SCIENTIA
Series A: *Mathematical Sciences*, Vol. 34 (2024), 23–26
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

INTEGRALS OF FUNCTIONS OF ROOTS OF x

ROBERT W. BENIM

Abstract. This note will show that the only integrals of the form $\int f(x^\alpha)\,dx$ that can be evaluated via the substitution $t = x^\alpha$ followed by repeated integration by parts are those where α is the reciprocal of a positive integer, $\alpha = \frac{1}{n}$, and f is an arbitrary function which can be antidifferentiated at least n times.

1. Motivation

Every student of mathematics in higher education has some familiarity with integration, and has likely had some exposure to integration tables. These can be found in most Calculus textbooks, such as [H2018] and [S2017]. At least one instance of a truly grand table of integrals can be found in [GR2015]. Despite this, we believe the result presented in this note is new and may be of interest to some.

This note will derive a formula for $\int f(x^{\frac{1}{n}})\,dx$ when we can antidifferentiate f at least n times. Assuming α is a real number, these are the only integrals of the form $\int f(x^\alpha)\,dx$ that can be solved with the substitution $t = x^\alpha$ followed by repeated integration by parts for an arbitrary function f which is n times anitdifferentiable. The derivation for this formula was motivated by the following integral, or ones like it, which can be found in a number of texts, including [H2018] and [S2017]:

$$\int \cos\left(\sqrt{x}\right) dx$$

To evaluate this integral, we can combine the techniques of substitution and integration by parts. Namely, let $t = \sqrt{x}$. It follows that $dt = \frac{1}{2\sqrt{x}}\,dx$. We can rearrange and apply the substitution again to get $2t\,dt = dx$. From this, we obtain the following integral:

$$\int \cos\left(\sqrt{x}\right)\,dx = \int 2t\cos(t)\,dt.$$

2000 *Mathematics Subject Classification.* 26A36.
Key words and phrases. Integration techniques.

This problem, and ones like it, are of a typical type presented in Calculus 2 courses on the topic of integration by parts. To that end, we will use the formula

$$\int u\,dv = uv - \int v\,du$$

where $u = 2t$ and $dv = \cos(t)\,dt$. This gives

$$\begin{aligned}\int 2t\cos(t)\,dt &= 2t\sin(t) - \int 2\sin(t)\,dt\\ &= 2t\sin(t) + 2\cos(t) + C\\ &= 2\sqrt{x}\sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C.\end{aligned}$$

2. Finding a more general formula

We now attempt to generalize this method. First, note that we could easily replace $\cos x$ with any function, $f(x)$, where we can find a second antiderivative. Let $f(x)$ be such a function, and choose $F(x)$ such that $F''(x) = f(x)$. Similar to the above, we have

$$\begin{aligned}\int f\left(\sqrt{x}\right)dx &= \int F''\left(\sqrt{x}\right)dx\\ &= 2\sqrt{x}F\left(\sqrt{x}\right) - 2F'\left(\sqrt{x}\right) + C.\end{aligned}$$

We generalize further, and in a more interesting direction, by answering the following question: Given the integral $\int f\left(x^{\alpha}\right)dx$, for which values of α will we be able to apply the substitution $t = x^{\alpha}$ and then evaluate using repeated integration by parts ?

When we had $\alpha = \frac{1}{2}$, we saw that

$$dt = \frac{1}{2\sqrt{x}}dx \longrightarrow dx = 2tdt.$$

Since the derivative of $2t$ was a constant, integration by parts successful in this example. We can show that this idea can be extended if $2t$ is replaced by other polynomials. We need to know what restriction this places on α. If $t = x^{\alpha}$ then

$$dt = \alpha x^{\alpha-1}dx \longrightarrow dx = \frac{1}{\alpha}x^{1-\alpha}dt = \frac{1}{\alpha}t^{\frac{1}{\alpha}-1}dt.$$

The restriction is $\frac{1}{\alpha} - 1$ must be a positive integer. (We can safely omit the case where $\frac{1}{\alpha} - 1 = 0$.) So, $\frac{1}{\alpha}$ needs to be an integer greater than or equal to 2, which means α needs to be the reciprocal of an integer which is greater than or equal to 2.

We now generalize the method of the original example to $\int f\left(x^{\frac{1}{n}}\right)dx$ where n is an integer at least 2 and we can find as many antiderivatives of $f(x)$ as needed. We apply the substitution $t = x^{\frac{1}{n}}$, which yields

$$dt = \frac{1}{n}x^{\frac{1}{n}-1}dx \longrightarrow dx = \frac{n}{x^{\frac{1}{n}-1}}dt = nt^{n-1}dt.$$

This yields the following:

$$\int f\left(x^{\frac{1}{n}}\right)dx = \int nt^{n-1}f(t)dt.$$

Choose $F(t)$ such that $F^{(n)}(t) = f(t)$. We now apply tabular integration (repeated integration by parts) with an initial choice of $u = nt^{n-1}$ and $dv = f(t)dt$.

u		dv
nt^{n-1}	$+$	$f(t) = F^{(n)}(t)$
$n(n-1)t^{n-2}$	$-$	$F^{(n-1)}(t)$
$n(n-1)(n-2)t^{n-3}$		$F^{(n-2)}(t)$
$\vdots$		$\vdots$
$n!t$	$\boldsymbol{(-1)^n}$	$F''(t)$
$n!$	$\boldsymbol{(-1)^{n+1}}$	$F'(t)$
0		$F(t)$

A term in the u column is multiplied by the entry in the dv column that is one row lower. Then, these products will be combined via an alternating sum beginning with addition. Lastly, we need to replace t with $x^{\frac{1}{n}}$. This yields the following formula:

Theorem 2.1. *Let n be an integer greater than or equal to 2. Suppose $f(x)$ can be antidifferentiated n times and choose $F(x)$ such that $F^{(n)}(x) = f(x)$. Then,*

$$\int f\left(x^{\frac{1}{n}}\right)dx = C + \sum_{k=1}^{n} \frac{(-1)^{k+1}n!}{(n-k)!} x^{1-\frac{k}{n}} F^{(n-k)}\left(x^{\frac{1}{n}}\right).$$

Applying the substitution $u = ax + b$ and the theorem above proves the following generalization.

Corollary 2.2. *Let n be an integer greater than or equal to 2. Suppose $f(x)$ can be antidifferentiated n times and choose $F(x)$ such that $F^{(n)}(x) = f(x)$. Then,*

$$\int f\left((ax+b)^{\frac{1}{n}}\right)dx = C + \sum_{k=1}^{n} \frac{(-1)^{k+1}n!}{a(n-k)!}(ax+b)^{1-\frac{k}{n}} F^{(n-k)}\left((ax+b)^{\frac{1}{n}}\right).$$

3. Examples

We illustrate our results with a few examples.

Example 3.1.

$$\int e^{x^{\frac{1}{5}}}dx = 5x^{\frac{4}{5}}e^{x^{\frac{1}{5}}} - 20x^{\frac{3}{5}}e^{x^{\frac{1}{5}}} + 60x^{\frac{2}{5}}e^{x^{\frac{1}{5}}} - 120x^{\frac{1}{5}}e^{x^{\frac{1}{5}}} + 120e^{x^{\frac{1}{5}}} + C$$

$$= \left(5x^{\frac{4}{5}} - 20x^{\frac{3}{5}} + 60x^{\frac{2}{5}} - 120x^{\frac{1}{5}} + 120\right) e^{x^{\frac{1}{5}}} + C$$

Example 3.2. We consider the integral

$$\int \frac{1}{1 + 4x^{\frac{1}{4}} + 6x^{\frac{1}{6}} + 4x^{\frac{3}{4}} + x}\, dx = \int \frac{1}{\left(1 + x^{\frac{1}{4}}\right)^4}\, dx.$$

This example requires antidifferentiating $\frac{1}{(1+u)^4}$ four times.

Then, the theorem gives:

$$\int \frac{1}{1 + 4x^{\frac{1}{4}} + 6x^{\frac{1}{6}} + 4x^{\frac{3}{4}} + x}\, dx = C - \frac{4}{3}\left(\frac{x^{\frac{1}{4}}}{1 + x^{\frac{1}{4}}}\right)^3 - 2\left(\frac{x^{\frac{1}{4}}}{1 + x^{\frac{1}{4}}}\right)^2 - 4\left(\frac{x^{\frac{1}{4}}}{1 + x^{\frac{1}{4}}}\right) + 4\ln\left(1 + x^{\frac{1}{4}}\right).$$

Example 3.3. We consider the integral $\int \frac{1}{\sqrt{1-(4x+3)^{\frac{2}{3}}}}\, dx$. This example requires antidifferentiating $\frac{1}{\sqrt{1-u^2}}$ three times.

Then, the corollary gives:

$$\int \frac{1}{\sqrt{1 - (4x+3)^{\frac{2}{3}}}}\, dx = \frac{3}{4}\left(\arcsin(4x+3)^{\frac{1}{3}} - (4x+3)^{\frac{1}{3}}\sqrt{1 - (4x+3)^{\frac{2}{3}}}\right) + C$$

References

[GR2015] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products. Edited by D. Zwillinger and V. Moll*, Academic Press, New York, 8th edition (2015)

[H2018] G. Hartman, T. Siemers, B. Heinold, and D. Chalishajar *APEX Calculus II, Version 4.0* (2018), p. 105, A17

[S2017] J. Stewert *Single Variable Essential Calculus* Second Edition *Brooks/Cole, Cengage Learning* (2017), p. 322, R6-R10

Received 18 07 2023, revised 02 01 2024

Department of Applied Mathematics,
University of Colorado-Boulder
Boulder, CO 80309,
USA

E-mail address: robert.benim@colorado.edu