

A note relating to Ramanujan's Bessel index integral

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ABSTRACT. Ramanujan's Bessel index integral

$$\int_{-\infty}^{\infty} J_{\mu-\xi}(t)J_{\nu+\xi}(t)d\xi$$

and several extensions are evaluated by an alternative method.

1. Introduction

In 1920, Ramanujan [3] introduced his unusual, and potentially important [4], Bessel integral

$$(1.1) \quad \int_{-\infty}^{\infty} d\xi J_{\mu-\xi}(t)J_{\nu-\xi}(t) = J_{\mu+\nu}(2t).$$

In this note, (1.1) is re-derived by a more pedestrian route and the method applied to obtain several similar index-integrals, some already tabulated [1, Chapter 17] and some apparently new. The idea is simply to employ Nielsen's series [4]

$$(1.2) \quad J_a(z)J_b(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(z/2)^{a+b+2m} (a+b+m+1)_m}{\Gamma(a+m+1)\Gamma(b+m+1)}$$

with an instance of

$$(1.3) \quad \int_{-\infty}^{\infty} \frac{F(\xi)}{\Gamma(a-\xi)\Gamma(b+\xi)} d\xi = \Phi(a, b)$$

a number of which were the principal subject of [3] and which can be found in [1], [2, Section 2.2]. Specifically, we shall examine the five examples, the first of which is known [1, 2, 3].

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	$F(z)$	$\Phi(a, b)$	Conditions
(i)	$\cos(cz)$	$\frac{[2 \cos(c/2)]^{a+b-2}}{\Gamma(a+b-1)} \cos[\frac{1}{2}c(b-a)]$	$ c < \pi, \operatorname{Re}(a+b) > 1$
(ii)	$\sin(cz)$	$\frac{[2 \cos(c/2)]^{a+b-2}}{\Gamma(a+b-1)} \sin[\frac{1}{2}c(b-a)]$	$ c < \pi, \operatorname{Re}(a+b) > 1$
(iii)	$\frac{\sin(n\pi z)}{\sin(\pi z)} \theta(z)$	$\frac{1 - (-1)^n}{2} \frac{2^{2a-2}}{\Gamma(2a-1)}$	$b = a, \operatorname{Re} a > \frac{1}{2}$
(iv)	$\frac{\cos^2(\pi z)}{\Gamma(a-z)\Gamma(b+z)}$	$\frac{\Gamma(2a+2b-3)}{2\Gamma(2a-1)\Gamma(2b-1)\Gamma^2(a+b-1)}$	$\operatorname{Re}(a+b) > \frac{3}{2}$
(v)	$P(z)e^{(2n\pi+\phi)iz}$	$\frac{[2 \cos(\phi/2)]^{a+b-2}}{\Gamma(a+b-1)} e^{i\phi(b-a)/2} I_n$ $P(x+1) = P(x)$	$ \phi < \pi, \operatorname{Re}(a+b) > 1$ $I_n = \int_0^1 P(x)e^{2n\pi ix} dx$

In most cases the resulting series is hypergeometric and can be summed.

2. Calculations and results

From (1.2) and (1.3) one has

$$(2.1) \quad J = \int_{-\infty}^{\infty} F(\xi) J_{a-\xi}(t) J_{b+\xi}(t) d\xi = \left(\frac{t}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{t}{2}\right)^{2m} \frac{\Gamma(a+b+2m+1)}{\Gamma(a+b+m+1)} \Phi(a+m+1, b+m+1).$$

For example (i), this easily reduces to

$$(2.2) \quad J_i = \left(\frac{t}{2}\right)^{a+b} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{t}{2}\right)^{2m} \frac{[2 \cos(c/2)]^{a+b+2m}}{\Gamma(a+b+1+m)} \cos[c(b-a)/2]$$

$$(2.3) \quad = \frac{(t \cos[\frac{1}{2}(b-a)] \cos(c/2))^{a+b}}{\Gamma(a+b+1)} {}_0F_1 \left(\begin{matrix} - \\ a+b+1 \end{matrix} \middle| - \left(t \cos \frac{c}{2}\right)^2 \right).$$

But

$$(2.4) \quad {}_0F_1 \left(\begin{matrix} - \\ A \end{matrix} \middle| -z^2 \right) = \Gamma(A) z^{1-A} J_{A-1}(2z),$$

so

$$(2.5) \quad \int_{-\infty}^{\infty} \cos(c\xi) J_{a-\xi}(t) J_{b+\xi}(t) d\xi = \cos[\frac{c}{2}(b-a)] J_{a+b}(2t \cos(\frac{c}{2}))$$

for $\operatorname{Re}(a+b) > 1$. Setting $c = 0$ in (2.5) one obtains Ramanujan's formula (1.1).

Similarly for (ii)

$$(2.6) \quad \int_{-\infty}^{\infty} \sin(c\xi) J_{a-\xi}(t) J_{b+\xi}(t) d\xi = \sin[\frac{c}{2}(b-a)] J_{a+b}(2t \cos(c/2)), \quad \operatorname{Re}(a+b) > 1,$$

and for example (iii)

$$(2.7) \quad \int_0^{\infty} \frac{\sin(n\pi x)}{\sin(\pi x)} J_{a-x}(t) J_{a+x}(t) dx = \frac{1 - (-1)^n}{2} J_{2a}(2t), \quad \operatorname{Re} a > \frac{1}{2}.$$

For example (iv) this procedure leads to

$$(2.8) \quad \int_0^{\infty} \frac{\cos^2(\pi x) J_{a-x}(2t) J_{a+x}(2t)}{\Gamma(b-x)\Gamma(b+x)} dx = \frac{t^{2a}}{2} {}_1F_1 \left(\begin{matrix} 2(a+b)+1 \\ a+b \end{matrix} \middle| -t^2 \right),$$

for $\operatorname{Re}(a+b) > \frac{3}{2}$. The right-hand side of (2.8) does not appear to reduce further.

Finally, for example (v) one has

$$(2.9) \quad \int_{-\infty}^{\infty} P(x)e^{(2\pi n+\phi)ix} J_{a-x}(t)J_{b+x}(t) dx = e^{(b-a)\phi i/2} J_{a+b}(2t \cos(\phi/2)) \int_0^1 P(u)e^{2n\pi iu} du$$

where P has period 1.

In view of the unproven formula (10.2) in [3], expressing a four Bessel function index integral as a hypergeometric series, it is possible that Ramanujan was aware of the method used here, but chose not to use it.

References

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