf{G} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ represents Catalan's constant ($\mathbf{G} \approx 0.9159655941$), while $\mathrm{e} \approx 2.71828182845$ stands for Euler's number. We define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ as the set of non-negative integers, where $n\mathbb{N}_0$ denotes all elements in $\mathbb{N}_0$ multiplied by n. Additionally, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ represent the sets of integers, rational, real, and complex numbers, respectively.

We denote the gamma [**1**, (5.2.1)], digamma [**1**, (5.2.2)], tetragamma, pentagamma, and hexagamma functions [**1**, §5.15] of argument z as $\Gamma(z)$, $\psi(z)$, $\psi_2(z)$, $\psi_3(z)$, and

$\psi_4(z)$, respectively. Here, $\psi_n(z)$ is the polygamma function, defined as the n-th derivative of $\ln\Gamma(z)$, and $n \in \mathbb{N}_0$. The digamma function can be expressed as [1, §1.7(6)]:

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+z}\right), \quad z \in \mathbb{C} \setminus -\mathbb{N}_0.$$

For positive integer values of z, the digamma function simplifies to [2, §1.7.1(9)]:

$$\psi(k+1) = -\gamma + H_k, \quad k \in \mathbb{N}. \tag{2.1}$$

The recurrence relation for the digamma function is given by [1, (5.5.2)]:

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{2.2}$$

The duplication formula for the $\psi(z)$ is [1, (5.5.8)]:

$$\psi\left(z + \frac{1}{2}\right) = 2\psi(2z) - \psi(z) - \ln 4, \quad z \in \mathbb{C} \setminus -\mathbb{N}_0. \tag{2.3}$$

The Lerch transcendent is defined as [1, (24.14.1)]:

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad |z| \leqslant 1, \Re s > 1, a \notin -\mathbb{N}_0.$$

The polygamma function can be expressed as $\psi_n(z) = (-1)^{n-1} n! \Phi(1, n+1, z)$, where $n \in \mathbb{N}$. The Dirichlet eta function is defined as $\eta(n) := \Phi(-1, n, 1)$, where $\Re n > 0$ [1, §1.12(2)]. The Riemann zeta function [1, §25.2] and the Hurwitz zeta function [1, §25.11] are, respectively, defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s},$$

where $z \notin -\mathbb{N}_0$, $\Re s > 1$. The domain $\Re s > 1$ of the Riemann and Hurwitz zeta functions can be extended to $s \in \mathbb{C} \setminus \{1\}$ through analytic continuation, using for instance, the Hermite integral representation for the Hurwitz zeta function [1, (25.11.29)]

$$\zeta(s,z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2\int_0^{\infty} \frac{\sin\left(s \arctan\left(x/z\right)\right)}{\left(x^2+z^2\right)^{\frac{s}{2}}\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x. \tag{2.4}$$

The relationship between the Dirichlet eta function and the Riemann zeta function is given by $\eta(n) = \left(1 - 2^{1-n}\right)\zeta(n)$ [2, §1.12(2)]. The polylogarithm function [1, §25.12(ii), §25.14.3], $\mathrm{Li}_s(z)$, is defined as: $\mathrm{Li}_s(z) = z\Phi(z,s,1)$, where $\Re s > 1$, $|z| \leqslant 1$. The dilogarithm function, $\mathrm{Li}_2(z)$, has the integral representation [1, (25.12.2)]

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\,\mathrm{d}t, \quad z \in \mathbb{C} \setminus (1,\infty). \tag{2.5}$$

The domain $|z| \leqslant 1$ can be extended to the entire complex plane through analytic continuation. This can be achieved using, for instance, the integral representation [1, (25.14.6)]

$$\mathrm{Li}_s(z) = \frac{1}{2}z + \int_0^{\infty} \frac{z^{t+1}}{(1+t)^s}\,\mathrm{d}t - 2z\int_0^{\infty} \frac{\sin\left(t\ln z - s\arctan t\right)}{\left(1+t^2\right)^{\frac{s}{2}}\left(e^{2\pi t}-1\right)}\,\mathrm{d}t, \quad \Re(s) > 0 \text{ if } z \in \mathbb{C} \setminus [1,\infty).$$

Throughout this work, we utilize the property that $\operatorname{Li}_s(z)$ is defined for all complex z.

3. Results

In this section, we present the main findings and outcomes of our study. We begin by deriving a closed form for (1.1) using known results. Afterwards, we present a novel generalization of this integral.

LEMMA 1 (Vălean). *The following results are valid:*

$$\sum_{k=1}^{\infty}\frac{H_k}{k^3}=\frac{\pi^4}{72}, \tag{3.1}$$

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k-1)^3}=-\frac{\pi^2}{4}+\frac{\pi^4}{64}+2\ln 2+\frac{7\zeta(3)}{4}-\frac{7\ln 2\zeta(3)}{4}, \tag{3.2}$$

$$\sum_{k=1}^{\infty}\frac{H_{2k-1}}{(2k-1)^3}=\frac{\pi^4}{45}+\frac{\pi^2}{24}\ln^2 2-\frac{\ln^4 2}{24}-\frac{7\ln 2\zeta(3)}{8}-\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{3.3}$$

$$\sum_{k=1}^{\infty}\frac{H_{2k}}{k^3}=-\frac{\pi^4}{15}-\frac{\pi^2}{3}\ln^2 2+\frac{\ln^4 2}{3}+7\ln 2\zeta(3)+8\operatorname{Li}_4\left(\frac{1}{2}\right). \tag{3.4}$$

Proof. The first series (3.1) follows from [**3**, §6.19, pp. 601, (6.149)] for the case $p=3$. Using $H_k=H_{k+1}-\frac{1}{k+1}$ in [**3**, §4.19, pp. 420, (4.102)] for the case $p=3$ and reindexing, the proof of (3.2) is complete. Adding (3.1) to the closed form of $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k}{k^3}$ provided in [**4**, §6.52, pp. 502], the proof of (3.3). Subtracting both series, the closed form of (3.4) is complete. □

REMARK 2. *The closed form for* (1.1) *is*

$$\int_0^1\frac{\operatorname{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z=\frac{\ln^4 2}{12}-\frac{\pi^2\ln^2 2}{12}-\frac{\pi^4}{60}+\frac{7\ln 2\zeta(3)}{4}+2\operatorname{Li}_4\left(\frac{1}{2}\right). \tag{3.5}$$

Proof. By utilizing the series representation of $\operatorname{Li}_2(z)$ for $|z|\leqslant 1$, we can express (1.1) as

$$\int_0^1\frac{\operatorname{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z=\sum_{k=1}^{\infty}\frac{1}{k^2}\int_0^1 z^{k-1}\ln(1+z)\,\mathrm{d}z. \tag{3.6}$$

Upon integrating term by term, we obtain

$$\int_0^1 z^{k-1}\ln(1+z)\,\mathrm{d}z=\frac{\ln 2}{k}-\frac{1}{k}\int_0^1\frac{z^k}{1+z}\,\mathrm{d}z=\frac{\ln 2}{k}-\frac{1}{2k}\left(\psi\left(\frac{k+2}{2}\right)-\psi\left(\frac{k+1}{2}\right)\right). \tag{3.7}$$

Applying (2.3) and subsequently employing (2.1) in (3.7), we derive

$$\int_0^1 z^{k-1}\ln(1+z)\,\mathrm{d}z=\frac{H_k-H_{\frac{k}{2}}}{k}. \tag{3.8}$$

Substituting (3.8) into (3.6) and applying (2.1) and (2.3) after splitting into odd and even parts, we obtain

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z &= \sum_{k=1}^{\infty}\frac{H_k - H_{\frac{k}{2}}}{k^3}\\ &= -2+\frac{7}{8}\sum_{k=1}^{\infty}\frac{H_k}{k^3}+\frac{7\ln 2\zeta(3)}{4}-\sum_{k=1}^{\infty}\frac{2H_{2k+1}-H_k}{(2k+1)^3}.\end{aligned}\tag{3.9}$$

Reindexing the series (3.2), we have

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k+1)^3}=\frac{\pi^4}{64}-\frac{7\ln 2}{4}\zeta(3).\tag{3.10}$$

Substituting (3.1), (3.3), and (3.10) into (3.9), we successfully conclude the proof of (3.5). Alternatively, we can employ the series representation of $\ln(1+z)$ for $|z|\leqslant 1$ to establish the proof of (3.5). This approach yields

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{\pi^2}{6k}-\frac{H_k}{k^2}\right)=\frac{\pi^4}{72}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k}{k^3}\\ &= \sum_{k=1}^{\infty}\frac{H_k}{k^3}-\frac{(-1)^{k-1}H_k}{k^3}=\frac{1}{4}\sum_{k=1}^{\infty}\frac{H_{2k}}{k^3}.\end{aligned}\tag{3.11}$$

By substituting (3.4) into (3.11), we readily conclude the proof of (3.5). Notably, this approach appears to be faster, as we immediately recognise $\frac{\pi^4}{72}$ as the closed form of $\sum_{k=1}^{\infty}\frac{H_k}{k^3}$. □

3.1. Generalization of the dilogarithmic integral and related integrals. In the following theorems, we provide the generalization (1.3), and the generalization of integrals related to (1.3). The closed forms for integrals presented in Theorems 3–9 are new and have not been presented elsewhere in the literature. In these theorems, we avoid computations of logarithm of negative real numbers.

THEOREM 3. *Let $a\in\mathbb{C}\smallsetminus(-\infty,-1)$. Then*

$$\begin{aligned}\int_0^1 \frac{\ln z\ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z+\frac{1}{2}\int_0^1\frac{\ln z\ln^2(1+az)}{z} &= -\frac{(\mathrm{Li}_2(-a))^2}{2}+\frac{\pi^2}{6}\mathrm{Li}_2(-a)\\ &\quad -2\,\mathrm{Li}_4(-a).\end{aligned}\tag{3.12}$$

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z+\frac{1}{2}\int_0^1\frac{\ln z\ln^2(1+az)}{z}\,\mathrm{d}z=-\frac{\pi^2}{6}\mathrm{Li}_2(-a)+\mathrm{Li}_4(-a).\tag{3.13}$$

Proof. In an effort to circumvent the computation of logarithm of negative real numbers, the initial part of the proof addresses the case where $a\in\mathbb{C}\smallsetminus(-\infty,0)$, while the subsequent section pertains to the scenario where $a\in\mathbb{C}\smallsetminus(-\infty,-1)\cup(0,\infty)$. In both cases, it follows that $a \in \mathbb{C} \setminus (-\infty, -1)$. Now, we begin by performing integration by parts, resulting in

$$\int_0^1 \frac{\operatorname{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z = \int_0^1 \frac{\ln z \ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z - a\int_0^1 \frac{\ln z \operatorname{Li}_2(z)}{1+az}\,\mathrm{d}z. \tag{3.14}$$

Vălean employed the Cauchy product of two series [5] to derive [3, §3.49, pp. 335, (3.333)]

$$(\operatorname{Li}_2(a))^2 = 4\sum_{k=1}^{\infty}\frac{a^k H_k}{k^3} + 2\sum_{k=1}^{\infty}\frac{a^k H_k^{(2)}}{k^2} - 6\sum_{k=1}^{\infty}\frac{a^k}{k^4}, \quad a \in \mathbb{C}, |a| \leqslant 1, \tag{3.15}$$

Utilizing (3.15), Vălean [3, §1.49, (1.218)] provided the closed form for the second resulting integral in (3.14), with a in (3.14) substituted with $-a$. This yields

$$\int_0^1 \frac{\ln z \operatorname{Li}_2(z)}{1+az}\,\mathrm{d}z = -\frac{(\operatorname{Li}_2(-a))^2}{2a} + \frac{\pi^2 \operatorname{Li}_2(-a)}{3a} - \frac{3\operatorname{Li}_4(-a)}{a}, \quad a \in \mathbb{C}\setminus(-\infty,-1)\cup\{0\}. \tag{3.16}$$

At this point, our focus narrows down to obtaining an expression for the first resulting integral in (3.14). By carrying out term-by-term integration, we deduce

$$\begin{aligned}\int_0^1 \frac{\ln z \ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}a^k}{k}\int_0^1 z^{k-1}\ln z\ln(1-z)\,\mathrm{d}z \\ &= -\operatorname{Li}_4(-a) - \sum_{k=1}^{\infty}\frac{(-1)^k a^k H_k}{k^3} + \sum_{k=1}^{\infty}\frac{(-1)^k a^k \psi_1(k)}{k^2}.\end{aligned} \tag{3.17}$$

By implementing the notation change $H_n^{(2)} = \pi^2/6 - \psi_1(n+1)$ in (3.15) and rearranging, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^k a^k \psi_1(k)}{k^2} = \frac{\pi^2}{6}\operatorname{Li}_2(-a) - 2\operatorname{Li}_4(-a) - \frac{1}{2}(\operatorname{Li}_2(-a))^2 + 2\sum_{k=1}^{\infty}\frac{(-1)^k a^k H_k}{k^3}. \tag{3.18}$$

Upon substituting (3.18) into (3.17), we derive

$$\begin{aligned}\int_0^1 \frac{\ln z \ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z &= -3\operatorname{Li}_4(-a) + \frac{\pi^2}{6}\operatorname{Li}_2(-a) - \frac{1}{2}(\operatorname{Li}_2(-a))^2 \\ &\quad + \sum_{k=1}^{\infty}\frac{(-1)^k a^k H_k}{k^3}.\end{aligned} \tag{3.19}$$

We employ the relationship [3, §4.6, pp. 504]

$$\sum_{k=1}^{\infty}\frac{H_k}{k+1}p^k = \frac{\ln^2(1-p)}{2p}, \quad p \in \mathbb{C}, |p| \leqslant 1, p \neq 0, 1. \tag{3.20}$$

By substituting p with $-p$ in (3.20), integrating over the interval from $p = 0$ to z, we deduce

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{(k+1)^2} z^{k+1} = \frac{1}{2} \int_0^z \frac{\ln^2(1+p)}{p} \, \mathrm{d}p, \quad z \in \mathbb{C} \setminus (-\infty, 0) \cup (1, \infty). \tag{3.21}$$

Subsequently, we perform a change of variable from z to t in (3.21), divide by t, and integrate once more from $t = 0$ to a, resulting in

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} H_k}{(k+1)^3} a^{k+1} = \frac{1}{2} \int_0^a \int_0^t \frac{\ln^2(1+p)}{p} \, \mathrm{d}p \, \mathrm{d}t. \tag{3.22}$$

Substituting [**4**, §1.4, pp. 3, (1.9)] into (3.22) and reindexing the series on the left-hand side, we arrive at

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} H_k a^k}{k^3} &= \int_0^a \frac{1}{t} \left(-\operatorname{Li}_3(-t) - \zeta(3) + \frac{\ln^3(1+t)}{3} + \operatorname{Li}_3\left(\frac{1}{1+t}\right) \right. \\ &\quad \left. + \ln(1+t) \operatorname{Li}_2\left(\frac{1}{1+t}\right) - \frac{1}{2} \ln t \ln^2(1+t) \right) \mathrm{d}t, \quad a \in \mathbb{C} \setminus (-\infty, 0) \cup (1, \infty). \end{aligned} \tag{3.23}$$

Through integration by parts, we obtain these two generalized results:

$$\begin{aligned} \int_0^a \frac{\zeta(3) - \operatorname{Li}_3\left(\frac{1}{1+t}\right)}{t} \, \mathrm{d}t &= \ln a \left(\zeta(3) - \operatorname{Li}_3\left(\frac{1}{1+a}\right) \right) - \ln a \ln(1+a) \operatorname{Li}_2\left(\frac{1}{1+a}\right) \\ &\quad - \operatorname{Li}_2\left(\frac{1}{1+a}\right) \operatorname{Li}_2(-a) + \ln a \ln(1+a) \operatorname{Li}_2(-a) \\ &\quad - \frac{1}{2} \ln^2(1+a) \operatorname{Li}_2(-a) + \frac{1}{2} \left(\operatorname{Li}_2(-a)\right)^2 + \frac{\ln^2 a}{2} \ln(1+a) \\ &\quad - \frac{\ln a}{3} \ln^3(1+a) - \frac{1}{6} \int_0^a \frac{\ln^3(1+t)}{t} \, \mathrm{d}t, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \int_0^a \frac{\ln(1+t) \operatorname{Li}_2\left(\frac{1}{1+t}\right)}{t} \, \mathrm{d}t &= -\operatorname{Li}_2\left(\frac{1}{1+a}\right) \operatorname{Li}_2(-a) + \operatorname{Li}_2(-a) \ln a \ln(1+a) \\ &\quad + \frac{1}{2} \left(\operatorname{Li}_2(-a)\right)^2 - \frac{\ln^2(1+a)}{2} \operatorname{Li}_2(-a) \\ &\quad - \frac{1}{2} \int_0^a \frac{\ln^3(1+t)}{t} \, \mathrm{d}t + \int_0^a \frac{\ln t \ln^2(1+t)}{t} \, \mathrm{d}t. \end{aligned} \tag{3.25}$$

By employing (3.24) and (3.25) within (3.23), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}H_k a^k}{k^3} &= -\operatorname{Li}_4(-a)+\ln a\left(\operatorname{Li}_3\left(\frac{1}{1+a}\right)-\zeta(3)\right)+\ln a\ln(1+a)\\ &\times\operatorname{Li}_2\left(\frac{1}{1+a}\right)+\ln a\ln(1+a)\left(\frac{\ln^2(1+a)}{3}-\frac{\ln a}{2}\right)\\ &+\frac{1}{2}\int_0^a\frac{\ln t\ln^2(1+t)}{t}\,\mathrm{d}t.\end{aligned}\tag{3.26}$$

Utilizing [4, §1.4, pp. 3, (1.9)] in the last integral in (3.26), we arrive at

$$\begin{aligned}\frac{1}{2}\int_0^a\frac{\ln t\ln^2(1+t)}{t}\,\mathrm{d}t &= \ln a\left(\zeta(3)-\operatorname{Li}_3\left(\frac{1}{1+a}\right)\right)+\frac{1}{2}\ln^2 a\ln(1+a)\\ &-\frac{1}{3}\ln a\ln^3(1+a)-\ln a\ln(1+a)\operatorname{Li}_2\left(\frac{1}{1+a}\right)\\ &+\frac{1}{2}\int_0^1\frac{\ln t\ln^2(1+at)}{t}\,\mathrm{d}t.\end{aligned}\tag{3.27}$$

Substituting (3.27) into (3.26), we deduce

$$\sum_{k=1}^{\infty}\frac{(-1)^k H_k a^k}{k^3}=\operatorname{Li}_4(-a)-\frac{1}{2}\int_0^1\frac{\ln z\ln^2(1+az)}{z}\,\mathrm{d}z,\quad a\in\mathbb{C}\smallsetminus(-\infty,0)\cup(1,\infty).\tag{3.28}$$

Likewise, for $a\in\mathbb{C}\smallsetminus(-\infty,0]\cup[1,\infty)$, we obtain using (3.20)

$$\int_0^a\frac{\ln z\ln^2(1-z)}{z}\,\mathrm{d}z=-2\ln a\operatorname{Li}_3(a)+2\operatorname{Li}_4(a)-2\sum_{k=1}^{\infty}\frac{H_k a^k}{k^3}+2\ln a\sum_{k=1}^{\infty}\frac{H_k a^k}{k^2}.\tag{3.29}$$

By making use of [3, §4.6, pp. 399, (4.36)]

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2}z^k=\zeta(3)+\operatorname{Li}_2(1-z)\ln(1-z)+\operatorname{Li}_3(z)-\operatorname{Li}_3(1-z)+\frac{1}{2}\ln z\ln^2(1-z),\tag{3.30}$$

in (3.29), we deduce

$$\sum_{k=1}^{\infty}\frac{H_k a^k}{k^3}=\operatorname{Li}_4(a)-\frac{1}{2}\int_0^1\frac{\ln z\ln^2(1-az)}{z}\,\mathrm{d}z,\quad a\in\mathbb{C}\smallsetminus(-\infty,0)\cup(1,\infty).\tag{3.31}$$

Substituting a with $-a$ in (3.31), we once again arrive at (3.28). However, this time, (3.28) is applicable for $a\in\mathbb{C}\smallsetminus(-\infty,-1)\cup(0,\infty)$. Further substituting (3.28) into (3.19), we derive

$$\begin{aligned}\int_0^1\frac{\ln z\ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z &= -2\operatorname{Li}_4(-a)+\frac{\pi^2}{6}\operatorname{Li}_2(-a)-\frac{1}{2}\left(\operatorname{Li}_2(-a)\right)^2\\ &-\frac{1}{2}\int_0^1\frac{\ln z\ln^2(1+az)}{z}\,\mathrm{d}z.\end{aligned}\tag{3.32}$$

Rearranging (3.32) concludes the proof of (3.12). Finally, by substituting (3.16) and (3.32) into (3.14), we complete the proof of (3.13). □

Remark 4. *In Theorem 3, we have the following special values:*

$$\int_0^1 \frac{\ln z \ln(1+z)\ln(1-z)}{z}\,\mathrm{d}z + \frac{1}{2}\int_0^1 \frac{\ln z \ln^2(1+z)}{z} = \frac{\pi^4}{480}, \tag{3.33}$$

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1+z)}{z}\,\mathrm{d}z + \frac{1}{2}\int_0^1 \frac{\ln z \ln^2(1+z)}{z}\,\mathrm{d}z = \frac{\pi^4}{240}, \tag{3.34}$$

$$\int_0^1 \frac{\mathrm{Li}_2(z)\ln(1-z)}{z}\,\mathrm{d}z + \frac{1}{2}\int_0^1 \frac{\ln z \ln^2(1-z)}{z}\,\mathrm{d}z = -\frac{\pi^4}{60}. \tag{3.35}$$

Theorem 5. *Let $a \in \mathbb{C} \setminus (-\infty, 0]$. Then*

$$\begin{aligned}\int_0^1 \frac{\ln z \ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z = &-\frac{\pi^4}{90} - \frac{(\mathrm{Li}_2(-a))^2}{2} + \frac{\pi^2}{6}\mathrm{Li}_2(-a) - \mathrm{Li}_4(-a)\\ &+ \frac{\pi^2}{12}\ln^2(1+a) + \frac{1}{3}\ln a \ln^3(1+a) - \frac{1}{4}\ln^4(1+a)\\ &+ \ln(1+a)\left(\mathrm{Li}_3\left(\frac{1}{1+a}\right) + \mathrm{Li}_3\left(\frac{a}{1+a}\right)\right)\\ &+ \mathrm{Li}_4\left(\frac{1}{1+a}\right) + \mathrm{Li}_4\left(\frac{a}{1+a}\right).\end{aligned}$$

Proof. We establish the proof by evaluating the second integral in (3.12). By algebraic substitutions, we have

$$\int_0^1 \frac{\ln z \ln^2(1+az)}{z}\,\mathrm{d}z = \int_{\frac{1}{1+a}}^1 \frac{\ln(1-z)\ln^2 z - \ln^3 z - \ln a \ln^2 z}{z(1-z)}\,\mathrm{d}z. \tag{3.36}$$

For the first resulting integral, we have

$$\begin{aligned}\int_{\frac{1}{1+a}}^1 \frac{\ln(1-z)\ln^2 z}{z(1-z)} = \int_0^1 \frac{\ln^2 z \ln(1-z)}{z}\,\mathrm{d}z - \int_0^{\frac{1}{1+a}} \frac{\ln^2 z \ln(1-z)}{z}\,\mathrm{d}z\\ + \int_0^{\frac{a}{1+a}} \frac{\ln z \ln^2(1-z)}{z}\,\mathrm{d}z.\end{aligned} \tag{3.37}$$

Using (3.30) and [**3**, §4.6, pp. 399, (4.38)]

(3.38)

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H_k}{k^3}z^k = \frac{\pi^4}{90} + \zeta(3)\ln(1-z) + \frac{\pi^2}{12}\ln^2(1-z) + \frac{1}{24}\ln^4(1-z) - \frac{1}{6}\ln z \ln^3(1-z)\\ - \ln(1-z)\mathrm{Li}_3(z) + 2\,\mathrm{Li}_4(z) - \mathrm{Li}_4(1-z) + \mathrm{Li}_4\left(\frac{z}{z-1}\right)\end{aligned}$$

in (3.29), we deduce

(3.39)
$$\int_0^z \frac{\ln t \ln^2(1-t)}{t}\,\mathrm{d}t = -\frac{\pi^4}{45} - 2\zeta(3)\ln(1-z) - \frac{\pi^2}{6}\ln^2(1-z) - \frac{1}{12}\ln^4(1-z) + \frac{1}{3}\ln z\ln^3(1-z) + 2\ln(1-z)\operatorname{Li}_3(z) - 2\operatorname{Li}_4(z) + 2\operatorname{Li}_4(1-z) - 2\operatorname{Li}_4\left(\frac{z}{z-1}\right) + 2\ln z\,\zeta(3) + 2\ln z\ln(1-z)\operatorname{Li}_2(1-z) - 2\ln z\operatorname{Li}_3(1-z) + \ln^2 z\ln^2(1-z).$$

The following results are obtained through straightforward term-by-term integration

$$\int_0^{\frac{1}{1+a}} \frac{\ln^3 z}{1-z}\,\mathrm{d}z = (\ln a - \ln(1+a))\ln^3(1+a) - 3\ln^2(1+a)\operatorname{Li}_2\left(\frac{1}{1+a}\right) - 6\ln(1+a)\operatorname{Li}_3\left(\frac{1}{1+a}\right) - 6\operatorname{Li}_4\left(\frac{1}{1+a}\right), \tag{3.40}$$

$$\int_0^{\frac{a}{1+a}} \frac{\ln^2(1-z)}{z}\,\mathrm{d}z = 2\zeta(3) - 2\operatorname{Li}_3\left(\frac{1}{1+a}\right) + (\ln a - \ln(1+a))\ln^2(1+a) - 2\ln(1+a)\operatorname{Li}_2\left(\frac{1}{1+a}\right). \tag{3.41}$$

By replacing z with $a/(a+1)$ in (3.39) and employing and (3.40), and further substituting the latter in (3.36) while taking into account (3.41), we ultimately arrive at

(3.42)
$$\int_0^1 \frac{\ln z\ln^2(1+az)}{z}\,\mathrm{d}z = \frac{\pi^4}{45} - \frac{\pi^2}{6}\ln^2(1+a) - \frac{2}{3}\ln a\ln^3(1+a) + \frac{1}{2}\ln^4(1+a) - 2\ln(1+a)\left(\operatorname{Li}_3\left(\frac{1}{1+a}\right) + \operatorname{Li}_3\left(\frac{a}{1+a}\right)\right) - 2\operatorname{Li}_4(-a) - 2\operatorname{Li}_4\left(\frac{1}{1+a}\right) - 2\operatorname{Li}_4\left(\frac{a}{1+a}\right), \quad a\in\mathbb{C}\setminus(-\infty,0].$$

Substituting (3.42) into (3.12), we thereby conclude the proof of Theorem 5. □

THEOREM 6. *Let $a\in\mathbb{C}\setminus(-\infty,0]$. Then*

(3.43)
$$\int_0^1 \frac{\operatorname{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z = -\frac{\pi^4}{90} - \frac{\pi^2}{6}\operatorname{Li}_2(-a) + 2\operatorname{Li}_4(-a) + \frac{\pi^2}{12}\ln^2(1+a) + \frac{1}{3}\ln a\ln^3(1+a) - \frac{1}{4}\ln^4(1+a) + \ln(1+a)\operatorname{Li}_3\left(\frac{1}{1+a}\right) + \ln(1+a)\operatorname{Li}_3\left(\frac{a}{1+a}\right) + \operatorname{Li}_4\left(\frac{1}{1+a}\right) + \operatorname{Li}_4\left(\frac{a}{1+a}\right).$$

Proof. Through the substitution of (3.42) into (3.13), we establish the proof of Theorem 6. □

REMARK 7. *Considering (3.19), we substitute $z = -a$ in (3.38), resulting in*

(3.44)
$$\sum_{k=1}^{\infty} \frac{(-1)^k H_k a^k}{k^3} = \frac{\pi^4}{90} + \zeta(3)\ln(1+a) + \frac{\pi^2}{12}\ln^2(1+a) + \frac{1}{24}\ln^4(1+a) - \frac{1}{6}\ln(-a)\ln^3(1+a) - \ln(1+a)\operatorname{Li}_3(-a) + 2\operatorname{Li}_4(-a) - \operatorname{Li}_4(1+a) + \operatorname{Li}_4\left(\frac{a}{a+1}\right).$$

While Vălean impressively provided different representations for (3.30) and (3.38), it is worth noting that the logarithms on the right-hand side of (3.30) and (3.38) avoid negative values for $z \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$. Consequently, (3.44) avoids logarithms of negative numbers for $a \in \mathbb{C} \setminus (-\infty, -1] \cup [0, \infty)$. In the other region where $a \in \mathbb{C} \setminus (-\infty, 0] \cup [1, \infty)$, by employing the relationships (3.28) and (3.42) that we have established, we arrive at

$$\sum_{k=1}^{\infty} \frac{(-1)^k H_k a^k}{k^3} = -\frac{\pi^4}{90} + \frac{\pi^2}{12}\ln^2(1+a) + \frac{1}{3}\ln a \ln^3(1+a) + \frac{1}{2}\ln^4(1+a) + \ln(1+a)\left(\operatorname{Li}_3\left(\frac{1}{1+a}\right) + \operatorname{Li}_3\left(\frac{a}{1+a}\right)\right) + 2\operatorname{Li}_4(-a) + \operatorname{Li}_4\left(\frac{1}{1+a}\right) + \operatorname{Li}_4\left(\frac{a}{1+a}\right), \tag{3.45}$$

*which is valid for all $z \in \mathbb{C}$, where $|z| \leqslant 1$ and $z \neq -1, 0$. Similarly, from (3.21) and [**4**, §1.4, pp. 3, (1.9)], we derive*

(3.46)
$$\sum_{k=1}^{\infty} \frac{(-1)^k H_k}{(k+1)^2} z^{k+1} = -\zeta(3) + \frac{1}{3}\ln^3(1+z) + \operatorname{Li}_3\left(\frac{1}{1+z}\right) + \ln(1+z)\operatorname{Li}_2\left(\frac{1}{1+z}\right) - \frac{1}{2}\ln z \ln^2(1+z).$$

Reindexing the series in the left-hand side of (3.46), we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k H_k}{(k+1)^2} z^{k+1} = -\sum_{k=1}^{\infty} \frac{(-1)^k H_k}{k^2} z^k + \operatorname{Li}_3(-z). \tag{3.47}$$

Substituting (3.47) into (3.46), we obtain

(3.48)

$$\sum_{k=1}^{\infty}\frac{(-1)^k H_k}{k^2}z^k=\zeta(3)-\frac{1}{3}\ln^3(1+z)+\mathrm{Li}_3(-z)-\mathrm{Li}_3\left(\frac{1}{1+z}\right)-\ln(1+z)\,\mathrm{Li}_2\left(\frac{1}{1+z}\right)+\frac{1}{2}\ln z\ln^2(1+z).$$

The representations (3.45) *and* (3.48)*, applicable to* $a\in\mathbb{C}\setminus(-\infty,0]\cup[1,\infty)$*, circumvents the need to compute logarithms of negative numbers. The advantage of* (3.45) *over* (3.44) *lies in its application to the evaluation of the integral in* (3.19)*. Specifically, when considering avoidance of logarithms of negative numbers and the analytic continuation of the polylogarithm function,* (3.45) *allows for a more general result, extending the domain of the integral from* $\mathbb{C}\setminus(-\infty,0]\cup[1,\infty)$ *to* $\mathbb{C}\setminus(-\infty,0]$*. Conversely,* (3.44) *still restricts the domain of the integral to* $\mathbb{C}\setminus(-\infty,-1]\cup[0,\infty)$*, avoiding logarithms of negative numbers while still accommodating the analytic continuation of the polylogarithm function. In light of these observations, we reveal two additional closed forms for the integrals in Theorems 5 and 6 in the following theorems.*

THEOREM 8. *Let* $a\in\mathbb{C}\setminus(-\infty,-1]\cup[0,\infty)$*. Then*

(3.49)

$$\begin{aligned}\int_0^1\frac{\ln z\ln(1+az)\ln(1-z)}{z}\,\mathrm{d}z&=\frac{\pi^4}{90}-\mathrm{Li}_4(-a)+\frac{\pi^2}{6}\mathrm{Li}_2(-a)-\frac{1}{2}\left(\mathrm{Li}_2(-a)\right)^2\\&+\zeta(3)\ln(1+a)+\frac{\pi^2}{12}\ln^2(1+a)+\frac{1}{24}\ln^4(1+a)\\&-\frac{1}{6}\ln(-a)\ln^3(1+a)-\ln(1+a)\,\mathrm{Li}_3(-a)\\&-\mathrm{Li}_4(1+a)+\mathrm{Li}_4\left(\frac{a}{a+1}\right).\end{aligned}$$

Proof. Upon substituting (3.44) into (3.19), we conclude the proof of Theorem 8. □

THEOREM 9. *Let* $a\in\mathbb{C}\setminus(-\infty,-1]\cup[0,\infty)$*. Then*

(3.50)

$$\begin{aligned}\int_0^1\frac{\mathrm{Li}_2(z)\ln(1+az)}{z}\,\mathrm{d}z&=\frac{\pi^4}{90}+2\,\mathrm{Li}_4(-a)-\frac{\pi^2}{6}\mathrm{Li}_2(-a)+\zeta(3)\ln(1+a)+\frac{\pi^2}{12}\ln^2(1+a)\\&+\frac{1}{24}\ln^4(1+a)-\frac{1}{6}\ln(-a)\ln^3(1+a)-\ln(1+a)\,\mathrm{Li}_3(-a)\\&-\mathrm{Li}_4(1+a)+\mathrm{Li}_4\left(\frac{a}{a+1}\right).\end{aligned}$$

Proof. Considering (3.31) and (3.44), we infer the following closed form

$$\begin{aligned}\int_0^1 \frac{\ln z \ln^2(1+az)}{z}\,\mathrm{d}z = &-\frac{\pi^4}{45} - 2\zeta(3)\ln(1+a) - \frac{\pi^2}{6}\ln^2(1+a) - \frac{1}{12}\ln^4(1+a)\\ &+\frac{1}{3}\ln(-a)\ln^3(1+a) + 2\ln(1+a)\operatorname{Li}_3(-a) - 2\operatorname{Li}_4(-a)\\ &+2\operatorname{Li}_4(1+a) - 2\operatorname{Li}_4\left(\frac{a}{a+1}\right), \quad a\in\mathbb{C}\setminus(-\infty,-1]\cup[0,\infty).\end{aligned} \tag{3.51}$$

By substituting (3.51) into (3.13), we conclude the proof of Theorem 9. □

In what follows, we apply our established identities to prove of Jonquière's inversion formula for specific cases.

Theorem 10 (Jonquière). *Let* $a \in \mathbb{C}\setminus(-\infty,-1]\cup[0,\infty)$. *Then*

$$\begin{aligned}\operatorname{Li}_4\left(\frac{z-1}{z}\right) + \operatorname{Li}_4\left(\frac{z}{z-1}\right) = &-\frac{7\pi^4}{360} - \frac{1}{4}\ln^2 z\ln^2(1-z) - \frac{\pi^2}{12}\ln^2(1-z) - \frac{1}{24}\ln^4(1-z)\\ &+\frac{\pi^2}{6}\ln z\ln(1-z) + \frac{1}{6}\ln z\ln^3(1-z) - \frac{\pi^2}{12}\ln^2 z\\ &+\frac{1}{6}\ln^3 z\ln(1-z) - \frac{1}{24}\ln^4 z.\end{aligned} \tag{3.52}$$

Proof. On the transformation $z \longrightarrow 1-z$, we have

$$\int_0^z \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t + \int_0^{1-z} \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t = \frac{1}{2}\ln^2 z\ln^2(1-z) + \int_0^1 \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t. \tag{3.53}$$

This yields

$$\int_0^z \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t + \int_0^{1-z} \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t = -\frac{\pi^4}{180} + \frac{1}{2}\ln^2 z\ln^2(1-z). \tag{3.54}$$

Utilizing (3.39) on the left-hand side of (3.54), we conclude the proof of Theorem 10. □

Remark 11. *Let us define a function* $F(z)$ *as*

$$F(z) = \int_0^z \frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t.$$

With this notation, (3.54) *can be expressed as*

$$F(z) + F(1-z) = -\frac{\pi^4}{180} + \frac{1}{2}\ln^2 z\ln^2(1-z). \tag{3.55}$$

The identity (3.52) *is not new; however, we can refer to Theorem 10 as a rediscovery, as the proof we provide is novel. Theorem 10 can be derived by substituting 4 for* m *and* $\frac{z}{z-1}$ *for* z *in Jonquière's inversion formula* [**2**, §1.11.1, pp. 31, (16)]. *Jonquière's inversion formula is derived from the Lerch transformation formula* [**2**, §1.11(7), pp. 29], *and the Lerch transformation formula is derived using the residue theorem (see* [**2**, §1.11, pp. 28]*). This demonstrates the uniqueness of our proof.*

THEOREM 12 (Jonquière). *Let* $z \in \mathbb{C} \setminus (-\infty, 0]$. *Then*

$$\operatorname{Li}_4\left(-\frac{1}{a}\right) + \operatorname{Li}_4(-a) = -\frac{7\pi^4}{360} - \frac{\pi^2}{12}\ln^2 a - \frac{\ln^4 a}{24}. \tag{3.56}$$

Proof. Define a function $G(z)$ as

$$G(a) = \int_0^1 \frac{\ln z \ln^2(1+az)}{z}\,\mathrm{d}z.$$

Now, using (3.36) and (3.37), we can establish the following relationship

$$\begin{aligned} G(a) + G\left(\frac{1}{a}\right) &= -\frac{2\pi^4}{45} - \int_0^{\frac{1}{1+a}} \frac{\ln^2 z \ln(1-z)}{z}\,\mathrm{d}z - \int_0^{\frac{a}{1+a}} \frac{\ln^2 z \ln(1-z)}{z}\,\mathrm{d}z \\ &\quad + \int_0^{\frac{a}{1+a}} \frac{\ln z \ln^2(1-z)}{z}\,\mathrm{d}z + \int_0^{\frac{1}{1+a}} \frac{\ln z \ln^2(1-z)}{z}\,\mathrm{d}z \\ &\quad - \int_{\frac{1}{1+a}}^1 \frac{\ln^3 z + \ln a \ln^2 z}{z(1-z)}\,\mathrm{d}z - \int_{\frac{a}{1+a}}^1 \frac{\ln^3 z - \ln a \ln^2 z}{z(1-z)}\,\mathrm{d}z. \end{aligned} \tag{3.57}$$

By applying (3.55) to (3.57) and subsequently utilizing (3.39)–(3.41) in (3.57), we derive

(3.58)

$$\begin{aligned} &\operatorname{Li}_4\left(-\frac{1}{a}\right) + \operatorname{Li}_4(-a) + \big(\ln a \ln(1+a) - \ln^2(1+a)\big)\left(\operatorname{Li}_2\left(\frac{1}{1+a}\right) + \operatorname{Li}_2\left(\frac{a}{1+a}\right)\right) \\ &= -\frac{7\pi^4}{360} - \frac{\pi^2}{12}\ln^2 a - \frac{\ln^4 a}{24} + \frac{\pi^2}{6}\ln a \ln(1+a) - \frac{\pi^2}{6}\ln^2(1+a) - 2\ln a \ln^3(1+a) \\ &\quad + \ln^2 a \ln^2(1+a) + \ln^4(1+a). \end{aligned}$$

Now, applying Euler's reflection formula [**1**, (25.12.6)] to the dilogarithms in (3.58), we conclude the proof of Theorem 12. Additionally, it is worth noting that the inversion formula can also be derived by substituting 4 for m and $-\frac{1}{z}$ for z in Jonquière's inversion formula [**2**, §1.11.1, pp. 31, (16)]. However, this newly presented proof offers the advantage of avoiding logarithms of negative real numbers. □

3.2. Infinite series involving the Hurwitz zeta function. In this subsection, we introduce a theorem for double infinite series with symmetric summands and apply it to derive new identities. The motivation for determining closed forms for these series stems from the fact that the harmonic series $\sum_{k=1}^{\infty} \frac{H_k}{k^3}$, as presented in Lemma 1, can also be proven as follows. By utilizing the series expression for $\psi_1(k)$, we initially obtain

$$\sum_{k=1}^{\infty} \frac{\psi_1(k)}{k^2} = \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} \frac{1}{k^2 j^2}.$$

Interchanging the order of summation, and then the roles of the dummy variables j and k, we obtain

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}\frac{1}{j^2k^2}=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\frac{1}{j^2k^2}=\sum_{k=1}^{\infty}\sum_{j=1}^{k}\frac{1}{j^2k^2}.$$

Consequently, this leads to

$$2\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}\frac{1}{j^2k^2}=\zeta(4)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j^2k^2}=\zeta(4)+\zeta^2(2).$$

Hence,

$$\sum_{k=1}^{\infty}\frac{\psi_1(k)}{k^2}=\frac{\zeta(4)+\zeta^2(2)}{2}=\frac{7\pi^4}{360}.$$

Utilizing (3.30), we derive

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H_k}{k^3}&=\zeta(4)+\int_0^1\frac{\left(\frac{\pi^2}{6}-\ln t\ln(1-t)-2\operatorname{Li}_2(t)\right)\ln(1-t)}{t}\,\mathrm{d}t+\frac{1}{2}\int_0^1\frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t\\&=\frac{\pi^4}{90}-\frac{1}{2}\int_0^1\frac{\ln t\ln^2(1-t)}{t}\,\mathrm{d}t=\frac{\pi^4}{90}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 t^{k-1}\ln t\ln(1-t)\,\mathrm{d}t\\&=\frac{\pi^4}{60}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{k^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\psi_1(k)}{k^2}=\frac{\pi^4}{72}.\end{aligned}$$

Theorem 13. *Let f be an arbitrary symmetric function of two variables such that*

(1) *if $f(k,j)=(-1)^{k+j}g(k,j)$, where $g(k,j)$ is positive, and $\sum_{k=1}^{\infty}f(k,k)$ converges, or*
(2) *if $f(k,j)$ is positive and $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}f(k,j)$ converges.*

Then $\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}f(k+j,j)$ is convergent, and

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}f(k+j,j)=\frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}f(k,j)+\sum_{k=1}^{\infty}f(k,k)\right). \tag{3.59}$$

Proof. Interchanging the order of summation, we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}f(k,j)=\sum_{k=1}^{\infty}\sum_{j=k}^{\infty}f(k,j). \tag{3.60}$$

Next, we interchange the roles of the dummy variables, resulting in

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j}f(k,j)=\sum_{k=1}^{\infty}\sum_{j=1}^{k}f(j,k). \tag{3.61}$$

Since by hypothesis $f(k,j)$ is symmetric, it follows that $f(j,k) = f(k,j)$, and thus, (3.61) can be expressed as

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j} f(k,j) = \sum_{k=1}^{\infty}\sum_{j=1}^{k} f(k,j). \tag{3.62}$$

Combining the two resulting expressions from (3.60) and (3.62), we deduce

$$\sum_{j=1}^{\infty}\sum_{k=1}^{j} f(k,j) = \frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) + \sum_{k=1}^{\infty} f(k,k)\right), \tag{3.63}$$

By reindexing the series on the left-hand side of (3.63), we have

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) - \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} f(k,j) = \frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) + \sum_{k=1}^{\infty} f(k,k)\right). \tag{3.64}$$

Shifting the index in the second series on the left-hand side of (3.63), we obtain

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k+j,j) = \frac{1}{2}\left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) + \sum_{k=1}^{\infty} f(k,k)\right). \tag{3.65}$$

Upon reindexing the second series on the left-hand side of (3.65) as

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k+j,j) = \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} f(k+j,j) - \sum_{k=1}^{\infty} f(k,k),$$

and rearranging, (3.59) follows. We must now demonstrate the conditions for convergence. To begin with the first condition, let $f(k,j) = (-1)^{k+j} g(k,j)$. It is evident that the symmetricity of f implies the symmetricity of g. Consider the following series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{k+j} g(k,j) = \sum_{j=1}^{\infty} g(k,k) - 2\sum_{1\leqslant j<k<\infty} g(k,j)$$

Since, according to our hypothesis, $g(k,j)$ is positive, we can write

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j) = \sum_{j=1}^{\infty} g(k,k) - 2\sum_{1\leqslant j<k<\infty} g(k,j) \leqslant \sum_{j=1}^{\infty} g(k,k) = \sum_{j=1}^{\infty} f(k,k).$$

By applying the comparison test, we can conclude that $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j)$ converges if $\sum_{j=1}^{\infty} f(k,k)$ converges. As the sum of two convergent series is itself convergent, we can now deduce from (3.59) that $\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} f(k+j,j)$ converges if $\sum_{k=1}^{\infty} f(k,k)$ converges. Moving on to the second condition, assume $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j)$ converges. It is clear that since $f(k,j)$ is positive, we can write

$$\sum_{k=1}^{\infty} f(k,k) \leqslant \sum_{j=1}^{\infty} f(k,k) + 2\sum_{1\leqslant j<k<\infty} f(k,j) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j).$$

By employing a similar argument, we can conclude that $\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} f(k+j,j)$ is indeed convergent, provided $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} f(k,j)$ converges. ☐

Corollary 14. *Let f be an arbitrary function such that*

(1) *if $f(k)=(-1)^k g(k)$, where $g(k)$ is positive, and $\sum_{k=1}^{\infty} f^2(k)$ converges, or*
(2) *if $f(k)$ is positive and $\sum_{k=1}^{\infty} f(k)$ converges.*

Then $\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} f(k+j)f(j)$ is convergent and

$$\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} f(k+j)f(j)=\frac{1}{2}\left(\left(\sum_{k=1}^{\infty} f(k)\right)^2+\sum_{k=1}^{\infty} f^2(k)\right). \tag{3.66}$$

Proof. Taking $f(k,j)=f(k)g(j)$ in Theorem 13, the symmetricity of $f(k,j)$ implies $f(k)=g(k)$. As such, the proof of Corollary 14 is complete. ☐

Remark 15. *Corollary 14 possesses the remarkable property of transforming a double infinite series into an expression that consists of the sum of the square of an infinite series and another infinite series. It is worth noting that the results established in Theorem 13 and Corollary 14 have not been previously presented elsewhere in the existing literature.*

We apply Corollary 14 in the following theorem.

Theorem 16. *Let $\Re(m)>1$, $r,s\in\mathbb{C}$, where $r\neq 0$, $rk\neq s$, for any positive integer k. Then*

$$\sum_{k=1}^{\infty}\frac{\zeta\left(m,\frac{rk-s}{r}\right)}{(rk-s)^m}=\frac{1}{2r^m}\left(\zeta^2\left(m,\frac{r-s}{r}\right)+\zeta\left(2m,\frac{r-s}{r}\right)\right). \tag{3.67}$$

Proof. By setting $f(k)=\frac{1}{(rk-s)^m}$ in Corollary 14, we conclude the proof of Theorem 16. ☐

Remark 17. *Theorem 16 does not appear in the DLMF* [**1**, §25.11(xi)] *and Prudnikov's book* [**6**, pp. 396–397], *where series involving the Hurwitz zeta function are discussed.*

Corollary 18. *Let m be any positive integer greater than 1, $r,s\in\mathbb{C}$, where $r\neq 0$, $rk\neq s$, for any positive integer k. Then*

$$\sum_{j=1}^{\infty}\frac{\psi_{m-1}\left(\frac{rj-s}{r}\right)}{(rj-s)^m}=\frac{(-1)^m}{2r^m}\left(\frac{\psi_{m-1}^2\left(\frac{r-s}{r}\right)}{(m-1)!}+\frac{(m-1)!}{(2m-1)!}\psi_{2m-1}\left(\frac{r-s}{r}\right)\right). \tag{3.68}$$

Proof. Theorem 16 reduces to Corollary 18, if we consider m as any positive integer greater than 1. ☐

Example 1. For any positive integer $m>1$, we have

$$\sum_{j=1}^{\infty}\frac{\psi_{m-1}\left(\frac{4j-1}{4}\right)}{(4j-1)^m}=(-1)^m 2^{-2m-1}\left(\frac{\psi_{m-1}^2\left(\frac{3}{4}\right)}{(m-1)!}+\frac{(m-1)!}{(2m-1)!}\psi_{2m-1}\left(\frac{3}{4}\right)\right). \tag{3.69}$$

For $m = 2, 3, 4, 5$, we have

$$\sum_{j=1}^{\infty} \frac{\psi_1\left(\frac{4j-1}{4}\right)}{(4j-1)^2} = 2\mathbf{G}^2 - \frac{\mathbf{G}\pi^2}{2} + \frac{\pi^4}{32} + \frac{\psi_3\left(\frac{3}{4}\right)}{192}, \tag{3.70}$$

$$\sum_{j=1}^{\infty} \frac{\psi_2\left(\frac{4j-1}{4}\right)}{(4j-1)^3} = -\frac{\pi^6}{64} + \frac{7\pi^3}{8}\zeta(3) - \frac{49\zeta^2(3)}{4} - \frac{\psi_5\left(\frac{3}{4}\right)}{7680}, \tag{3.71}$$

$$\sum_{j=1}^{\infty} \frac{\psi_3\left(\frac{4j-1}{4}\right)}{(4j-1)^4} = \frac{\psi_3^2\left(\frac{3}{4}\right)}{3072} + \frac{\psi_7\left(\frac{3}{4}\right)}{430080}, \tag{3.72}$$

$$\sum_{j=1}^{\infty} \frac{\psi_4\left(\frac{4j-1}{4}\right)}{(4j-1)^5} = -\frac{25\pi^{10}}{768} + \frac{155\pi^5}{8}\zeta(5) - 2883\zeta^2(5) - \frac{\psi_9\left(\frac{3}{4}\right)}{30965760}. \tag{3.73}$$

Corollary 19. *Let m be any positive integer greater than 1. Then*

(3.74)
$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\psi_{2m-2}\left(\frac{4j-1}{4}\right)}{(4j-1)^{2m-1}} = &-\frac{|E_{2m-2}|}{8}\left(1-2^{2m-1}\right)\pi^{2m-1}\zeta(2m-1)\\ &- \frac{\left(1-2^{2m-1}\right)^2(2m-2)!}{8}\zeta^2(2m-1) - \frac{E_{2m-2}^2}{32(2m-2)!}\pi^{4m-2}\\ &- \frac{(2m-2)!}{2^{4m-1}(4m-3)!}\psi_{4m-3}\left(\frac{3}{4}\right),\end{aligned}$$

where E_m are the Euler numbers.

Proof. Olaikhan [7, §1.20.6, §1.20.7, pp. 62–63] expressed $\psi_{2a}\left(\frac{3}{4}\right)$ in terms of $\psi_{2a}\left(\frac{1}{4}\right)$, and $\psi_{2a}\left(\frac{1}{4}\right)$ in terms of the Euler numbers E_a, with a as a positive integer. By employing both of these expressions, we derive

$$\psi_{2m-2}\left(\frac{3}{4}\right) = 2^{2m-2}\left(\left(1-2^{2m-1}\right)(2m-2)!\zeta(2m-1) + \frac{\pi^{2m-1}}{2}|E_{2m-2}|\right). \tag{3.75}$$

By replacing m with $2m-1$ and subsequently substituting (3.75) into (3.69), we conclude the proof of Corollary 19. ☐

Theorem 20. *Let m be any positive integer greater than 1, $r, s \in \mathbb{C}$, where $r \neq 0$, $rk \neq s$, for any positive integer k. Then*

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{p=0}^{m-1} \frac{1}{r^p(rk-s)^{m-p}}\binom{m}{p}\sin\left(\frac{\pi}{2}(m-p)\right)\int_0^{\infty}\frac{x^{m-p}}{(r^2x^2+(rk-s)^2)^m(e^{2\pi x}-1)}\,\mathrm{d}x\\ = \frac{1}{4r^{3m}}\zeta^2\left(m, \frac{r-s}{r}\right) - \frac{1}{2r^{3m}(m-1)}\zeta\left(2m-1, \frac{r-s}{r}\right).\end{aligned}$$

Proof. Employing Hermite's integral representation (2.4) for $\zeta(s,z)$ in Theorem 16, we have

$$\sum_{k=1}^{\infty}\frac{1}{(rk-s)^m}\left(\frac{\left(\frac{rk-s}{r}\right)^{-m}}{2}+\frac{\left(\frac{rk-s}{r}\right)^{1-m}}{m-1}\right) = \frac{1}{2r^m}\zeta\left(2m,\frac{r-s}{r}\right)+\frac{1}{r^m(m-1)}\zeta\left(2m-1,\frac{r-s}{r}\right). \tag{3.76}$$

Define the last integral in (2.4) as

$$f(m,z)=\int_0^\infty \frac{\sin\left(m\arctan\left(x/(z)\right)\right)}{\left(x^2+z^2\right)^{\frac{m}{2}}\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x.$$

By De Moivre's theorem [1, (4.21.34)], we have

$$f(m,z)=\Im\int_0^\infty \frac{\left(\cos\left(\arctan\left(\frac{x}{z}\right)\right)+\mathrm{i}\sin\left(\arctan\left(\frac{x}{z}\right)\right)\right)^m}{\left(x^2+z^2\right)^{\frac{m}{2}}\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x,$$

where $\mathrm{i}=\sqrt{-1}$. By the binomial theorem, we have

$$f(m,z)=\Im\int_0^\infty\sum_{p=0}^{m}\binom{m}{p}\frac{\left(\cos\left(\arctan\left(\frac{x}{z}\right)\right)\right)^p\mathrm{i}^{m-p}\left(\sin\left(\arctan\left(\frac{x}{z}\right)\right)\right)^{m-p}}{\left(x^2+z^2\right)^{\frac{m}{2}}\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x.$$

Interchanging summation and integration, we have

$$f(m,z)=\Im\sum_{p=0}^{m}\binom{m}{p}\int_0^\infty\frac{\left(\cos\left(\arctan\left(\frac{x}{z}\right)\right)\right)^p\mathrm{i}^{m-p}\left(\sin\left(\arctan\left(\frac{x}{z}\right)\right)\right)^{m-p}}{\left(x^2+z^2\right)^{\frac{m}{2}}\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x.$$

Using the trigonometric identity $1+\tan^2\theta=\sec^2\theta$, we deduce

$$\cos\left(\arctan\left(\frac{x}{z}\right)\right)=\frac{z}{\sqrt{x^2+z^2}},\qquad \sin\left(\arctan\left(\frac{x}{z}\right)\right)=\frac{x}{\sqrt{x^2+z^2}}.$$

Therefore, we can express $f(m,z)$ as

$$f(m,z)=\Im\sum_{p=0}^{m}\binom{m}{p}z^p\mathrm{i}^{m-p}\int_0^\infty\frac{x^{m-p}}{\left(x^2+z^2\right)^m\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x.$$

Since

$$\Im\mathrm{i}^{m-p}=\Im e^{\mathrm{i}\frac{\pi}{2}(m-p)}=\Im\left(\cos\left(\frac{\pi}{2}(m-p)\right)+\mathrm{i}\sin\left(\frac{\pi}{2}(m-p)\right)\right)=\sin\left(\frac{\pi}{2}(m-p)\right),$$

we conclude that

$$f(m,z)=\sum_{p=0}^{m-1}\binom{m}{p}z^p\sin\left(\frac{\pi}{2}(m-p)\right)\int_0^\infty\frac{x^{m-p}}{\left(x^2+z^2\right)^m\left(\mathrm{e}^{2\pi x}-1\right)}\,\mathrm{d}x. \tag{3.77}$$

Replacing z in (3.77) with $\frac{rk-s}{r}$, utilizing the latter and (3.76) to derive an integral representation for $\zeta\left(m,\frac{rk-s}{r}\right)$, and further substituting the result in Theorem 16, we conclude the proof of Theorem 20. □

Since $\sin\left(\frac{\pi}{2}(m-p)\right)=0,\pm1$, for integer values of m and p, Theorem 20 can be rewritten for odd and even values of m, as follows.

Corollary 21. *Let m be any positive integer, $r,s \in \mathbb{C}$, where $r \neq 0$, $rk \neq s$, for any positive integer k. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m-1}\frac{(-1)^{m+p-1}}{r^{2p+1}(rk-s)^{2m-2p-1}}\binom{2m}{2p+1}\int_0^{\infty}\frac{x^{2m-2p-1}}{(r^2x^2+(rk-s)^2)^{2m}(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{1}{4r^{6m}}\zeta^2\left(2m,\frac{r-s}{r}\right)-\frac{1}{2r^{6m}(2m-1)}\zeta\left(4m-1,\frac{r-s}{r}\right).$$

Corollary 22. *Let m be any positive integer, $r,s \in \mathbb{C}$, where $r \neq 0$, $rk \neq s$, for any positive integer k. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m}\frac{(-1)^{m+p}}{r^{2p}(rk-s)^{2m-2p+1}}\binom{2m+1}{2p}\int_0^{\infty}\frac{x^{2m-2p+1}}{(r^2x^2+(rk-s)^2)^{2m+1}(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{1}{4r^{6m+3}}\zeta^2\left(2m+1,\frac{r-s}{r}\right)-\frac{1}{4r^{6m+3}m}\zeta\left(4m+1,\frac{r-s}{r}\right).$$

Theorem 20 reduces directly to an expression in the Riemann zeta function if we set $r=1$, $s=0$ and $r=2$, $s=1$. This yields the following corollaries.

Corollary 23. *Let m be any positive integer greater than 1. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m-1}\frac{1}{k^{m-p}}\binom{m}{p}\sin\left(\frac{\pi}{2}(m-p)\right)\int_0^{\infty}\frac{x^{m-p}}{(x^2+k^2)^m(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{1}{4}\zeta^2(m)-\frac{1}{2(m-1)}\zeta(2m-1).$$

Corollary 24. *Let m be any positive integer greater than 1. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m-1}\frac{1}{2^p(2k-1)^{m-p}}\binom{m}{p}\sin\left(\frac{\pi}{2}(m-p)\right)\int_0^{\infty}\frac{x^{m-p}}{(4x^2+(2k-1)^2)^m(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=2^{-3m-2}\left(2^m-1\right)^2\zeta^2(m)-\frac{2^{-3m-1}\left(2^{2m-1}-1\right)}{m-1}\zeta(2m-1).$$

As in Corollaries 21 and 22, we provide expressions for Corollaries 23 and 24 for odd and even values of m.

Corollary 25. *Let m be any positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m-1}\frac{(-1)^{m+p-1}}{k^{2m-2p-1}}\binom{2m}{2p+1}\int_0^{\infty}\frac{x^{2m-2p-1}}{(x^2+k^2)^{2m}(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{2^{4m-4}}{(2m)!^2}B_{2m}^2\pi^{4m}-\frac{1}{2(2m-1)}\zeta(4m-1),$$

where B_m are the Bernoulli numbers.

Example 2. For $m=1,2,3$, we have

$$\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{(x^2+k^2)^2(e^{2\pi x}-1)}\,\mathrm{d}x=\frac{\pi^4}{288}-\frac{\zeta(3)}{4},$$

$$\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{\left(x^2+k^2\right)^4\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty}\frac{x^3}{\left(x^2+k^2\right)^4\left(e^{2\pi x}-1\right)}\,\mathrm{d}x=\frac{\pi^8}{129600}-\frac{\zeta(7)}{24},$$

$$6\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{\left(x^2+k^2\right)^6\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-20\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty}\frac{x^3}{\left(x^2+k^2\right)^6\left(e^{2\pi x}-1\right)}\,\mathrm{d}x$$
$$+6\sum_{k=1}^{\infty}\frac{1}{k^5}\int_0^{\infty}\frac{x^5}{\left(x^2+k^2\right)^6\left(e^{2\pi x}-1\right)}\,\mathrm{d}x=\frac{\pi^{12}}{3572100}-\frac{\zeta(11)}{10}.$$

Corollary 26. *Let m be any positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m}\frac{(-1)^{m+p}}{k^{2m-2p+1}}\binom{2m+1}{2p}\int_0^{\infty}\frac{x^{2m-2p+1}}{(x^2+k^2)^{2m+1}(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{1}{4}\zeta^2(2m+1)-\frac{1}{4m}\zeta(4m+1).$$

Example 3. For $m=1,2,3$, we have

$$3\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{\left(x^2+k^2\right)^3\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty}\frac{x^3}{\left(x^2+k^2\right)^3\left(e^{2\pi x}-1\right)}\,\mathrm{d}x=\frac{\zeta^2(3)}{4}-\frac{\zeta(5)}{4},$$

$$5\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{\left(x^2+k^2\right)^5\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-10\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty}\frac{x^3}{\left(x^2+k^2\right)^5\left(e^{2\pi x}-1\right)}\,\mathrm{d}x$$
$$+\sum_{k=1}^{\infty}\frac{1}{k^5}\int_0^{\infty}\frac{x^5}{\left(x^2+k^2\right)^5\left(e^{2\pi x}-1\right)}\,\mathrm{d}x=\frac{\zeta^2(5)}{4}-\frac{\zeta(9)}{8},$$

$$7\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty}\frac{x}{\left(x^2+k^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-35\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^{\infty}\frac{x^3}{\left(x^2+k^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x$$
$$+21\sum_{k=1}^{\infty}\frac{1}{k^5}\int_0^{\infty}\frac{x^5}{\left(x^2+k^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x-\sum_{k=1}^{\infty}\frac{1}{k^7}\int_0^{\infty}\frac{x^7}{\left(x^2+k^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x$$
$$=\frac{\zeta^2(7)}{4}-\frac{\zeta(13)}{12}.$$

Corollary 27. *Let m be any positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m-1}\frac{(-1)^{m+p-1}}{2^{2p+1}(2k-1)^{2m-2p-1}}\binom{2m}{2p+1}\int_0^{\infty}\frac{x^{2m-2p-1}}{(4x^2+(2k-1)^2)^{2m}(e^{2\pi x}-1)}\,\mathrm{d}x$$
$$=\frac{2^{-2m-4}\left(2^{2m}-1\right)^2}{(2m)!^2}B_{2m}^2\pi^{4m}-\frac{2^{-6m-1}\left(2^{4m-1}-1\right)}{2m-1}\zeta(4m-1),$$

where B_m are the Bernoulli numbers.

Example 4. For $m = 1, 2, 3$, we have

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{(4x^2+(2k-1)^2)^2(e^{2\pi x}-1)}\,\mathrm{d}x = \frac{\pi^4}{1024} - \frac{7\zeta(3)}{128},$$

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{\left(4x^2+(2k-1)^2\right)^4(e^{2\pi x}-1)}\,\mathrm{d}x - 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \int_0^{\infty} \frac{x^3}{\left(4x^2+(2k-1)^2\right)^4(e^{2\pi x}-1)}\,\mathrm{d}x = \frac{\pi^8}{589824} - \frac{127\zeta(7)}{24576},$$

$$\frac{3}{16}\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{\left(4x^2+(2k-1)^2\right)^6(e^{2\pi x}-1)}\,\mathrm{d}x - \frac{5}{2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \int_0^{\infty} \frac{x^3}{\left(4x^2+(2k-1)^2\right)^6(e^{2\pi x}-1)}\,\mathrm{d}x + 3\sum_{k=1}^{\infty} \frac{1}{(2k-1)^5} \int_0^{\infty} \frac{x^5}{\left(4x^2+(2k-1)^2\right)^6(e^{2\pi x}-1)}\,\mathrm{d}x = \frac{\pi^{12}}{235929600} - \frac{2047\zeta(11)}{2621440}.$$

Corollary 28. *Let m be any positive integer. Then*

$$\sum_{k=1}^{\infty}\sum_{p=0}^{m} \frac{(-1)^{m+p}}{2^{2p}(2k-1)^{2m-2p+1}} \binom{2m+1}{2p} \int_0^{\infty} \frac{x^{2m-2p+1}}{(4x^2+(2k-1)^2)^{2m+1}(e^{2\pi x}-1)}\,\mathrm{d}x = 2^{-6m-5}\left(2^{2m+1}-1\right)^2\zeta^2(2m+1) - \frac{2^{-6m-4}\left(2^{4m+1}-1\right)}{2m}\zeta(4m+1).$$

Example 5. For $m = 1, 2, 3$, we have

$$\frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{\left(4x^2+(2k-1)^2\right)^3(e^{2\pi x}-1)}\,\mathrm{d}x - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \int_0^{\infty} \frac{x^3}{\left(4x^2+(2k-1)^2\right)^3(e^{2\pi x}-1)}\,\mathrm{d}x = \frac{49\zeta^2(3)}{2048} - \frac{31\zeta(5)}{2048},$$

$$\frac{5}{16}\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{\left(4x^2+(2k-1)^2\right)^5(e^{2\pi x}-1)}\,\mathrm{d}x - \frac{5}{2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \int_0^{\infty} \frac{x^3}{\left(4x^2+(2k-1)^2\right)^5(e^{2\pi x}-1)}\,\mathrm{d}x + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^5} \int_0^{\infty} \frac{x^5}{\left(4x^2+(2k-1)^2\right)^5(e^{2\pi x}-1)}\,\mathrm{d}x = \frac{961\zeta^2(5)}{131072} - \frac{511\zeta(9)}{262144},$$

$$\frac{7}{64}\sum_{k=1}^{\infty} \frac{1}{2k-1} \int_0^{\infty} \frac{x}{\left(4x^2+(2k-1)^2\right)^7(e^{2\pi x}-1)}\,\mathrm{d}x$$

$$
\begin{aligned}
&-\frac{35}{16}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\int_0^{\infty}\frac{x^3}{\left(4x^2+(2k-1)^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x\\
&+\frac{21}{4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^5}\int_0^{\infty}\frac{x^5}{\left(4x^2+(2k-1)^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x\\
&-\sum_{k=1}^{\infty}\frac{1}{(2k-1)^7}\int_0^{\infty}\frac{x^7}{\left(4x^2+(2k-1)^2\right)^7\left(e^{2\pi x}-1\right)}\,\mathrm{d}x=\frac{16129\zeta^2(7)}{8388608}-\frac{8191\zeta(13)}{25165824}.
\end{aligned}
$$

4. Conclusion

In this work, we have introduced several new theorems that provide closed forms for generalized integrals and series. Furthermore, we have demonstrated the application of our double infinite series transformation formula in deriving new identities, which have been expressed using well-known numbers such as the Euler and Bernoulli numbers. One that we find interesting are the simplest cases of Theorem 20:

$$
\begin{aligned}
&\int_0^{\infty}\frac{x}{(x^2+1)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\frac{1}{2}\int_0^{\infty}\frac{x}{(x^2+4)^2(e^{2\pi x}-1)}\,\mathrm{d}x\\
&\quad+\frac{1}{3}\int_0^{\infty}\frac{x}{(x^2+9)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\frac{1}{4}\int_0^{\infty}\frac{x}{(x^2+16)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\cdots\\
&=\frac{\pi^4}{288}-\frac{\zeta(3)}{4},
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\infty}\frac{x}{(4x^2+1)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\frac{1}{3}\int_0^{\infty}\frac{x}{(4x^2+9)^2(e^{2\pi x}-1)}\,\mathrm{d}x\\
&\quad+\frac{1}{5}\int_0^{\infty}\frac{x}{(4x^2+25)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\frac{1}{7}\int_0^{\infty}\frac{x}{(4x^2+49)^2(e^{2\pi x}-1)}\,\mathrm{d}x+\cdots\\
&=\frac{\pi^4}{1024}-\frac{7\zeta(3)}{128}.
\end{aligned}
$$

Interested readers can further explore the transformation formulas outlined in Theorem 13 and Corollary 14 to potentially uncover additional new results.

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