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A note on integration by parts

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Abstract. A systematic method to choose terms in the integration by parts formula is presented.

1. How to choose factors when integrating by parts

The evaluation of a definite integral using the method of integration by parts

$$\int u\,dv = uv - \int v\,du \tag{1.1}$$

is one of the basic techniques taught in the elementary courses. Naturally this requires to make a choice on the functions u and v. This is sometimes intuitively clear: if you want to integrate $x \sin x$, then choose $u = x$ and $dv = \sin x\,dx$. The other possible choice ($u = \sin x$ and $dv = x\,dx$) leads to the integration of $x^2 \cos x$ and this seems more complicated than the original one.

One of the authors has been involved in a project dealing with the table of integrals by I. S. Gradshteyn and I. M. Ryzhik [**1**]. It is part of this project to provide proofs of all the entries in this table. Several papers with these evaluations have appeared in this journal, starting with [**2**].

Many of the entries in table of integrals appear, in principle, to be suitable to the method of integration by parts. For example, entry 2.117.1 in I. S. Gradshteyn and I. M. Ryzhik [**1**]

$$\int \frac{dx}{x^n z^m} = -\frac{1}{(n-1)ax^{n-1}z^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z^m} \tag{1.2}$$

with $z = a + bx$ fits the profile. Some trial and error will convince the reader that it is not obvious is how to pick u and dv in (1.1), in order to prove (1.2).

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A natural solution to this question is the subject of this note. Begin by imposing the relations

$$u\,dv = \frac{dx}{x^n z^m} \quad \text{and} \quad uv = -\frac{1}{(n-1)ax^{n-1}z^{m-1}} \tag{1.3}$$

and hope that the term $v\,du$ will match the right-hand side of (1.2). Note that if we were not given this right-hand side to check, we would not know a priori to guess the form of uv. Dividing both equations in (1.3) one obtains an expression for dv/v free of u, namely

$$\frac{dv}{v} = -(n-1)\left[\frac{1}{x} - \frac{b}{a+bx}\right]. \tag{1.4}$$

Integration gives $v = z^{n-1}x^{1-n}$ and from the product uv in (1.3) it follows that

$$u = -\frac{1}{(n-1)az^{m+n-1}}. \tag{1.5}$$

With these choices of u and v, one verifies that

$$v\,du = \frac{b(m+n-1)}{(n-1)ax^{n-1}z^m} \tag{1.6}$$

and then integration by parts establishes the desired formula (1.2).

This technique is useful in the proofs of many entries of [**1**]. The reader is encouraged to try it.

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References

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