SCIENTIA
Series A: *Mathematical Sciences*, Vol. 33 (2023), 13–20
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The integral of Wallis

Victor H. Moll

ABSTRACT. The evaluation of one of the earliest definite integrals and its connection to an infinite product for π is presented.

1. Introduction

One of the challenging problems in Integral Calculus courses presented to students is the evaluation of the integral

$$W_n = \int_0^\infty \frac{dx}{(x^2+1)^n}. \tag{1.1}$$

The goal of this note is to describe a variety of interesting mathematical findings connected to W_n. In particular, we have tried to avoid unmotivated steps and to place ourselves as a student trying to find an expression for W_n alone. The goal of this note is mostly pedagogical. We hope to interest the reader to look into the question of evaluation of integrals and to see beautiful connections with many areas of Mathematics.

2. A blind request from `Mathematica`

A first direct request from `Mathematica` to evaluate W_n produces the answer

$$\frac{\sqrt{\pi}\texttt{Gamma}[-\frac{1}{2}+n]}{2\,\texttt{Gamma}[n]} \tag{2.1}$$

and the restriction `if` $\text{Re}[n] > \frac{1}{2}$. For the moment, assume that the reader is not acquainted with the `Gamma` function and ignore the specific answer. The restriction condition is clear: the integrand is a continuous function, but since the interval of integration is infinite, problems with the convergence of the integral can occur at infinity. For large x,

$$\frac{1}{(x^2+1)^n} \sim \frac{1}{x^{2n}}, \tag{2.2}$$

2000 *Mathematics Subject Classification.* 33-01.
Key words and phrases. Integrals, Wallis, π.

so W_n has the same convergence properties as

$$\int_a^\infty \frac{1}{x^{2n}}\,dx = \frac{1}{1-2n} x^{1-2n}\Big|_a^\infty \tag{2.3}$$

and this expression, when n is real, is finite precisely when $n > \frac{1}{2}$. In the case of $n \in \mathbb{C}$, this condition becomes $\text{Re}[n] > \frac{1}{2}$, as stated by `Mathematica`. The gamma function, which actually gives the value of W_n, is discussed in Section 5.

3. An empirical prediction of the answer

In this section we describe how to predict a (possible) expression for W_n. Using `Mathematica` one can obtain the value of W_n for any specific `fixed value` of n. For instance, the request for W_1 is written as

$$\texttt{Integrate}\Big[(x^2+1)^{-1}, \{x, 0, \texttt{Infinity}\}\Big], \tag{3.1}$$

and one obtains $W_1 = \dfrac{\pi}{2}$ almost immediately. This is the basic value

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}. \tag{3.2}$$

A similar request yields the values

$$W_2 = \frac{\pi}{4}, \quad W_3 = \frac{3\pi}{16}, \quad W_4 = \frac{5\pi}{32}, \quad W_5 = \frac{35\pi}{256}, \quad W_6 = \frac{63\pi}{512}. \tag{3.3}$$

The data above suggests that W_n is a rational multiple of π. This motivates the definition of

$$W_n^{(1)} = \frac{W_n}{\pi}. \tag{3.4}$$

and this yields $W_1^{(1)} = \frac{1}{2}$ and converts (3.3) into

$$W_2^{(1)} = \frac{1}{4}, \quad W_3^{(1)} = \frac{3}{16}, \quad W_4^{(1)} = \frac{5}{32}, \quad W_5^{(1)} = \frac{35}{256}, \quad W_6^{(1)} = \frac{63}{512}. \tag{3.5}$$

The denominators of the list (3.5) are all powers of 2. The can be extracted with the `Mathematical` command `Denominator[W1[n]]`. Indeed, the command

$$\texttt{D1}[n] = \texttt{Denominator[W1[n]]} \tag{3.6}$$

can be used to create the table

$$\texttt{Table[D1[n]:}\ \ \{\texttt{n,1,6}\ \}\ \texttt{]} \tag{3.7}$$

and obtain

$$\texttt{List}_1 = \{2, 4, 16, 32, 256, 512\}, \tag{3.8}$$

and the corresponding exponent is obtained by defining

$$\texttt{D2}[n] = \texttt{Log}[2, \texttt{D1}[n]], \tag{3.9}$$

which produces

$$\texttt{List}_2 = \{1, 2, 4, 5, 8, 9\}. \tag{3.10}$$

Now comes an important step in this process. One needs to `guess` an upper bound for the expression `D2`$[n]$. There is no rule for this, but observing the list (3.10) it is reasonable (after staring at data for some time) that a good guess is

$$\texttt{D2}[n] \leqslant 2n. \tag{3.11}$$

The reader should enlarge the list (3.10), say to include 100 values, and confirm the guess (3.11). This analysis motivates the definition

$$W_n^{(2)} = 2^{2n} W_n^{(1)}. \tag{3.12}$$

One expects $W_n^{(2)}$ to be an integer. Indeed, `Mathematica` gives the values

$$\texttt{List}_3 = \{2,\ 4,\ 12,\ 40,\ 140,\ 504\}. \tag{3.13}$$

Of course, the data above suggests that one could have multiplied by 2^{2n-1} in (3.12), but this is too fine of a point to make in the beginning of the guessing process. It is also convenient to check the guess that $W_n^{(2)}$ is an integer for larger values of n. This is indeed the case; for instance,

$$W_{100}^{(2)} = 45501766158845869932363908079137770791208336520308209468000.$$

The next step is to guess an expression for $W_n^{(2)}$. Starting with the (empirical) assumption that $W_n^{(2)}$ is an integer, one may try to obtain its prime factorization. `Mathematica` produces this factorization with the command

$$\texttt{FactorInteger[W2[n]]}. \tag{3.14}$$

Starting with small numbers, one gets

$$W_{10}^{(2)} = 97240 = 2^3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \tag{3.15}$$

and then moving to larger values

$$\begin{aligned} W_{100}^{(2)} = 2^5 \cdot 3 \cdot 5^3 \cdot 11 \cdot 13^2 \cdot 17 \cdot 37 \cdot \cdot 53 \cdot 59 \cdot 61 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131 \cdot \\ \cdot 137 \cdot 139 \cdot 149 \cdot 151 \cdot 157 \cdot 163 \cdot 167 \cdot 173 \cdot 179 \cdot 181 \cdot 191 \cdot 193 \cdot 197. \end{aligned}$$

At this point one needs to observe the factorization and try to see patterns. This is difficult when prime factorizations are produced. Therefore, we begin with small observations and see where this leads us to. In the first example, note that $W_{10}^{(2)}$ is divisible by primes near 20 (of course, 19 is missing) and $W_{100}^{(2)}$ is divisible by primes near 200. As a matter of fact, all primes between 101 and 197 appear in the factorization of $W_{100}^{(2)}$ (again $199 = 2 \cdot 100 - 1$ does not appear, but we ignore this observation for the moment). Now comes an important point: one needs to find an expression of n divisible by all the primes up to $2n$. A natural choice is $(2n)!$, leading to the definition

$$W_n^{(3)} = \frac{(2n)!}{W_n^{(2)}}. \tag{3.16}$$

(Using `Mathematica` one checks that $W_n^{(2)}/(2n)!$ produces reciprocals of integers, this explains the form of the quotient in $W_n^{(3)}$). The hope is that $W_n^{(3)}$ is a simpler function of n. of course, *there are no guarantees that this is true*. Continuing with the computed examples, we find that

$$W_{10}^{(3)} = 25019559936000 = 2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 19$$

and

$$\begin{aligned} W_{100}^{(3)} = & 173324671532880653580390144222695268724837440814422718670 \\ & 897057471243261927297008184971545217021822920829338577168170188 \\ & 779649359287574811 79\ 242181748240683975268904911610142218446731 \\ & 656044497681568678019974 2838\ 319321039350151609988997837566147295 17 \\ & 611415172671003993047040000000000\ 0000000000000000000000000000000000 \end{aligned}$$

with prime factorization

$$\begin{aligned} W_{100}^{(3)} = 2^{192} \cdot 3^{96} \cdot 5^{46} \cdot 7^{32} \cdot 11^{18} \cdot 13^{14} \cdot 17^{10} \cdot 19^{10} \cdot 23^8 \cdot 29^6 \cdot 31^6 \cdot 37^4 \cdot 41^4 \cdot 43^4 \cdot 47^4 \cdot 53^2 \\ \cdot 59^2 \cdot 61^2 \cdot 67^2 \cdot 71^2 \cdot 73^2 \cdot 79^2 \cdot 83^2 \cdot 89^2 \cdot 97^2 \cdot 199. \end{aligned}$$

From this single example we see that 199 stays as a factor, but all the primes between 101 and 200 have disappeared and that the primes near 100 have even exponents in the prime factorization. This motives the definition

$$W_n^{(4)} = \frac{W_n^{(3)}}{n!^2}. \tag{3.17}$$

`Mathematica` now produces

$$W_{100}^{(4)} = \frac{199}{100} \tag{3.18}$$

and it seems that the work above has payed off. Naturally a single example could be just a coincidence. The next example

$$W_{200}^{(4)} = \frac{399}{200} \tag{3.19}$$

gives us hope and suggests the final definition

$$W_n^{(5)} = \frac{n}{2n-1} W_n^{(4)}. \tag{3.20}$$

`Mathematica` can now be used to produce a list of the values of $W_n^{(5)}$. It shows that $W_n^{(5)} = 1$ for all computed values.

Now unwind all the definitions given here to see that an (empirical) expression for the Wallis integral in (1.1) is

$$W_n = \frac{\pi}{2^{2n-1}} \binom{2n-2}{n-1}. \tag{3.21}$$

4. A proof using recurrences

Now that we have produced a candidate for the integral (1.1) in the form (3.21) it is time to look for a rigorous proof. Given the definition

$$W_n = \int_0^\infty \frac{dx}{(x^2+1)^n}, \tag{4.1}$$

there are few options on how to proceed. In a lecture, the speaker stated `if you are stuck in a problem, integrate by parts`. Perhaps we should try that advise. Let $u = (x^2+1)^{-n}$ and $dv = dx$ to produce

$$W_n = \frac{x}{(x^2+1)^n}\Big|_0^\infty + 2n \int_0^\infty \frac{x^2\,dx}{(x^2+1)^{n+1}}. \tag{4.2}$$

In the new integral, since x^2+1 is the fundamental object, it is convenient to write $x^2 = (x^2+1) - 1$ to produce

$$\int_0^\infty \frac{x^2\,dx}{(x^2+1)^{n+1}} = \int_0^\infty \frac{dx}{(x^2+1)^n} - \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} = W_n - W_{n+1}. \tag{4.3}$$

Replacing in (4.2) and observing that the boundary terms vanishes, it leads to

$$W_{n+1} = \frac{2n-1}{2n} W_n. \tag{4.4}$$

This recurrence and the initial value $W_1 = \pi/2$ can be used to prove the evaluation (3.21) by induction.

On the other hand, the value guessed for W_n can be used to simplify the inductive proof. Define

$$T_n = \frac{2^{2n-1}}{\pi\binom{2n-2}{n-1}} W_n, \tag{4.5}$$

that is, divide the unknown expression W_n by the value guessed for it. Then (4.4) becomes

$$T_{n+1} = \left(\frac{2n-1}{2n} \cdot \frac{\pi\binom{2n-2}{n-1}}{2^{2n-1}} \cdot \frac{2^{2n+1}}{\pi\binom{2n}{n}} \right) T_n. \tag{4.6}$$

and simplifying the factor in braces leads to

$$T_{n+1} = T_n. \tag{4.7}$$

To prove $T_n \equiv 1$ is truly a one-line proof.

5. The gamma function

Motivated by the value given by `Mathematica`

$$W_n = \frac{\sqrt{\pi}\Gamma(-\frac{1}{2}+n)}{2\,\Gamma(n)}, \tag{5.1}$$

an online search of the gamma function produces the definition

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt \tag{5.2}$$

that can be interpreted as a simple example of the Laplace transform

$$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{5.3}$$

for the power function $f_x(t) = t^{x-1}$. Indeed, a simple scaling gives

$$\mathcal{L}f_x(s) = s^{-x}\Gamma(x). \tag{5.4}$$

The integral (5.2) converges for $x > 0$ (or $\operatorname{Re} x > 0$ if one takes $x \in \mathbb{C}$). Integration by parts gives the functional equation

$$\Gamma(x+1) = x\Gamma(x) \tag{5.5}$$

and an induction argument shows that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. Therefore, the gamma function interpolates the factorials. This function allows us to compute values like

$$\left(-\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-1/2}\,dt. \tag{5.6}$$

The change of variables $t = s^2$ gives

$$\int_0^\infty e^{-t}t^{-1/2}\,dt = 2\int_0^\infty e^{-s^2}\,ds = \int_{-\infty}^\infty e^{-s^2}\,ds. \tag{5.7}$$

The value of this last integral is $\sqrt{\pi}$, familiar to students from the elementary courses in Statistics. This yields the spectacular formula

$$\left(-\frac{1}{2}\right)! = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{5.8}$$

6. A trigonometric version

The change of variables $x = \tan\varphi$ gives

$$W_n = \int_0^{\pi/2} \cos^{2n-2}\varphi\,d\varphi. \tag{6.1}$$

This motivates the introduction of the integral

$$I_n = \int_0^{\pi/2} \cos^n\varphi\,d\varphi. \tag{6.2}$$

Observe that

$$I_n = \int_0^{\pi/2} \sin^n\varphi\,d\varphi. \tag{6.3}$$

by letting $\varphi \mapsto \pi/2 - \varphi$. It is convenient to double the interval of integration and work with

$$D_n = \int_0^{\pi} \sin^n\varphi\,d\varphi. \tag{6.4}$$

Integration by parts produces the recurrence

$$D_n = \frac{n-1}{n}D_{n-2} \tag{6.5}$$

and from here it follows that $D_0 = \pi$ determines D_{2n} and $D_1 = 2$ does the same for D_{2n+1}. Indeed, writing the recurrence (6.5) according to parity

$$D_{2n} = \frac{2n-1}{2n} D_{2n-2} \quad \text{and} \quad D_{2n+1} = \frac{2n}{2n+1} D_{2n-1}. \tag{6.6}$$

and iteration yields

$$D_{2n} = \pi \prod_{k=1}^{n} \frac{2k-1}{2k} \quad \text{and} \quad D_{2n+1} = 2 \prod_{k=1}^{n} \frac{2k}{2k+1}. \tag{6.7}$$

The inequalities $\sin^{2n+1} x \leqslant \sin^{2n} x \leqslant \sin^{2n-1} x$ shows that

$$D_{2n+1} \leqslant D_{2n} \leqslant D_{2n-1} \tag{6.8}$$

and therefore

$$1 \leqslant \frac{D_{2n}}{D_{2n+1}} \leqslant \frac{D_{2n-1}}{D_{2n+1}} \leqslant \frac{2n+1}{2n} \tag{6.9}$$

proving that

$$\lim_{n \to \infty} \frac{D_{2n}}{D_{2n+1}} = 1. \tag{6.10}$$

This produces one of the earliest analytic expressions for π:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdots \tag{6.11}$$

The reader will find in [**2, 5**] more information about π and the list [**1, 3, 4, 6, 7, 9**] [**8**] includes some of the many papers written about this number. We wish the reader happy hunting.

References

[1] T. Amdeberhan, O. Espinosa, V. Moll, and A. Straub. Wallis-Ramanujan-Schur-Feynman. *Amer. Math. Monthly*, 117:618–632, 2010.

[2] J. M. Borwein and P. B. Borwein. *Pi and the AGM- A study in analytic number theory and computational complexity*. Wiley, New York, 1st edition, 1987.

[3] J. Gurland. On Wallis' formula. *Amer. Math. Monthly*, 63:643–645, 1956.

[4] P. Levrie and W. Daems. Evaluating the probability integral using Wallis's product formula for π. *Amer. Math. Mon.*, 116:538–541, 2009.

[5] V. Moll. *Numbers and Functions. Special Functions for Undergraduates*, volume 65 of *Student Mathematical Library*. American Mathematical Society, 2012.

[6] T. J. Osler. The union of Vieta's and Wallis' product for pi. *Amer. Math. Monthly*, 106:774–776, 1999.

[7] P. Paule and V. Pillwein. Automatic improvements of Wallis' inequality. In T. Ida et al., editor, *SYNASC 2010, 12th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing*, pages 12–16. IEEE Computer Society, 2011.

[8] J. Wallis. *Arithmetica infinitorum*. Oxford, England, 1656.

[9] J. Wastlund. An elementary proof of the Wallis product formula for pi. *Amer. Math. Monthly*, 114:914–917, 2007.

Received 21 02 2023, revised 28 03 2023

Department of Mathematics,
Tulane University, New Orleans,
LA 70118,
USA