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A family of definite integrals

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Abstract. An investigation into a family of definite integrals containing log functions will be undertaken in this paper. It will be shown that Euler sums play an important part in the solution of these integrals and may be represented as a BBP type formula. In a special case we prove that the corresponding log integral can be represented as a linear combination of the product of zeta functions and the Dirichlet beta function.

1. Introduction Preliminaries and Notation

The motivation for this paper was inspired by the recent work of Henry and Moll [**4**]. In that paper the authors compiled a list of integrals related to special functions such as the Dirichlet beta function. In this paper we investigate a family of integrals with logarithmic integrand containing some parameters. It will be shown that the solution of this family of integrals may be expressed as a BBP- type representation and include some classical constants such as the Riemann zeta function and the Dirichlet beta function. In particular we investigation a family of integrals of the type

$$I\left(a,q\right)=\int_{x\in(\alpha,\beta)}\frac{\ln^{p}\left(x\right)}{1+x^{2}}\log\left(1+x^{2(2q+1)}\right)dx, \tag{1.1}$$

where $p\in\mathbb{N}_0$, $q\geqslant-\frac{1}{2}$ and for the two domains of $x\in(0,1)$ and $x\in(0,\infty)$. We shall represent the resulting integral (1.1) in closed form in terms of special functions including the Riemann zeta function and harmonic numbers. Other related papers dealing with Euler sums are [**7**], [**8**], [**9**] and the excellent books [**15**] and [**16**]. The following special functions will be used in the analysis of the integral (1.1). The polylogarithm function $\mathrm{Li}_p(z)$ is, for $|z|\leqslant 1$

$$\mathrm{Li}_p(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^p}. \tag{1.2}$$

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The well known result

$$\zeta(z)+\eta(z)=2\lambda(z)$$

connects the zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, with the alternating zeta function $\eta(z)$ and the odd zeta function $\lambda(z)$. The zeta function has a simple pole at $z=1$. The Dirichlet beta function, $\beta(z)$ or Dirichlet L function is given by, see Finch [**3**]

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^z};\ z>0 \tag{1.3}$$

where $\beta(2)=G$ is Catalan's constant. The Dirichlet beta function can be represented in powers of π at positive odd integer values of z, such that

$$\beta(2m+1)=\frac{(-1)^m E(2m)}{2^{2m+2}(2m)!}\pi^{2m+1}$$

where $E(\cdot)$ are the Euler numbers generated by

$$\frac{1}{\cosh z}=\frac{2e^z}{e^{2z}+1}=\sum_{n=0}^{\infty}\frac{E(n)\,z^n}{n!}.$$

The Dirichlet beta function can be analytically extended to the whole complex plane, has no singularities in the complex plane and is given by the functional equation

$$\beta(1-z)=\left(\frac{2}{\pi}\right)^z\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z).$$

The Euler beta function,

$$\begin{aligned} B(x,y) &= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},\ \text{for } \operatorname{Re}x>0,\ \operatorname{Re}y>0 \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{x-1}(\cos\theta)^{y-1}\,d\theta, \end{aligned} \tag{1.4}$$

and the Gamma function,

$$\Gamma(z)=\int_0^{+\infty}t^{z-1}e^{-t}dt.$$

For real values of x, $\psi(x)$ is the digamma (or psi) function defined by

$$\psi(x):=\frac{d}{dx}\{\log\Gamma(x)\}=\frac{\Gamma'(x)}{\Gamma(x)}\ .$$

We know that for $n\geqslant 1,\ \psi(n+1)-\psi(1)=H_n$ with $\psi(1)=-\gamma$, where γ is the Euler Mascheroni constant and $\psi(n)$ is the digamma function. The polygamma function

$$\psi^{(k)}(z)=\frac{d^k}{dz^k}\{\psi(z)\}=(-1)^{k+1}\,k!\sum_{r=0}^{\infty}\frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1)=\psi^{(k)}(z)+\frac{(-1)^k\,k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$H_z^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(z+1), \; z \neq \{-1,-2,-3,...\}.$$

$$= \frac{(-1)^m}{m!}\int_0^1 \frac{(1-t^z)}{1-t}\ln^m t \, dz \tag{1.5}$$

We expect that integrals of the type (1.1) may be represented by Euler sums and therefore in terms of special functions such as the Riemann zeta function. A search of the current literature has found some examples for the representation of the log-log integrals in terms of Euler sums, see [**1**] and [**17**]. The following papers [**10**], [**11**], [**12**] and [**13**] also examined some integrals in terms of Euler sums. Some examples will be given highlighting specific cases of the integrals, some of which are not amenable to a computer mathematical package.

2. Analysis of Integrals

Consider the following.

THEOREM 1. *Let $(p,q) \in \mathbb{N}_0$, the following integral,*

$$I(p,q) = \int_0^1 \frac{\ln^p(x)}{1+x^2}\ln\left(1+x^{2(2q+1)}\right)dx \tag{2.1}$$

$$= (-1)^p \, p! \sum_{n\geqslant 1}(-1)^{n+1} H_n \sum_{j=0}^{2q}\frac{(-1)^j}{(2(2q+1)n+2j+1)^{p+1}} \tag{2.2}$$

where H_n are harmonic numbers.

PROOF. For $x \in (0,1)$ and from

$$\ln\left(1+x^{2(2q+1)}\right) = \sum_{n\geqslant 1}(-1)^{n+1}\frac{x^{2(2q+1)n}}{n}$$

it follows that

$$\frac{\ln\left(1+x^{2(2q+1)}\right)}{1+x^2} = \sum_{n\geqslant 1}(-1)^{n+1} H_n \sum_{j=0}^{2q}(-1)^j x^{2(2q+1)n+2j}$$

and therefore

$$\frac{\ln^p(x)\ln\left(1+x^{2(2q+1)}\right)}{1+x^2} = \sum_{n\geqslant 1}(-1)^{n+1} H_n \sum_{j=0}^{2q}(-1)^j x^{2(2q+1)n+2j}\ln^p(x).$$

Integrating both sides for $x \in (0,1)$, we have

$$\int_0^1 \frac{\ln^p(x)\ln\left(1+x^{2(2q+1)}\right)}{1+x^2}dx = \sum_{n\geqslant 1}(-1)^{n+1} H_n \sum_{j=0}^{2q}(-1)^j \int_0^1 x^{2(2q+1)n+2j}\ln^p(x)\,dx$$

$$= (-1)^p\, p! \sum_{n\geqslant 1} (-1)^{n+1}\, H_n \sum_{j=0}^{2q} \frac{(-1)^j}{(2\,(2q+1)\,n+2j+1)^{p+1}}$$

and this is the BBP-type representation for the integral (2.1). □

The next corollary deals with an alternative representation for the integral (2.1).

Corollary 1. For $p \in \mathbb{N}_0$,and $q \in \mathbb{R} \geqslant -\frac{1}{2}$ then

$$I\,(p,q) = \frac{(-1)^p\, p!}{2^{2p+2}} \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{1}{4}} - H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{3}{4}} \right). \tag{2.3}$$

Also

$$I\,(p,q) = \int_0^{\frac{\pi}{4}} \ln^p(\tan\theta)\,(\ln 2 - (2q+1)\ln(1+\cos(2\theta)))\,d\theta$$

$$+ \int_0^{\frac{\pi}{4}} \ln^p(\tan\theta) \ln\left(\sum_{r=0}^{q} \binom{2q+1}{2r} (\cos(2\theta))^{2r} \right) d\theta, \tag{2.4}$$

where $H^{(p+1)}_{n\left(q+\frac{1}{2}\right)}$ are harmonic numbers of order $p+1$.

Proof. A Taylor series expansion of

$$\ln\left(1 + x^{2(2q+1)}\right) = \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} x^{2(2q+1)n}$$

allows us to write

$$\begin{aligned} I\,(p,q) &= \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \sum_{r\geqslant 0} (-1)^r \int_0^1 x^{2r+2(2q+1)n} \ln^p(x)\,dx \\ &= (-1)^p\, p! \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \sum_{r\geqslant 0} \frac{(-1)^r}{(2r+1+2\,(2q+1)\,n)^{p+1}} \end{aligned}$$

$$= (-1)^p\, p! \sum_{n\geqslant 1} \frac{(-1)^{n+1}(-1)^p}{np!2^{2p+2}} \left(\psi^{(p)}\left(n\left(q+\frac{1}{2}\right) + \frac{3}{4} \right) - \psi^{(p)}\left(n\left(q+\frac{1}{2}\right) + \frac{1}{4} \right) \right).$$

From the identity (1.5) we obtain the required identity

$$I\,(p,q) = \frac{(-1)^p\, p!}{2^{2p+2}} \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} \left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{1}{4}} - H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{3}{4}} \right).$$

The integral (2.4) is obtained by substituting $x = \sin\theta$ and then using the trigonometric identity

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}.$$

□

REMARK 1. For $(p,q) \in \mathbb{N}_0$, we see from (2.2) and (2.3) the remarkable Euler sum identity

$$\begin{aligned} &\sum_{n\geqslant 1}(-1)^{n+1}H_n\sum_{j=0}^{2q}\frac{(-1)^j}{(2(2q+1)n+2j+1)^{p+1}} \\ =\; &\frac{1}{2^{2p+2}}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{1}{4}}-H^{(p+1)}_{n\left(q+\frac{1}{2}\right)-\frac{3}{4}}\right)\end{aligned}$$

The next corollary deals with a negative exponent in the second log term of the integral (2.1).

COROLLARY 2. Let $a = 2(2q+1)$ then

$$\begin{aligned} I(p,-a) &= \int_0^1 \frac{\ln^p(x)}{1+x^2}\ln\left(1+x^{-2(2q+1)}\right)dx \\ &= I(p,a) - a\begin{cases} \frac{(-1)^m E(2m)\pi^{2m+1}}{2^{2(m+1)}}, & \text{for } p=2m-1,\ m\in\mathbb{N} \\ -(2m+1)!\beta(2m+2), & \text{for } p=2m,\ m\in\mathbb{N}\end{cases} \end{aligned} \tag{2.5}$$

where $E(\cdot)$ are the Euler, secant or Zig numbers given as A000364 in the On-line Encyclopedia of Integer sequences [**6**], and $\beta(\cdot)$ are the Dirichlet Beta functions (1.3), and where $G = \beta(2)$ is known as Catalan's constant.

PROOF. Now

$$\begin{aligned} I(p,-a) &= \int_0^1 \frac{\ln^p(x)}{1+x^2}\ln\left(1+x^{-a}\right)dx \\ &= \int_0^1 \frac{\ln^p(x)}{1+x^2}\left(\ln(1+x^a)-a\ln(x)\right)dx \\ &= I(p,a) - a\int_0^1 \frac{\ln^{p+1}(x)}{1+x^2}dx. \end{aligned}$$

to evaluate the integral

$$J(p+1) = \int_0^1 \frac{\ln^{p+1}(x)}{1+x^2}dx \tag{2.6}$$

we employ a Taylor series expansion of the integrand, integrate, for $x \in (0,1)$ and we have

$$\int_0^1 \frac{\ln^{p+1}(x)}{1+x^2} = \frac{(-1)^{p+1}(p+1)!}{2^{2(p+2)}}\left(\zeta\left(p+2,\frac{1}{4}\right)-\zeta\left(p+2,\frac{3}{4}\right)\right)$$

where $\zeta(p+2,\cdot)$ is the classical Hurwitz zeta function defined by

$$\zeta(p,b)=\sum_{n\geqslant 0}\frac{1}{(n+b)^p},\ \operatorname{Re}p>1.$$

Considering the odd and even values of p, we have

$$\int_0^1\frac{\ln^{p+1}(x)}{1+x^2}=\begin{cases}\frac{(-1)^m E(2m)\pi^{2m+1}}{2^{2(m+1)}}, \text{ for } p=2m-1,\ m\in\mathbb{N}\\ -(2m+1)!\beta(2m+2)\,, \text{ for } p=2m,\ m\in\mathbb{N}\end{cases},$$

and (2.5) is obtained. $\square$

Some examples follow.

Example 1. For $p\in\mathbb{N},\ q=-\frac{1}{2}$,

$$\begin{aligned}I\left(p,-\tfrac{1}{2}\right) &= \ln 2\int_0^1\frac{\ln^p x}{1+x^2}dx=J(p)\ln 2\\ &= (-1)^p\,p!\ln 2\sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)^{p+1}}.\\ &= (-1)^p\,p!\beta(p+1)\ln 2.\end{aligned}$$

For $p\in 0,\ q=-\frac{1}{4}$,

$$\begin{aligned}I\left(0,-\tfrac{1}{4}\right) &= \int_0^1\frac{\ln(1+x)}{1+x^2}dx=\frac{\pi}{8}\ln 2\\ &= \frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H_{\frac{n}{4}-\frac{1}{4}}-H_{\frac{n}{4}-\frac{3}{4}}\right),\end{aligned}$$

this integral is also evaluated by Bradley [**2**]. Also from (2.5)

$$\int_0^1\frac{\ln\left(1+x^{-1}\right)}{1+x^2}dx=\frac{\pi}{8}\ln 2+G$$

where $G=\sum\limits_{n\geqslant 0}\frac{(-1)^{n+1}}{(2n+1)^2}.$ is Catalan's constant. It is interesting to note that G may be represented as the sum of positive terms, in the following way. Let

$$X=\int_0^1\frac{\tanh^{-1}\sqrt{1-x^2}}{\sqrt{1-x^2}}dx$$

and by the substitution $x=\sin\theta$

$$X=\int_0^{\frac{\pi}{2}}\tanh^{-1}(\cos\theta)\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\ln(1+\cos\theta)-\ln(1-\cos\theta)\right)d\theta,$$

these integrals are evaluated by Bradley [**2**], so that

$$X=\frac{1}{2}\left(2G-\frac{\pi}{2}\ln 2+2G+\frac{\pi}{2}\ln 2\right)=2G.$$

Now by a Taylor series expansion

$$X=\sum_{n\geqslant 0}\int_0^{\frac{\pi}{2}}\frac{\cos^{2n+1}\theta}{2n+1}d\theta=\sum_{n\geqslant 0}\frac{B\left(\frac{1}{2},n+1\right)}{2\left(2n+1\right)},$$

in which case the identity

$$G=\frac{1}{4}\sum_{n\geqslant 0}\frac{B\left(\frac{1}{2},n+1\right)}{2n+1}=\sum_{n\geqslant 0}\frac{\left(-1\right)^{n+1}}{\left(2n+1\right)^{2}.}=L\left(2,\chi\right)$$

follows, where χ is the non-principal character of modulo 4 and $B\left(\cdot,\cdot\right)$ is the classical Euler Beta integral (1.4).

For $p=0,q=0$

$$\begin{aligned}I\left(0,0\right) &= \int_0^1\frac{\ln\left(1+x^2\right)}{1+x^2}dx=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan^2\theta\right)d\theta=-2\int_0^{\frac{\pi}{4}}\ln\left(\cos\theta\right)d\theta\\ &= \frac{\pi}{2}\ln 2-G.\end{aligned}$$

We also have,

$$\int_0^1\frac{\ln\left(1+x^{-2}\right)}{1+x^2}dx=\frac{\pi}{2}\ln 2+G.$$

For $p=0,q=\frac{1}{4}$

$$\begin{aligned}I\left(0,\tfrac{1}{4}\right) &= \int_0^1\frac{\ln\left(1+x^3\right)}{1+x^2}dx=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan^3\theta\right)d\theta\\ &= \frac{\pi}{8}\ln 2+\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)-\frac{5}{3}G\\ &= \frac{1}{4}\sum_{n\geqslant 1}\frac{\left(-1\right)^{n+1}}{n}\left(H_{\frac{3n}{4}-\frac{1}{4}}-H_{\frac{3n}{4}-\frac{3}{4}}\right)\end{aligned}$$

We also have,

$$\int_0^1\frac{\ln\left(1+x^{-3}\right)}{1+x^2}dx=\frac{\pi}{8}\ln 2+\frac{\pi}{3}\ln\left(2+\sqrt{3}\right)+\frac{4}{3}G.$$

For $p=0,q=\frac{1}{2}$

$$I\left(0,\tfrac{1}{2}\right)=\int_0^1\frac{\ln\left(1+x^4\right)}{1+x^2}dx=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan^4\theta\right)d\theta$$

and using the identity $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$, we have

$$\begin{aligned}
I\left(0,\tfrac{1}{2}\right) &= \int_0^{\frac{\pi}{4}} \left(\ln\left(3+\cos\left(4\theta\right)\right) - 2\ln\left(1+\cos\left(2\theta\right)\right)\right) d\theta \\
&= -\frac{\pi}{4}\ln\left(6-4\sqrt{2}\right) - 2G + \frac{\pi}{4}\ln 4 \\
&= \frac{\pi}{4}\ln\left(6+4\sqrt{2}\right) - 2G \\
&= \frac{1}{4}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H_{\frac{n}{2}-\frac{1}{4}} - H_{\frac{n}{2}-\frac{3}{4}}\right),
\end{aligned}$$

and

$$\int_0^1 \frac{\ln\left(1+x^{-4}\right)}{1+x^2}dx = \frac{\pi}{4}\ln\left(6+4\sqrt{2}\right) + 2G.$$

For $p=0, q=1$

$$\begin{aligned}
I\left(0,1\right) &= \int_0^1 \frac{\ln\left(1+x^6\right)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan^6\theta\right) d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(\ln\left(2+6\cos^2\left(2\theta\right)\right) - 3\ln\left(1+\cos\left(2\theta\right)\right)\right) d\theta \\
&= \frac{\pi}{4}\ln 2 + \frac{\pi}{2}\ln\left(\frac{3}{2}\right) - 3G + \frac{3\pi}{4}\ln 2 \\
&= \frac{\pi}{2}\ln 6 - 3G \\
&= \sum_{n\geqslant 1}(-1)^{n+1} H_n\left(\frac{1}{6n+1} - \frac{1}{6n+3} + \frac{1}{6n+5}\right),
\end{aligned}$$

and

$$\int_0^1 \frac{\ln\left(1+x^{-6}\right)}{1+x^2}dx = \frac{\pi}{2}\ln 6 + 3G.$$

For $p=0, q=2$

$$
\begin{aligned}
I(0,2) &= \int_0^1 \frac{\ln\left(1+x^{10}\right)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan^{10}\theta\right)d\theta \\
&= \int_0^{\frac{\pi}{4}} \ln\left(2+20\cos^2(2\theta)+10\cos^4(4\theta)\right)d\theta - 5\left(G-\frac{\pi}{2}\ln 2\right) \\
&= -\frac{\pi}{4}\ln\left(\frac{8}{5}\left(9-4\sqrt{5}\right)\right) - 5\left(G-\frac{\pi}{2}\ln 2\right) \\
&= \frac{\pi}{4}\ln\left(\frac{20}{5+8\beta}\right) - 5G \\
&= \sum_{n\geqslant 1}(-1)^{n+1}H_n\left(\frac{1}{10n+1}-\frac{1}{10n+3}+\frac{1}{10n+5}-\frac{1}{10n+7}+\frac{1}{10n+9}\right),
\end{aligned}
$$

where $\beta = \frac{1-\sqrt{5}}{2}$, and

$$\int_0^1 \frac{\ln\left(1+x^{-10}\right)}{1+x^2}dx = \frac{\pi}{4}\ln\left(\frac{20}{5+8\beta}\right) + 5G.$$

For $p=1, q=0$

$$
\begin{aligned}
I(1,0) &= \int_0^1 \frac{\ln x\ \ln\left(1+x^2\right)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln(\tan\theta)\ln\left(1+\tan^2\theta\right)d\theta \\
&= -2\int_0^{\frac{\pi}{4}} \ln(\tan\theta)\ln(\cos\theta)\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \left(2\ln^2(\cos\theta) - 2\ln(\cos\theta)\ln(\sin\theta)\right)d\theta \\
&= -\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n}{(2n+1)^2} \\
&= -G\ln 2 - \frac{\pi^3}{64} - \frac{\pi}{16}\ln^2 2 \\
&\quad -2i\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right) + \frac{5\pi^2}{192}\ln 2 - \frac{1}{48}\ln^3 2 - \frac{35}{64}\zeta(3)\right).
\end{aligned}
$$

From Lewin ([**5**], p.164,296) we have that

$$\mathrm{Re}\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) = \frac{1}{48}\ln^3 2 + \frac{35}{64}\zeta(3) - \frac{5\pi^2}{192}\ln 2$$

and therefore

$$I\left(0,1\right)=2\,\mathrm{Im}\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)-G\ln 2-\frac{\pi^3}{64}-\frac{\pi}{16}\ln^2 2.$$

Sofo and Nimbran [**14**] have shown that the imaginary part of the trilogarithm:

$$\begin{aligned} W\left(3\right) &:= \mathrm{Im}\,\mathrm{Li}_3\left(\frac{1\pm i}{2}\right)=\sum_{n\geqslant 1}\frac{\sin\left(\frac{n\pi}{4}\right)}{2^{\frac{n}{2}}\,n^3} \\ &= \sum_{n\geqslant 1}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{2}{\left(4n-3\right)^3}+\frac{2}{\left(4n-2\right)^3}+\frac{1}{\left(4n-1\right)^3}\right), \end{aligned}$$

and finally we have

$$I\left(0,1\right)=2W\left(3\right)-G\ln 2-\frac{\pi^3}{64}-\frac{\pi}{16}\ln^2 2.$$

Also, we have

$$\int_0^1\frac{\ln x\ \ln\left(1+x^{-2}\right)}{1+x^2}dx=2W\left(3\right)-G\ln 2-\frac{9\pi^3}{64}-\frac{\pi}{16}\ln^2 2.$$

For $p=1, q=-\frac{1}{4}$

$$I\left(1,-\tfrac{1}{4}\right)=\int_0^1\frac{\ln x\ \ln\left(1+x\right)}{1+x^2}dx=\frac{3\pi}{32}\ln^2 2-2G\ln 2+\frac{11\pi^3}{128}-3W\left(3\right),$$

and

$$\int_0^1\frac{\ln x\ \ln\left(1+x^{-1}\right)}{1+x^2}dx=\frac{3\pi}{32}\ln^2 2-2G\ln 2-\frac{5\pi^3}{128}-3W\left(3\right).$$

For $p=2, q=-\frac{3}{8}$

$$\begin{aligned} I\left(2,-\tfrac{3}{8}\right) &= \int_0^1\frac{\ln^2\left(x\right)\ \ln\left(1+\sqrt{x}\right)}{1+x^2}dx=\int_0^{\frac{\pi}{4}}\ln^2\left(\tan\theta\right)\ln\left(1+\sqrt{\tan\theta}\right)d\theta \\ &= \frac{\pi^3}{64}\ln\left(39202+27720\sqrt{2}\right)-\frac{\pi^2}{3}G-2\beta\left(4\right) \\ &= \frac{1}{32}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H^{(3)}_{\frac{n}{8}-\frac{1}{4}}-H^{(3)}_{\frac{n}{8}-\frac{3}{4}}\right), \end{aligned}$$

and

$$\int_0^1\frac{\ln^2\left(x\right)\ \ln\left(1+x^{-\frac{1}{2}}\right)}{1+x^2}dx=\frac{\pi^3}{64}\ln\left(39202+27720\sqrt{2}\right)-\frac{\pi^2}{3}G+\beta\left(4\right).$$

For $p=4, q=-\frac{1}{4}$

$$\begin{aligned} I\left(4,-\tfrac{1}{4}\right) &= \int_0^1 \frac{\ln^4(x)\ \ln(1+x)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln^4(\tan\theta)\ln(1+\tan\theta)\,d\theta \\ &= \frac{7\pi^4}{30}G+\frac{5\pi^5}{128}\ln 2+2\pi^2\beta(4)-48\beta(6) \\ &= \frac{3}{128}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H^{(5)}_{\frac{n}{4}-\frac{1}{4}}-H^{(5)}_{\frac{n}{4}-\frac{3}{4}}\right), \end{aligned}$$

and

$$\int_0^1 \frac{\ln^4(x)\ \ln\left(1+x^{-1}\right)}{1+x^2}dx = \frac{7\pi^4}{30}G+\frac{5\pi^5}{128}\ln 2+2\pi^2\beta(4)+60\beta(6).$$

For $p=2, q=0$

$$\begin{aligned} I(2,0) &= \int_0^1 \frac{\ln^2 x\ \ln\left(1+x^2\right)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \ln^2(\tan\theta)\ln\left(1+\tan^2\theta\right)d\theta \\ &= -2\int_0^{\frac{\pi}{4}} \ln^2(\tan\theta)\ln(\cos\theta)\,d\theta \\ &= 2\sum_{n\geqslant 1}\frac{(-1)^{n+1}H_n}{(2n+1)^3} = \frac{1}{32}\sum_{n\geqslant 1}\frac{(-1)^{n+1}}{n}\left(H^{(3)}_{\frac{n}{2}-\frac{1}{4}}-H^{(3)}_{\frac{n}{2}-\frac{3}{4}}\right) \\ &= \frac{7\pi\zeta(3)}{8}+\frac{\pi^3}{8}\ln 2-6\beta(4), \end{aligned}$$

also

$$\int_0^1 \frac{\ln^2 x\ \ln\left(1+x^{-2}\right)}{1+x^2}dx = \frac{7\pi\zeta(3)}{8}+\frac{\pi^3}{8}\ln 2+6\beta(4).$$

In the next theorem we consider the integral (2.1) on the positive half plane $x \geqslant 0$.

THEOREM 2. *For* $p\in\mathbb{N},\ q\geqslant -\frac{1}{2}$

$$Y(p,q) = \int_0^\infty \frac{\ln^p(x)}{1+x^2}\ln\left(1+x^{2(2q+1)}\right)dx \tag{2.7}$$

$$= \begin{cases} \frac{(2q+1)(-1)^m E(2m)\pi^{2m+1}}{2^{2m+1}}, & \textit{for } p=2m-1,\ m\in\mathbb{N} \\ 2I(2m,q)+2(2q+1)(2m+1)!\beta(2m+2), & \textit{for } p=2m,\ m\in\mathbb{N} \end{cases}, \tag{2.8}$$

where

$$f(p,q,x)=\frac{\ln^{p}(x)}{1+x^{2}}\ln\left(1+x^{2(2q+1)}\right), \tag{2.9}$$

$E(2m)$ are the Euler or secant numbers and $\beta(2m+2)$ are the Dirichlet Beta functions.

Proof. We begin with

$$Y(p,q)=\int_{0}^{\infty}\frac{\ln^{p}(x)}{1+x^{2}}\ln\left(1+x^{2(2q+1)}\right)dx=\int_{0}^{\infty}f(p,q,x)\,dx$$

and put

$$Y(p,q)=\int_{0}^{\infty}f(p,q,x)\,dx=\int_{0}^{1}f(p,q,x)\,dx+\int_{1}^{\infty}f(p,q,x)\,dx,$$

we notice that $f(p,q,x)$ is continuous, bounded and differentiable on the interval $x\in(0,1]$, with $\lim_{x\to0^{+}}f(p,q,x)=\lim_{x\to1}f(p,q,x)=0$, then

$$\int_{0}^{\infty}f(p,q,x)\,dx=\int_{0}^{1}f(p,q,x)\,dx+(-1)^{p}\int_{0}^{1}\frac{\ln^{p}(t)}{1+t^{2}}\left(\ln\left(1+t^{2(2q+1)}\right)-(2q+1)\ln(t)\right)dt$$

where we have made the transformation $xt=1$. Collecting integrals, we have

$$\begin{aligned}Y(p,q)&=(1+(-1)^{p})\,I(p,q)-2(-1)^{p}(2q+1)\int_{0}^{1}\frac{\ln^{p+1}(x)}{1+x^{2}}dx\\&=(1+(-1)^{p})\,I(p,q)-2(-1)^{p}(2q+1)\,J(p+1)\end{aligned}$$

where $J(p+1)$ is given by (2.6). We can consider the two separate cases of p odd and even such that

$$Y(p,q)=\begin{cases}\frac{(2q+1)(-1)^{m}E(2m)\pi^{2m+1}}{2^{2m+1}}, \text{ for } p=2m-1,\ m\in\mathbb{N}\\ 2I(2m,q)+2(2q+1)(2m+1)!\beta(2m+2), \text{ for } p=2m,\ m\in\mathbb{N}\end{cases},$$

and the proof is finished. □

Some examples follow.

Example 2. For $p=2,q=-\frac{3}{8}$

$$\begin{aligned}Y\left(2,-\tfrac{3}{8}\right)&=\int_{0}^{\infty}\frac{\ln^{2}(x)\ \ln(1+\sqrt{x})}{1+x^{2}}dx\\&=\frac{\pi^{3}}{32}\ln\left(39202+27720\sqrt{2}\right)-\frac{2\pi^{2}}{3}G-\beta(4).\end{aligned}$$

For $p = 4, q = -\frac{1}{4}$

$$\begin{aligned} Y\left(4,-\tfrac{1}{4}\right) &= \int_0^\infty \frac{\ln^4(x)\ \ln(1+x)}{1+x^2}dx \\ &= \frac{7\pi^4}{15}G + \frac{5\pi^5}{64}\ln 2 + 4\pi^2\beta(4) + 24\beta(6). \end{aligned}$$

For $p = 3, q = 1$

$$Y(3,1) = \int_0^\infty \frac{\ln^3(x)\ \ln\left(1+x^6\right)}{1+x^2}dx = \frac{15\pi^5}{32}.$$

Concluding Remarks. We have carried out a systematic study of a family of integrals containing $\log - \log$ functions in terms of Euler sums. We believe most of our results are new in the literature and have given many examples some of which are not amenable to a mathematical computer package.

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